On Ternary Quadratic Diophantine Equation

\[ 3(x^2 + y^2) - 5xy + x + y + 1 = 15z^2 \]

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Abstract

The ternary quadratic equation representing non-homogeneous cone given by \( 3(x^2 + y^2) - 5xy + x + y + 1 = 15z^2 \) is analyzed for its non-zero distinct integer points on it. The different patterns of integer points satisfying the cone under consideration are obtained. A few interesting relations between the solutions and special number patterns are presented.

Keywords: Ternary non-homogeneous quadratic, integral solutions

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1. Introduction

The ternary quadratic Diophantine equations offer an unlimited field for research due to their variety [1, 20]. For an extensive review of various problems, one may refer [2 – 19]. This communication concerns with yet another interesting ternary quadratic equation \( 3(x^2 + y^2) - 5xy + x + y + 1 = 15z^2 \) representing a cone for determining its infinitely many non-zero integral points. Also, a few interesting relations among the solutions are presented.

NOTATIONS:

- \( T_{m,n} \) - Polygonal number of rank \( n \) with size \( m \).
- \( P^n_{m} \) - Pyramidal number of rank \( n \) with size \( m \).
- \( C_{m,n} \) - Centered Polygonal number of rank \( n \) with size \( m \).
- \( CP_{m,n} \) - Centered Pyramidal number of rank \( n \) with size \( m \).
- \( Pr_{n} \) - Pronic number of rank \( n \).
- \( S_n \) - Star number of rank \( n \).

2.METHOD OF ANALYSIS:

The ternary quadratic diophantine equation to be solved for its non-zero distinct integral solutions is

\[ 3(x^2 + y^2) - 5xy + x + y + 1 = 15z^2 \]  \hspace{1cm} (1)

The substitution of linear transformations \( v = u + v \), \( y = u - v \) in (1) leads to

\[ (u + 1)^2 + 11v^2 = 15z^2 \]  \hspace{1cm} (3)

Different patterns of solutions of (1) are presented below.

2.1 PATTERN : 1

Write 15 as

\[ 15 = (2 + i\sqrt{11})(2 - i\sqrt{11}) \]  \hspace{1cm} (4)

Assume \( z = a^2 + 11b^2 \) \hspace{1cm} (5)

where \( a, b \) are non-zero distinct integers. Using (4) & (5) in (3), and applying the method of factorization, define

\[ (u + 1) + i\sqrt{11}v = (2 + i\sqrt{11})(a + i\sqrt{11}b)^2 \]  \hspace{1cm} (6)

Equating the real imaginary parts, we have

\[ u = u(a,b) = 2a^2 - 22b^2 - 22ab - 1 \]  \hspace{1cm} (7)

\[ v = v(a,b) = a^2 - 11b^2 + 4ab \]  \hspace{1cm} (8)

Substituting the above values of \( u \) & \( v \) in equation (2), the values of \( x \) and \( y \) are given by

\[ x = x(a,b) = 3a^2 - 33b^2 - 18ab - 1 \]  \hspace{1cm} (9)

\[ y = y(a,b) = a^2 - 11b^2 - 26ab - 1 \]  \hspace{1cm} (10)
Thus (5), (7) & (8) represent a non-zero distinct integral solutions of (1) in two parameters.

**PROPERTIES:**
1. $x(a,b) - 6t_{a,b-1} = -4$(mod15)
2. $6[x(a,1) - y(a,1)] - 2S_a = -14$(mod60)
3. $z(1,2^n) - 45J_{2n} + 6J_n + 2j_n = 6$

**2.2 PATTERN 2:**

Instead of (4), we write 15 as

$$15 = \frac{(7 + i\sqrt{11})(7 - i\sqrt{11})}{4}$$

(9)

Following the procedure presented above, the corresponding non-zero distinct integer solutions to (1) are obtained as

$$x = x(a,b) = 4a^2 - 44b^2 - 4ab - 1$$

(10)

$$y = y(a,b) = 3a^2 - 33b^2 - 18ab - 1$$

(11)

along with (5)

**PROPERTIES:**
1. $x(2^n,l) = 12J_{2n} - 6J_n - 2j_n - 41$.
2. $x(2^n,l) - y(2^n,l) = j_{2n} + 7j_n + 21J_n - 12$.
3. $3x(a,l) = 2S_a - 137$.

