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On Ternary Quadratic Diophantine Equation $3(x^2+y^2)-5xy+x+y+1=15z^2$

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Abstract

The ternary quadratic equation representing non-homogeneous cone given by $3(x^2+y^2)-5xy+x+y+1=15z^2$ is analyzed for its non-zero distinct integer points on it. The different patterns of integer points satisfying the cone under consideration are obtained. A few interesting relations between the solutions and special number patterns are presented.

Keywords: **Ternary non-homogeneous quadratic, integral solutions**

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1. Introduction

The ternary quadratic Diophantine equations offer an unlimited field for research due to their variety [1, 20]. For an extensive review of various problems, one may refer [2 – 19]. This communication concerns with yet another interesting ternary quadratic equation $3(x^2+y^2)-5xy+x+y+1=15z^2$ representing a cone for determining its infinitely many non-zero integral points. Also, a few interesting relations among the solutions are presented.

NOTATIONS:

$T_{m,n}$ - Polygonal number of rank n with size m.

P_n^m - Pyramidal number of rank n with size m.

$Ct_{m,n}$ - Centered Polygonal number of rank n with size m.

$CP_{m,n}$ - Centered Pyramidal number of rank n with size m.

Pr_n - Pronic number of rank n.

S_n - Star number of rank n.

2.METHOD OF ANALYSIS:

The ternary quadratic diophantine equation to be solved for its non-zero distinct integral solutions is

$$3(x^2+y^2)-5xy+x+y+1=15z^2 \quad (1)$$

The substitution of linear transformations

$$x=u+v,\ y=u-v \quad (2)$$

in (1) leads to

$$(u+1)^2+11v^2=15z^2 \quad (3)$$

Different patterns of solutions of (1) are presented below.

2.1 PATTERN : 1

Write 15 as

$$15=(2+i\sqrt{11})(2-i\sqrt{11}) \quad (4)$$

Assume z= a^2+11b^2 (5)

where a, b are non-zero distinct integers.

Using (4) & (5) in (3), and applying the method of factorization,define

$$\left[(u+1)+i\sqrt{11}v\right]=(2+i\sqrt{11})(a+i\sqrt{11}b)^2 \quad (6)$$

Equating the real imaginary parts, we have

$$u=u(a,b)=2a^2-22b^2-22ab-1$$

$$v=v(a,b)=a^2-11b^2+4ab$$

Substituting the above values of u & v in equation (2), the values of x and y are given by

$$x=x(a,b)=3a^2-33b^2-18ab-1 \quad (7)$$

$$y=y(a,b)=a^2-11b^2-26ab-1 \quad (8)$$

Thus (5), (7) & (8) represent a non-zero distinct integral solutions of (1) in two parameters.

PROPERTIES:

$1. x(a,b) - 6t_{3,a-1} \equiv -4 \pmod{15}$

$2. 6[x(a,1) - y(a,1)] - 2S_a \equiv -14 \pmod{60}$

$3. z(1,2^n) - 45J_{2n} + 6J_n + 2j_n = 6$

2.2 PATTERN 2:

Instead of (4), we write 15 as

$$15 = \frac{(7+i\sqrt{11})(7-i\sqrt{11})}{4} \quad (9)$$

Following the procedure presented above, the corresponding non-zero distinct integer solutions to (1) are obtained as

$$x = x(a,b) = 4a^2 - 44b^2 - 4ab - 1 \quad (10)$$

$$y = y(a,b) = 3a^2 - 33b^2 - 18ab - 1 \quad (11)$$

along with (5)

PROPERTIES:

$1. x(2^n,1) = 12J_{2n} - 6J_n - 2j_n - 41.$

$2. x(2^n,1) - y(2^n,1) = j_{2n} + 7j_n + 21J_n - 12.$

$3. 3x(a,1) = 2S_a - 137.$

2.3 PATTERN : 3

Equation (3) can be written as

$$(u+1)^2 = 15z^2 - 11v^2 \quad (12)$$

Introducing the linear transformations,

$$z = X + 11T \ \& \ v = X + 15T \quad (13)$$

in (12), we get

$$X^2 = 165T^2 + \left[\frac{u+1}{2}\right]^2 \quad (14)$$

which is satisfied by

$$\left.\begin{aligned} T(r,s) &= 2rs \\ u(r,s) &= 330r^2 - 2s^2 - 1 \\ X(r,s) &= 165r^2 + s^2 \end{aligned}\right\} \quad (15)$$

