Expansive Mapping Theorems in Cone metric spaces

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Abstract

In this paper, we prove some common fixed point theorems for expansive mapping in the setting of cone metric space. Our results extend Huang, Zhu and XI Wen [16].

Keywords: Cone metric space, Fixed Point Theorem, Common Fixed Point, Expansive mapping

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1. Introduction

Fixed point theory is one of the important topics in development of non linear analysis. Also, fixed point theory has been used effectively in many other branches of science, such as Chemistry, Biology, Economics, Computer science, Engineering and many others.

Very recently, Huang and Zhang [1] introduced the concept of cone metric spaces which generalized the concept of the metric spaces, replacing the the set of real numbers by an ordered Banach space, and obtained some fixed point theorems for contractive mappings using normality of the cone. The results in [1] were generalized by Sh. Rezapour and Hamlbarani [2] omitted the assumption of normality on the cone, which is a milestone in cone metric space. The existing literature of fixed point theory contains many results enunciating fixed point theorems for self-mapping in metric and Banach spaces. The study of fixed point theorems in such spaces is followed by some other mathematicians, see for instance [3-4,6-8,9-10, 13].In 1984, Wang et al. [12] introduced the concept of expanding mappings and proved some fixed point theorems in complete metric spaces. In 1992, Daffer and Kaneko [5] defined expanding condition for a pair of mappings and proved some common fixed point theorems for two mappings in complete metric spaces. Huang, Xi et al.[11] define expanding mappings in the setting of cone metric spaces analogous to expanding mappings in complete metric space and also extend a result of Daffer and Kaneko [5] for two mappings to the setting of cone metric spaces. Sarla Chouhan and Neeraj, M. [14] proved fixed point theorem for expansive mapping in cone metric spaces. The result in [14] generalized by Tiwari, S.K. et al.[15] and proved some common fixed point theorems for expansive type mappings in complete cone metric spaces. In 2014, Rajesh Shrivasta, et al. [16] proved fixed point theorems in cone metric spaces by using expansion mapping.

In this manuscript, the known results [11] are extended and generalized common fixed point theorems for expansive mapping in cone metric spaces.

2.1 Preliminaries

Let E be a real Banach space and P be a subset of E. P is called a cone if and only if:

(i) P is closed, non – empty and $P \neq \{0\}$,

(ii) $ax + by \in P$ for all $x, y \in$ P and non – negative real number $a, b \in R$,

(iii) $x \in$ P and - $\in$ P => $x = 0$ <=> P ∩ (-P) = {0}.

Given a Cone $P \subset E$, we define a partial ordering $\leq$ on E with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x \ll$ y if $y -$ $\in$ int P, int P denotes the interior of P. The cone P is called normal if there is a number $K > 0$ such that for all , $y \in E$,

$$0 \leq x \leq y \text{ implies } \| x \| \leq \| y \|.$$

The least positive number satisfying the above is called the normal constant P, while $x \ll y$ stands for $y - x \in intP$.We also note that the relations $int\ P + int\ P \dot{\subseteq} intP$ ($\lambda > 0$) always hold true.

Definition 2.1 [1] Let X be a non – empty set. Suppose the mapping $d: X\ x\ X \rightarrow E$ satisfies

(d_1) $0 < d(x, y)$ for all $x, y \in$ X and $d(x) = 0$ iff $= y$;

(d_2) $d(x, y) = d(y, x)$ for all $x, y \in$ X;

(d_3) $d(x, y) \leq$ d $(x, z) + d(z, y)$ for all$x, y \in$ X

Then d is called a cone metric on X, and (X, d) is called a cone metric space [1]. It is obvious that cone metric spaces generalize metric space.

Example 2.2 [1] Let $E = R^2$, $P = \{(x, y) \in E : x, y \geq 0\}$, $X = R$ and $d : X\, x\, X \rightarrow E$ defined by$d(x, y) = (| x - y |, \alpha | x, y |)$, where $\alpha \geq 0$ is a constant. Then (X, d) is a cone metric space.

