Cube Difference Labeling Of Some Cyclerelated Graphs

G.Amuda, S.Meena
Lecturer in Mathematics, Govt., college for women(A),Kumbakonam,Tamil Nadu,India.
Associate Professor of Mathematics, Govt.,Arts and science college, Chidambaram.

Abstract
A graph \( G = (V, E) \) with \( p \) vertices and \( q \) edges is said to admit cube difference labeling, if there exists a bijection \( f: V(G) \rightarrow \{0,1,2,\ldots,p-1\} \) such that the induced function \( f^*: E(G) \rightarrow \mathbb{N} \) given by \( f^*(uv) = |f(u)^3 - f(v)^3| \) for every \( uv \in E(G) \) are all distinct. A graph which admits cube difference labeling is called cube difference graph. In this paper we prove that some classes of graphs like Double Triangular Snake, Umbrella Graph, \( P_n(Q_{sn}) \) graph, \( C_n(Q_{sn}) \) graphs are cube difference graphs.

Keywords: Cube difference labeling, Cube difference graph.

INTRODUCTION
All graphs in this paper are simple finite undirected and nontrivial graph \( G = (V, E) \) with vertex set \( V \) and the edge set \( E \). For graph theoretic terminology, we refer to Harary [2]. A dynamic survey on graph labeling is regularly updated by Gallian [3] and it is published by Electronic Journal of Combinatorics. Vast amount of literature is available on different types of graph labeling and more than 1000 research papers have been published so far in past three decades. The square difference labeling is previously defined by V. Ajitha, S. Arumugam and K. A. Germina [1]. The concept of cube difference labeling was first introduced by J. Shiama and it was proved in [7] that many standard graphs like \( P_n \), \( C_n \), complete graphs, ladder, lattice grids, wheels, comb, star graphs, crown, dragon, coconut trees and shell graphs are cube difference Labeling. And also some references [4]-[6]. Some graphs like cycle cactus graph, special tree and a New Key graph [8] can also be investigated for the cube difference.

Definition:1.1: Let \( G = (V(G), E(G)) \) be a graph. \( G \) is said to be cube difference labeling if there exist a bijection \( f: V(G) \rightarrow \{0,1,2,\ldots,p-1\} \) such that the induced function \( f^*: E(G) \rightarrow \mathbb{N} \) given by \( f^*(uv) = |f(u)^3 - f(v)^3| \) is injective.

Definition:1.2: An umbrella graph \( U(m,n) \) to be a graph obtained by joining a path \( p_n \) with the central vertex of a fan \( f_n \).

Definition:1.3: Let \( G = P_n(Q_{sn}) \) is a graph. Let \( V(G) = \{u_1,u_2,\ldots,u_n,v_1,v_2,\ldots,v_{mn},w_1,w_2,\ldots,w_{mn}\} \) be the vertices of the graph and \( E(G) = \{u_{i+1}v_i,1 \leq i \leq n\} \cup \{u_iv_{2i},1 \leq i \leq n\} \cup \{w_{2i}v_{2i},1 \leq i \leq n\} \). Let \( G_1,G_2,\ldots,G_n \) be \( m \) copies of \( C_4 \) and \( P_n : u_1,u_2,\ldots,u_n \) be a path. The \( P_n(Q_{sn}) \) is \( 4mn + (n-1) \) copies of \( P_2 \).

Definition:1.4: Let \( G = C_n(Q_{sn}) \) is a graph. let \( V(G) = \{u_1,u_2,\ldots,u_n,v_1,v_2,\ldots,v_{mn},w_1,w_2,\ldots,w_{mn}\} \) be the vertices of the graph and \( E(G) = \{u_{i+1}v_i,1 \leq i \leq n-1\} \cup \{u_iv_{2i},1 \leq i \leq n\} \cup \{u_iv_{2i},1 \leq i \leq n\} \cup \{u_{i+1}v_i,1 \leq i \leq n\} \). Let \( G_1,G_2,\ldots,G_n \) be \( m \) copies of \( C_4 \) and Let \( C_n : u_1,u_2,\ldots,u_n \) be a cycle.
MAIN RESULT

Theorem: 1

The Double Triangular Snake $G_n$ is a Cube difference graph.

