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# Integer Solutions of the Binary Quadratic Equation $x^2-5xy+y^2+33x=0$

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**Abstract:**

The binary quadratic equation $x^2-5xy+y^2+33x=0$ is studied for it's non-trivial integral solutions. The recurrence relations satisfied by the solutions x and y are given. A few interesting properties among the solutions are presented.

***Keywords*** *: Binary quadratic equation, Integer solutions.*

**MSC subject classification: 11D09**.

## 1. INTRODUCTION

The binary quadratic Diophantine equations (both homogeneous and non homogeneous) are rich in variety $[1-6]$.In [7-23] the binary quadratic non-homogeneous equations representing hyperbolas respectively are studied for their non-zero integral solutions.

However, in [12] it is shown that the hyperbola represented by $3x^2+xy=14$ has only finite number of integral points. These results have motivated us to search for infinitely many non-zero integral solutions of yet another interesting binary quadratic equation given by $x^2-5xy+y^2+33x=0$.The recurrence relations satisfied by the solutions x and y are given. Also a few interesting properties among the solutions are exhibited.

## 2. METHOD OF ANALYSIS:

The Diophantine equation representing the binary quadratic equation

$$x^2-5xy+y^2+33x=0 \tag{1}$$

is satisfied by the following non-zero distinct integer pairs

$(11,11),\ (11,44),\ (-33,-165),\ (176,44),\ (-725,-165)$.

However, we have a pattern of solutions for (1), which is illustrated below:

**2.1 Solving (1) for y:**

$$y=\frac{5x\pm\sqrt{25x^2-4(x^2+33x)}}{2} \tag{2}$$

Let $x=2X$ (3)

Then $y=5X\pm\sqrt{21X^2-66X}$ (4)

Assuming $21X^2-66X=\beta^2$ (5)

(4) Becomes $y=5X\pm\beta$ (6)

Arranging (5), we have

$$(21X-33)^2=21\beta^2+33^2 \tag{7}$$

By taking $21X-33=S$ (8)

we have the Pellian equation

$$S^2=21\beta^2+33^2 \tag{9}$$

Now, consider the Pellian equation

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$$S^2 = 21\beta^2 + 1 \tag{10}$$

whose general solution $\left(\tilde{S}_n, \tilde{\beta}_n\right)$ is given by

$$\tilde{S}_n = \frac{1}{2}\left[\left(55+12\sqrt{21}\right)^{n+1} + \left(55-12\sqrt{21}\right)^{n+1}\right]$$

$$\tilde{\beta}_n = \frac{1}{2\sqrt{21}}\left[\left(55+12\sqrt{21}\right)^{n+1} - \left(55-12\sqrt{21}\right)^{n+1}\right], \quad n = 0,1,2....$$

Thus, the general solution $\left(S_n, \beta_n\right)$ of (9) is obtained as

$$S_n = \frac{33}{2}\left[\left(55+12\sqrt{21}\right)^{n+1} + \left(55-12\sqrt{21}\right)^{n+1}\right] \tag{11}$$

$$\beta_n = \frac{33}{2\sqrt{21}}\left[\left(55+12\sqrt{21}\right)^{n+1} - \left(55-12\sqrt{21}\right)^{n+1}\right], \quad n = 0,1,2.... \tag{12}$$

In view of (3), (8) and (11), we have

$$x_n = \frac{22}{7}\left(\frac{f}{2}+1\right) \tag{13}$$

where $f = \left[\left(55+12\sqrt{21}\right)^{n+1} + \left(55-12\sqrt{21}\right)^{n+1}\right]$ (14)

Again in view of (3), (6) and (12), we have

$$y_n = \frac{55}{7}\left(\frac{f}{2}+1\right) \pm 33\left(\frac{g}{2\sqrt{21}}\right) \tag{15}$$

where $g = \left[\left(55+12\sqrt{21}\right)^{n+1} - \left(55-12\sqrt{21}\right)^{n+1}\right]$ (16)

Our aim is to get integer solutions to (1), which is obtained for $n = 0,2,4,....$

Hence we have

$$x_{2n} = \frac{22}{7}\left(\frac{F}{2}+1\right) \tag{17}$$

$$y_{2n} = \frac{55}{7}\left(\frac{F}{2}+1\right) \pm 33\left(\frac{G}{2\sqrt{21}}\right) \tag{18}$$

where $F = \left[\left(55+12\sqrt{21}\right)^{2n+1} + \left(55-12\sqrt{21}\right)^{2n+1}\right]$ and (19)

$$G = \left[\left(55+12\sqrt{21}\right)^{2n+1} - \left(55-12\sqrt{21}\right)^{2n+1}\right] \tag{20}$$

$$n = 0,1,2....$$

Equations (17) and (18) together will give the distinct integral solutions of (1).

The above values of $x_{2n}$ and $y_{2n}$ satisfy respectively the following recurrence relations.

$$x_{2n+4} - 12098x_{2n+2} + x_{2n} = -38016, \tag{21}$$

$$y_{2n+4} - 12098y_{2n+2} + y_{2n} = -95040, \tag{22}$$

$$n = 0,1,2....$$

A few numerical examples are given below:

| n | $x_{2n}$ | $y_{2n}$ |
|---|---|---|
| 0 | 176 | 44,836 |
| 1 | 2091056 | 436436,10018844 |
| 2 | 25297557296 | 5279907644,121207878836 |

**2.2 Some relations satisfied by the solutions (17) and (18) are as follows**:

1. Both the values of x, y is positive and even
2. $y_{2n} - x_{2n} \equiv 0 (\bmod 66)$
3. $x_n + y_n \equiv 0 (\bmod 11)$

4.The following expressions are nasty number

(i) $150 x_{2n+1} - 60 y_{2n+1}$

(ii) $66[7(5x_{2n+1} - 2y_{2n+1})^2 - 44]$

5. $7(5x_{2n+1} - 2y_{2n+1})^2 - 44 = 0 (\bmod 11)$

**3. Conclusion**

We have solved the equation (1) using the method of solvable for y. One may use the method of solvable for x to get the solution to (1) and obtain the corresponding properties.

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