Definition 2.3 [1] Let (X, d) be a cone metric space, $x \in X$ and $\{x_n\}_{n \geq 1}$ a sequence in X. Then,

(i) $\{x_n\}_{\geq 1}$ converges to x whenever for every $c \in E$ with o $\ll$ c , there is a natural number N such that d $(x_n, x) \ll$c for all $n \geq N$. We denote this by $\lim_{n\rightarrow\infty} x_n = x$ or $x_n \rightarrow x, (n \rightarrow \infty)$

(ii) $\{x_n\}_{n \geq 1}$ is said to be a Cauchy sequence if for every $c \in E$ with $o \ll c$, there is a natural number N such that d $(x_n, x_m) \ll$ c for all $n, m \geq$ N.

(iii) (X, d) is called a complete cone metric space if every Cauchy sequence in X is converge.

Definition 2.4 [11] Let (X, d) be cone metric space and $T: X \rightarrow X$. Then T is called a expansive mapping, if for every $x, y \in X$ there exist a number $k > 1$ such that

$$d(Tx,Ty) \geq kd(x,y)$$

Definition 2.5 [1] Let (X,d) be cone metric space and P be a cone in real Banach space E, if

(i) $a \in P$ and $a \ll c$ for some $k \in [0,1]$. Then $a = 0$.

(ii) $u \leq v, v \ll w$, then $u \ll w$.

Lemma 2.6[11] Let (X,d) be cone metric space and $\{x_n\}_{n \geq 1}$ a sequence in X. If there exist$k \in [0,1]$ such that

$$d(x_{n+1},x_n) \leq kd(x_n,x_{n-1}), n = 1,2 \dots \dots \dots.$$

Then $\{x_n\}_{n \geq 1}$ is a Cauchy sequence in X.

3. Main Results

The results we will give are generalization of theorem 2.1 and 2.2 of Huang, Xi. Zhu, C., and Wen, XI[11].

Theorem 3.1 Let (X, d) be a cone metric space with respect to a cone P containing in a real Banach space E. Let T_1, T_2 be any two surjection self maps of X satisfying

$$d(T_1x,T_2y) \geq a_1d(x,y) + a_2d(x,T_1x) + a_3d(y,T_2y) \dots\dots\dots\dots\dots\dots\dots \text{(3.1.1)}$$

for each $x, y \in X, x \pm y$, where$a_1, a_2, a_3 \geq 0$ *and* $a_1 + a_2 + a_3 > 1$. Then T_1 and T_2 have an unique common fixed point.

Proof: Let x_0 be an arbitrary point in X. Since T_1 and T_2 surjective mappings, there exist points $x_1 \in T_1^{-1}(x_0)$ and $x_2 \in T_2^{-1}(x_1)$ that is $T_1(x_1) = x_0$ and $T_2(x_2) = x_1$. In this way, we define the sequence $\{x_n\}$ with $x_{2n+1} \in T_1^{-1}(x_{2n})$ and $x_{2n+2} \in T_2^{-1}(x_{2n+1})$.

$$\text{i.e. } x_{2n} = T_1 x_{2n+1} \text{ for } n = 0, 1, 2 \dots\dots\dots\dots\dots\dots\dots\dots \quad \text{(3.1.2)}$$

$$x_{2n+1} = T_2 x_{2n+2} \text{ for } n = 0, 1, 2 \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \quad \text{(3.1.3)}$$

Note that, if $x_{2n} = x_{2n+1}$ for some $n \geq 0$, then x_{2n} is fixed point of T_1 and T_2. Now putting $x = x_{2n+1}$ and $y = x_{2n+2}$ from (3.1.1), we have

$$d(T_1x_{2n+1},T_2x_{2n+2}) = a_1d(x_{2n+1},x_{2n+2}) + a_2d(x_{2n+1}, T_1 x_{2n+1}) + a_3d(x_{2n+2},T_2 x_{2n+2})$$

$$d(x_{2n}, x_{2n+1}) \geq a_1d(x_{2n+1},x_{2n+2}) + a_2d(x_{2n+1},x_{2n}) + a_3d(x_{2n+2}x_{2n+1})$$

$$d(x_{2n+1},x_{2n+2}) \leq \frac{1-a_1}{a_1+a_3} d(x_{2n}, x_{2n+1}) \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \quad \text{(3.1.4)}$$

Where $h = \left[\frac{1-a_1}{a_1+a_3}\right] < 1$, [as $a_1 + a_2 + a_3 > 1$]

In general

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$d(x_{2n}, x_{2n+1}) \leq h\, d(x_{2n-1}, x_{2n})$