Proof

Let $G$ be the Double Triangular Snake and let $V(G) = \{u_1,u_2, \ldots, u_n,v_1,v_2,\ldots,v_n, w_1,w_2,\ldots,w_n\}$ be the vertices of the graph and $E(G) = \{u_{i+1}u_i/1 \leq i \leq n\} \cup \{v_{i+1}v_i/1 \leq i \leq n\} \cup \{w_{i+1}w_i/1 \leq i \leq n\}$ be the edges of the graph.

Let $|V(G)| = 2n+2$ and $|E(G)| = 4n-1$.

Define the vertex labeling $f: V(G) \to \{0,1,2,\ldots,p-1\}$

$$f(u_i) = i-1, 1 \leq i \leq n$$

$$f(v_i) = i+n-1, 1 \leq i \leq n-3$$

$$= i+k, 1 \leq i \leq k$$

$$f(w_i) = 2k+i, 1 \leq i \leq k$$

and the induced edge labeling function $f: E(G) \to \mathbb{N}$ defined by

$$f(uv) = |f(u)|^3 - |f(v)|^3$$

for every $uv \in E(G)$

are all distinct such that $f(ei) \neq f(ej)$ for every $ei \neq ej$.

The edge sets are

$E_1 = \{u_{i+1}u_i/1 \leq i \leq n-1\}$

$E_2 = \{v_{i+1}v_i/1 \leq i \leq n-1\}$

$E_3 = \{v_{i+1}v_i/1 \leq i \leq n-1\}$

$E_4 = \{w_{i+1}w_i/1 \leq i \leq n-1\}$

$E_5 = \{w_{i+1}w_i/1 \leq i \leq n-1\}$

and the edge labeling are

In $E_1$

$$f^*(u_{i+1}u_i) = \bigcup_{i=1}^n |f(u_i)^3 - f(u_{i+1})^3|$$

$$= \bigcup_{i=1}^n |(i-1)^3 - i^3|$$
\[ V = \bigcup_{i=1}^{n} |3i^2 - 3i + 1| \]
\[ = \{1, 7, 19, \ldots, 3n^2 - 3n + 1\} \]

In \( E_2 \)
\[ f^*(v_i u_i) = \bigcup_{i=1}^{n-1} |f(v_i)^3 - f(u_i)^3| \]
\[ = \bigcup_{i=1}^{n-1} |k^3 + 3i(k + 1) + 3i(k^2 - 1) + 1| \]
\[ = \{64, 124, 208, \ldots\} \]

In \( E_3 \)
\[ f^*(v_i u_{i+1}) = \bigcup_{i=1}^{n-1} |f(v_i)^{23} - f(u_{i+1})^3| \]
\[ = \bigcup_{i=1}^{n-1} |k^3 + 3k^2 + i(k + i)| \]
\[ = \{63, 117, 189, \ldots\} \]

In \( E_4 \)
\[ f^*(w_i u_i) = \bigcup_{i=1}^{n-1} |f(w_i)^3 - f(u_i)^3| \]
\[ = \bigcup_{i=1}^{n-1} |(i + 2k)^3 - (i - 1)^3| \]
\[ = \bigcup_{i=1}^{n-1} |8k^3 + 3i(2k + 1) + 3i(4k^2 - 1) + 1| \]
\[ = \{343, 511, 721, \ldots\} \]

In \( E_5 \)
\[ f^*(w_i u_{i+1}) = \bigcup_{i=1}^{n-1} |f(w_i)^3 - f(u_{i+1})^3| \]
\[ = \bigcup_{i=1}^{n-1} |8k^2 + 6ki(2k + i)| \]
\[ = \{342, 504, 702, \ldots\} \]

Here the edges are distinct.