Substituting the values of (15) in (13) and using (2), the corresponding integer solutions of (1) are given by

$$x = x(r,s) = 495r^2 - s^2 + 30rs - 1 \quad (16)$$

$$y = y(r,s) = 165r^2 - 3s^2 - 30rs - 1 \quad (17)$$

$$z = z(r,s) = 165r^2 + s^2 + 22rs \quad (18)$$

PROPERTIES:

$1. x(1,s) - 3y(1,s) + 2t_{12,s} \equiv -4 \pmod{68}$

$2. \frac{1}{2}[x(1,n) - y(1,n)]2t_{3,n-1} \equiv 10 \pmod{31}$

$3. x(2^n,1) - y(2^n,1) = 990J_{2n} + 12J_n + 4j_n + 227$

Note:

In addition to (13), one may also consider the linear transformations $z = X - 11T, v = X - 15T$ .Following the method presented above , different set of solutions are obtained.

2.4 PATTERN : 4

Consider (3) as

$$(u+1)^2 - 4v^2 = 15(z^2 - v^2) \quad (19)$$

Write (19) in the form of ratio as

$$\frac{(u+1)+2v}{z+v} = \frac{15(z-v)}{(u+1)-2v} = \frac{A}{B}, B \neq 0$$

which is equivalent to the following two equations

$$B(u+1) + (2B-A)v - Az = 0$$

$$A(u+1) + (15B-2A) - 15Bz = 0$$

On employing the method of cross multiplication, we get

$$\left.\begin{aligned} u &= 2A^2 - 30AB + 30B^2 - 1 \\ v &= A^2 - 15B^2 \end{aligned}\right\} \quad (20)$$

$$z = A^2 - 4AB + 15B^2 \quad (21)$$

Substituting the values of u and v from (20) in (2), the non-zero distinct integral values of x, y are given by

$$\left.\begin{aligned} x &= x(A,B) = 3A^2 + 15B^2 - 30AB - 1 \\ y &= y(A,B) = A^2 + 45B^2 - 30AB - 1 \end{aligned}\right\} \quad (22)$$

Thus (21) and (22) represent non-zero distinct integral solutions of (1) in two parameters.

PROPERTIES:

Each of the following expressions represents a Nasty number

$i) 2[3z(1,\beta) - y(1,\beta)]$

$ii) 2[y(a,-a) + 1]$

$2. 18x(1,\beta) - 26y(1,\beta) + 60z(1,\beta) = 96$

Note:

(19) can also be expressed in the form of ratio in two different ways as follows.

$$(i). \frac{(u+1)+2v}{3(z+v)} = \frac{5(z-v)}{(u+1)-2v} = \frac{A}{B}, B \neq 0.$$

$$(ii). \frac{(u+1)+2v}{5(z+v)} = \frac{3(z-v)}{(u+1)-2v} = \frac{A}{B}, B \neq 0.$$

Repeating the analysis as above the corresponding integer solutions along with properties are presented below.

Solutions of (i):

$x = x(A,B) = -9A^2 - 5B^2 + 30AB - 1$

$y = y(A,B) = -3A^2 - 15B^2 + 30AB - 1$

$z = z(A,B) = 3A^2 + 5B^2 - 4AB$

PROPERTIES :

1. $x(1,2\alpha - 5\alpha^2) - y(1,2\alpha - 5\alpha^2) - 14t_{3,(-5\alpha^2+2\alpha-1)}$ is a perfect square.

2. $5x(1,n) + z(1,n) \equiv 27 \pmod{34}$

3. $x(1,n) + y(1,n) + z(1,n) + 22t_{3,n-1} \equiv 19 \pmod{22}$

Solutions of (ii):

$x = x(A,B) = 15A^2 + 3B^2 - 30AB - 1$

$y = y(A,B) = 5A^2 + 9B^2 - 30AB - 1$

$z = z(A,B) = -5A^2 - 3B^2 + 4AB$

PROPERTIES :

1. $x(2^n,1) = 45J_{2n} - 45J_n - 15j_n + 17$

2. $x(2^n,1) - y(2^n,1) = 30J_{2n} + 4$

3. $6x(a,1) - 15S_a \equiv 87 \pmod{90}$

3.CONCLUSION:

In this paper, we have presented six different patterns of non-zero distinct integer solutions of the non-homogeneous cone given by $3(x^2 + y^2) - 5xy + x + y + 1 = 15z^2$.To conclude, one may search for patterns of non-zero integer distinct solutions and their corresponding properties for other choices of ternary quadratic diophantine equations.

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