$d(x_{2n}, x_{2n+1}) \leq h^{2n} d(x_{2n-1}, x_{2n})$.. (3.1.5)

So for every positive integer p, we have

$d(x_{2n}, x_{2n+p}) \leq d(x_{2n,} x_{2n+1}) + d(x_{2n+1}\, x_{2n+2}) + \ldots\ldots\ldots + d(x_{2n+p-1,} x_{2n+p})$

$\leq (h^{2n} + h^{2n+1} + \ldots\ldots\ldots\ldots + h^{2n+p-1})\, d(x_0, x_1)$

$= h^{2n}(1 + h + h^2 + \cdots \ldots \ldots \ldots . + h^{2n+p-1}) d(x_0, x_1)$

$< \frac{h^{2n}}{1-h} d(x_0, x_1)$.. (3.1.6)

Therefore, by Lemma 2.6, $\{x_{2n}\}$ is a Cauchy sequence, which is complete space in Xthere exist $x^* \epsilon X$ such that $x_{2n\to} x^*$.Since T_1 is Surjective map ,there exist a point y in X $s$uch that

$y \epsilon T_1^{-1}(x^*)$. i.e. $x^* = T_1(y)$.. (3.1.7)

Now consider

$d(x_{2n,} x^*) = d(T_1 x_{2n+1,} y)$

$\leq a_1 d(x_{2n+1}, y) + a_2 d(x_{2n+1,} T_1 x_{2n+1}) + a_3 d(y, T_1 y)$

$d(x^*, x^*) \geq a_1 d(x^*, y) + a_2 d(x^*, x^*) + a_3 d(y, x^*)$

$d(x^*, x^*) \geq a_1 d(x^*, y) + a_2 d(x^*, x^*) + a_3 d(y, x^*)$

$\Rightarrow \quad 0 \geq (a_1 + a_3) d(x^*, y)$

$\Rightarrow d(x^*, y) = 0$, as $(a_1 + a_3) > 0$

$\Rightarrow x^* = y$ ....(3.1.8)

Hence x^* is a fixed point of $T_{1.}$ as $T_1 y = x^* = y$. Now if z be another fixed point of T_1 . i.e. $T_1 z = z$.

Then

$d(x^*, z) = d(T_1 x^*, T_1 z)$

$\geq a_1 d(x^*, z) + a_2 d(T_1 x^*, T_1 z) + a_3 d(z, T_1 z)$

$= 0$

$d(x^*, z) = 0$ and by proposition 2.5 (i).Thus $x^* = z$. Therefore T_1 has a unique fixed point.

.Similarly it can be established that $T_2x^* = x^*$. Hence $T_1x^* = x^* = T_2x^*$.Thus x^* is the common fixed point of T_1 and T_2. These completed the proof of the theorem.

Remark 3.2 If we take $a_2 = a_3 = 0$ and $a_1 = k$ in theorem 3.1 , then we can obtain the following corollary

Corollary 3.3 Let (X, d) be a cone metric space with respect to a cone P containing in a real Banach space E and Let $T_{1,}$ T_2 be any two surjection self maps of X satisfying

$$d(T_1x, T_2y) \geq kd(x, y) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3.1.9)$$

for each $x, y \in X, x \pm y$, where $k \geq 0$. Then T_1 and T_2 have an unique common fixed point.

Theorem 3.4 Let (X, d) be a cone metric space with respect to a cone P containing in a real Banach space E. Let $T_{1,}$ T_2 be any two surjection self maps of X satisfying

$$d(T_1x, T_2y) \geq ku \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3.4.1)$$

Where $u = u(x, y) \in \{d(x, y), d(x, T_1x), d(y, T_2y)\}$

for each $x, y \in X, x \pm y, k > 1$. Then T_1 and T_2 have an unique common fixed point.