Hence the Double triangular snake graph admits a cube difference labeling.

**Theorem 2**

The Umbrella graph is a cube difference labeling.

**Proof:**

Let \( U(G) \) be the umbrella graph and the vertex is: \( u_1, u_2, \ldots, u_n \). Let \( v_1, v_2, \ldots, v_p \) be the another vertex.

Let \( |V(G)| = n \) and
\[ |E(G)| = 3n \]

Define the vertex labeling \( f: E \rightarrow \{0, 1, 2, \ldots, p-1\} \)

\[ f(u_i) = i - 1, \quad 1 \leq i \leq n \]

\[ f(v_i) = i + 2, \quad 1 \leq i \leq n + 1 \]

and the induced edge labeling function \( f: E \rightarrow \mathbb{N} \) defined by

\[ f(uv) = |[f(u)]^3 - [f(v)]^3| \]

for every \( uv \in E(G) \)

are all distinct such that \( f(e_i) \neq f(e_j) \) for every \( e_i \neq e_j \)

The edge sets are

\[ E_1 = \{ u_i u_{i+1} / 1 \leq i \leq n - 1 \} \]

\[ E_2 = \{ v_i v_{i+1} / 1 \leq i \leq n \} \]

\[ E_3 = \{ v_i u_n / 1 \leq i \leq n + 1 \} \]

And the edge labeling are

\[ f^*(E_1) = \bigcup_{i=1}^{n-1} |f(u_i)^3 - f(u_{i+1})^3| \]

\[ = \bigcup_{i=1}^{n-1} |2 - 3i^2| \]

\[ = \{1, 7, \ldots, 3n^2 - 9n + 7\} \]

\[ f^*(E_2) = \bigcup_{i=1}^{n} |f(v_i)^3 - f(v_{i+1})^3| \]

\[ = \bigcup_{i=1}^{n} |(i + 2)^3 - (i + 3)^3| \]

\[ = \bigcup_{i=1}^{n} |-3i^2 - 15i - 19| \]

\[ = \{37, 61, 91, \ldots, (3n^2 + 15n + 19)\} \]

\[ f^*(E_3) = \bigcup_{i=1}^{n+1} |f(v_i)^3 - f(u_n)^3| \]

\[ = \bigcup_{i=1}^{n+1} |i^3 + 6i(i + 2) - n^3 + 3n(n - 1 + 9)| \]

\[ = \{19, 56, 117, \ldots, 12n^2 + 24n + 28\} \]

Here the edges are distinct.

Hence the umbrella graph admits a cube difference labeling.
For example:

\[
\begin{array}{ccccccc}
V_1 & 37 & V_2 & 61 & V_3 & 91 & V_4 \\
3 & 4 & 5 & 6 & 19 & 56117208 & 2 \\
1 & u_2 & 0 & u_1 \\
\end{array}
\]

**Theorem 3**

The graph \( G = P_n (QS_n) \) is a cube difference labeling \((n \geq 1, m \geq 1)\).

**Proof:**

Let \( G = P_n (QS_n) \) be a graph. Let \( V(G) = \{u_1,u_2, \ldots,u_n,v_1,v_2,\ldots,v_{mn},w_1,w_2,\ldots,w_{mn}\} \) be the vertices of the graph and 
\[
E(G) = \{u_iu_{i+1}/1 \leq i \leq n-1\} \cup \{u_iv_{2i},w_iv_{2i}/1 \leq i \leq n\} \cup \{u_iv_{2i},v_iv_{2i}/1 \leq i \leq n\}.
\]

Let \( G_1,G_2,\ldots,G_m \) be m copies of \( C_4 \) and \( P_n : u_1,u_2,\ldots,u_n \) be a path. The \( P_n (QS_n) \) is \( 4mn + (n-1) \) copies of \( P_2 \).