**2.3 PATTERN : 3**

Equation (3) can be written as

$$(u + 1)^2 = 15z^2 - 11v^2$$

(12)

Introducing the linear transformations,

$$z = X + 11T \text{ & } v = X + 15T$$

(13)

in (12), we get

$$X^2 = 165T^2 + \left[\frac{u + 1}{2}\right]^2$$

(14)

which is satisfied by

$$T(r,s) = 2rs$$

$$u(r,s) = 330r^2 - 2s^2 - 1$$

(15)

$$X(r,s) = 165r^2 + s^2$$

Substituting the values of (15) in (13) and using (2), the corresponding integer solutions of (1) are given by

$$x = x(r,s) = 495r^2 - s^2 + 30rs - 1$$

(16)

$$y = y(r,s) = 165r^2 - 3s^2 - 30rs - 1$$

(17)

$$z = z(r,s) = 165r^2 + s^2 + 22rs$$

(18)

**PROPERTIES:**

1. $x(1,s) - 3y(1,s) + 2t_{1,2,s} = -4$(mod68)
2. $\frac{1}{2}[x(1,n) - y(1,n)] = 10$(mod31)
3. $x(2^n,l) - y(2^n,l) = 990J_{2n} + 12J_n + 4j_n + 227$

**Note:**

In addition to (13), one may also consider the linear transformations $z = X - 11T, v = X - 15T$. Following the method presented above, different set of solutions are obtained.

**2.4 PATTERN : 4**

Consider (3) as

$$(u + 1)^2 - 4v^2 = 15(z^2 - v^2)$$

(19)

Write (19) in the form of ratio as

$$\frac{u + 1 + 2v}{z + v} = \frac{15(z - v)}{(u + 1 - 2v)} = \frac{A}{B}, B \neq 0$$

which is equivalent to the following two equations

$$B(u + 1) + (2B - A)v - Az = 0$$

$$A(u + 1) + (15B - 2A)v - 15Bz = 0$$

On employing the method of cross multiplication, we get

$$u = 2A^2 - 30AB + 30B^2 - 1$$

$$v = A^2 - 15B^2$$

(20)

Substituting the values of $u$ and $v$ from (20) in (2), the non-zero distinct integral values of $x$, $y$ are given by

$$x = x(A,B) = 3A^2 + 15B^2 - 30AB - 1$$

$$y = y(A,B) = A^2 + 45B^2 - 30AB - 1$$

(21)

Thus (21) and (22) represent non-zero distinct integral solutions of (1) in two parameters.

**PROPERTIES:**

Each of the following expressions represents a Nasty number

1) $2[3z(1,\beta) - y(1,\beta)]$
2) $2[y(a,-a) + 1]$

$2.18x(1,\beta) - 26y(1,\beta) + 60z(1,\beta) = 96$

**Note:**

(19) can also be expressed in the form of ratio in two different ways as follows.

(i) $\frac{(u + 1) + 2v}{3(z + v)} = \frac{5(z - v)}{(u + 1 - 2v)} = \frac{A}{B}, B \neq 0.$

(ii) $\frac{(u + 1) + 2v}{5(z + v)} = \frac{3(z - v)}{(u + 1 - 2v)} = \frac{A}{B}, B \neq 0.$
Repeating the analysis as above the corresponding integer solutions along with properties are presented below.

**Solutions of (i):**

\[
\begin{align*}
x &= x(A, B) = -9A^2 - 5B^2 + 30AB - 1 \\
y &= y(A, B) = -3A^2 - 15B^2 + 30AB - 1 \\
z &= z(A, B) = 3A^2 + 5B^2 - 4AB
\end{align*}
\]

**PROPERTIES:**

- \(1.x(1, 2\alpha - 5\alpha^2) - y(1, 2\alpha - 5\alpha^2) - 14t_{3,(-5\alpha^2+2\alpha-1)}^t\) is a perfect square.
- \(2.5x(1, n) + z(1, n) \equiv 27 \pmod{34}\)
- \(3.x(1, n) + y(1, n) + z(1, n) + 10t_{3, n-1}^t \equiv 19 \pmod{22}\)

**Solutions of (ii):**

\[
\begin{align*}
x &= x(A, B) = 15A^2 + 3B^2 - 30AB - 1 \\
y &= y(A, B) = 5A^2 + 9B^2 - 30AB - 1 \\
z &= z(A, B) = -5A^2 - 3B^2 + 4AB
\end{align*}
\]

**PROPERTIES:**

- \(1.x(2^n, 1) = 45J_{2n} - 45J_{n} - 15j_{n} + 17\)
- \(2.x(2^n, 1) - y(2^n, 1) = 30J_{2n} + 4\)
- \(3.6x(a, l) - 15S_{a} \equiv 87 \pmod{90}\)

**3. CONCLUSION:**

In this paper, we have presented six different patterns of non-zero distinct integer solutions of the non-homogeneous cone given by

\[3(x^2 + y^2) - 5xy + x + y + 1 = 15z^2.\]

To conclude, one may search for patterns of non-zero integer distinct solutions and their corresponding properties for other choices of ternary quadratic diophantine equations.

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