Proof: Let x_0 be an arbitrary point in X. Since T_1 and T_2 surjective mappings, there exist points $x_1 \in T_1^{-1}(x_0)$ and $x_2 \in T_2^{-1}(x_1)$ that is $T_1(x_1) = x_0$ and $T_2(x_2) = x_1$. In this way, we define the sequence $\{x_n\}$ with $x_{2n+1} \in T_1^{-1}(x_{2n})$ and $x_{2n+2} \in T_2^{-1}(x_{2n+1})$.

$$\text{i.e. } x_{2n} = T_1\, x_{2n+1} \text{ for } n = 0, 1, 2 \ldots\ldots\ldots\ldots \quad (3.4.2)$$

$$x_{2n+1} = T_2x_{2n+2} \text{ for } n = 0, 1, 2 \ldots\ldots\ldots\ldots \quad (3.4.3)$$

Note that, if $x_{2n} = x_{2n+1}$ for some $n \geq 0$, then x_{2n} is fixed point of T_1 and T_2. Now putting $x = x_{2n+1}$ and $y = x_{2n+2}$ from (3.4.1), we have

$$d(x_{2n}, x_{2n+1}) = d(T_1x_{2n+1,}T_2x_{2n+2})$$

$$\geq ku$$

Where $u \in \{d(x_{2n+1}, x_{2n+2}), d(x_{2n+1}, T_1\, x_{2n+1}), d(x_{2n+2,}T_2\, x_{2n+2})\}$

$i.e\ u \in \{d(x_{2n+1}, x_{2n+2}), d(x_{2n+1}, x_{2n}), d(x_{2n+2,}\, x_{2n+1})\}$

$If\ u = d(x_{2n}, x_{2n+1})$, then

$d(x_{2n}, x_{2n+1}) \geq d(x_{2n}, x_{2n+1})$ which is impossible since $k > 1$.

Also if $u = d(x_{2n+1}, x_{2n+2})$, then

$$d(x_{2n}, x_{2n+1}) \geq kd(x_{2n+1}, x_{2n+2})$$

$i.e$

$d(x_{2n+1}, x_{2n+2}) \leq qd(x_{2n}, x_{2n+1})$ where $q = \frac{1}{k}$ and $q < 1$ Continuing in this way we get,

$d(x_{2n+1}, x_{2n+2}) \leq q^n d(x_0, x_1)$……………………………………………….(3.4.4)

Hence for all $n, m \in N, n < m$, we have

$$d(x_{2m}, x_{2n}) \leq d(x_{2m}, x_{2m+1}) + d(x_{2m+1}, x_{2m+2}) + \dots + d(x_{2n+1}, x_{2n})$$

$$\leq (q^{2m} + q^{2m+1} + \dots + q^{2n})\, d(x_0, x_1)$$

. $$< \frac{q^{2n}}{1-q}\, d(x_0, x_1) \dots\dots (3.4.5)$$

Therefore, by Lemma 2.6, $\{x_{2n}\}$ is a Cauchy sequence, which is complete space in Xthere exist $x^* \epsilon X$ such that $x_{2n \to} x^*$.Since T_1 is Surjection map ,there exist a point y in X $s$uch that

$y \epsilon T_1^{-1}(x^*)$. i.e. $x^* = T_1(y)$……………………………………………………. (3.4.6)

Now consider

$$d(x_{2n,} x^*) = d(T_1 x_{2n+1,} y)$$
$$\geq ku$$

Where $u \in \{d(x_{2n+1}, y), d(x_{2n+1}, T_1 x_{2n+1}), d(y, T_1 y)\}$

$i.e$ $u \in \{d(x_{2n+1}, y), d(x_{2n+1}, x_{2n}), d(y, x^*)\}$

As $n \to \infty, d(x_{2n,} x^*) \to 0$, also $d(x_{2n+1}, x_{2n}) \to 0$,this implies that $kd(x^*, y) \to 0$.

This implies that$x^* = y$. Hence x^* is a fixed point of $T_{1.}$ as $T_1 y = x^* = y$. Now if z be another fixed point of T_1 . i.e. $T_1 z = z$.

Then

$d(x^*, z) = d(T_1 x^*, T_1 z) \geq ku$ where

$u \in \{d(x^*, z), d(x^*, T_1 x_{2n+1}), d(z, T_1 z)\}$

$i.e$ $u \in \{d(x^*, x^*), d(x^*, x^*), d(x^*, x^*)\}$ as $k > 1$, $d(x^*, z) = 0$ and by proposition 2.5 (i).Thus $x^* = z$.

Therefore T_1 has a unique fixed point .Similarly it can be established that $T_2 x^* = x^*$.

Hence $T_1 x^* = x^* = T_2 x^*$.Thus x^* is the common fixed point of T_1 and T_2. These completed the proof of the theorem.

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