Let \( |V(G)| = 3mn + n \) and \( |E(G)| = 4mn + (n-1) \).

Define the vertex labeling \( f:E \to \{0,1,\ldots,p-1\} \)

\[
f(u_i) = i-1 \quad 1 \leq i \leq n \\
f(v_{2ik+i}) = (3k+1)n+(i-1), \quad 1 \leq i \leq n-1, \quad k=0,1,2,3,\ldots,n \\
f(w_{2ik+i}) = 3n(k+1)+(i-1), \quad 1 \leq i \leq n-1, \quad k=0,1,2,3,\ldots,n \\
f(v_i) = n+i-1 \text{ for } k=0 \\
f(w_i) = 3n+i-1 \text{ for } k=0
\]

and the induced edge labeling function 
\( f:E \to \mathbb{N} \) defined by
\( f(uv) = \left| (f(u))^3 - (f(v))^3 \right| \) for every \( uv \in E(G) \)

are all distinct such that \( f(e_i) \neq f(e_j) \) for every \( e_i \neq e_j \)

The edge sets are

\[ E_1 = \{u_{i-1}u_i \mid 1 \leq i \leq n-1\} \]
\[ E_2 = \{u_{2i-1}u_i \mid 1 \leq i \leq n\} \]
\[ E_3 = \{u_{2i} \mid 1 \leq i \leq n\} \]
\[ E_4 = \{w_{2i-1}u_i \mid 1 \leq i \leq n\} \]
\[ E_5 = \{w_{2i} \mid 1 \leq i \leq n\} \]

And the edge labeling are

**In** \( E_1 \)

\[ f^*(u_{i-1}u_i) = \bigcup_{\ell=1}^{n-1} \left| (1 - \ell)^3 - \ell^3 \right| \]
\[ = \bigcup_{\ell=1}^{n-1} (1 - 3\ell(1 - \ell)) \]
\[ = \{1, 7, \ldots, 1 - 3n(1 - n)\} \]

**In** \( E_2 \)

\[ f^*(u_{2i-1}u_i) = \bigcup_{\ell=1}^{n} \left| (\ell - 1)^3 - (\ell + 2\ell - 2)^3 \right| \]
\[ = \bigcup_{\ell=1}^{n} \left| i(7\ell + 9 - 12n) + i(-3 - 6n2 + 12n) + (-n3 + 6n2 - 12n + 7) \right| \]
\[ = \{8, 63, \ldots, (-26n3 + 51n^2 - 33n + 7)\} \]

**In** \( E_3 \)

\[ f^*(u_{2i}) = \bigcup_{\ell=1}^{n} \left| (\ell - 1)^3 - (\ell + 2\ell - 1)^3 \right| \]
\[ = \bigcup_{\ell=1}^{n} \left| -7\ell^3 + 9\ell^2 - 3\ell - n3 - 6n2 - 12n \ell + 3n2 - 12n \ell - 3n \right| \]
\[ = \{27, 124, \ldots, (-26n^3 + 24n^2 - 6n)\} \]

**In** \( E_4 \)

\[ f^*(w_{2i-1}) = \bigcup_{\ell=1}^{n} \left| (\ell - 1)^3 - (n + 2\ell - 2)^3 \right| \]
\[ \mathcal{E}_5 = \bigcup_{i=1}^{n} \left| 26n^3 - 7i3 - 21n2 + 21n2i - 3ni2 + 6ni + 21i2 - 3n - 21i + 7 \right| \]
\[ = \bigcup_{i=1}^{n} \left| 2(7i + 21 - 3n) + i(21n2 + 6n - 21) + (26n3 - 21n2 - 3n + 7) \right| \]
\[ = \{208, 279, \ldots, (37n^3 + 6n^2 - 24n + 7)\} \]

**In \( E_5 \)**

\[ f^*(w, v) = \bigcup_{i=1}^{n} \left| f(w_i)^3 - f(v_i)\right|^3 \]
\[ = \bigcup_{i=1}^{n} \left| (3n + i - 1)^3 - (n + 2i - 1)^3 \right| \]
\[ = \bigcup_{i=1}^{n} \left| 26n^3 + 21n2i - 3ni2 - 7i3 - 24n2 - 6ni + 9i2 + 6n - 3i \right| \]
\[ = \bigcup_{i=1}^{n} \left| 2(7i + 9 - 3n) + i(21n2 - 6n - 3) + (26n3 - 24n2 + 6n) \right| \]
\[ = \{189, 218, \ldots, (37n^3 - 21n^2 + 3n)\} \]

Here the edges are distinct.

Hence the \( P_d(QS_n) \) graph admits a cube difference labeling.

**For example;**

The graph \( P_2(QS_2) \) is a cube difference labeling.

**Solution:**

If \( n \geq 1 \) and \( m \geq 1 \)
Theorem: 4

The graph $G = C_n (QS_n)$ is a cube difference labeling.

Proof:

Let $G = C_n (QS_n)$ be a graph. Let $V(G) = \{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_{mn}, w_1, w_2, \ldots, w_{mn}\}$ be the vertices of the graph and $E(G) = \{u_i u_i+1/1 \leq i \leq n-1 \} \cup \{u_i v_{2i-1}, w_i v_{2i-1}/1 \leq i \leq n \} \cup \{u_i v_{2i}, w_i v_{2i}/1 \leq i \leq n \}$. 

Let $G_1, G_2, \ldots, G_n$ be $m$ copies of $C_4$ and let $C_n : u_1, u_2, \ldots, u_n$ be a cycle. Let $|V(G)| = 3mn + n$ and $|E(G)| = 4mn + n$. 

Define the vertex labeling $f: E \rightarrow \{0,1,\ldots,p-1\}$

$f(u_i) = i-1$, $1 \leq i \leq n$

$f(v_{2nk+i}) = (3k+1)n+(i-1)$, $1 \leq i \leq n-1$, $k=0,1,2,3,\ldots,n$

$f(w_{nk+1}) = 3n(k+1)+(i-1)$, $1 \leq i \leq n-1$, $k=0,1,2,3,\ldots,n$

$f(v_i) = n+i-1$ for $k=0$ and $f(w_i) = 3n+i-1$ for $k=0$

and the induced edge labeling function

$f: E \rightarrow N$ defined by

$f(uv) = \left| [f(u)]^3 - [f(v)]^3 \right|$ for every $uv \in E(G)$

are all distinct such that $f(e_i) \neq f(e_j)$ for every $e_i \neq e_j$

The edge sets are

$E_1 = \{u_i u_{i+1}/1 \leq i \leq n-1\}$

$E_2 = \{u_i v_{2i}/1 \leq i \leq n\}$

$E_3 = \{u_i v_{2i+1}/1 \leq i \leq n\}$

$E_4 = \{w_i v_{2i}/1 \leq i \leq n\}$

$E_5 = \{w_i v_{2i+1}/1 \leq i \leq n\}$

and the edge labeling are

In $E_1$

$f^*(u_i u_{i+1}) = \bigcup_{i=1}^{n-1} \left| (1 - i)^3 - i^3 \right|$

$= \bigcup_{i=1}^{n-1} \left| (1 - 3i(1 - i) \right|$

$= \{1,7,\ldots,1-3n(1-n)\}$

In $E_2$
\begin{align*}
f^*(u_i v_{2i}) &= U_i^{n} \left| f(u_i)^3 - f(v_{2i-1})^3 \right| \\
&= U_i^{n} \left| (i - 1)^3 - (n + 2i - 2)^3 \right| \\
&= U_i^{n} \left| -7i^3 + 21i^2 - 21i + 7 - n^3 - 6n^2i - 12ni^2 + 6n^2 + 24ni - 12n \right| \\
&= U_i^{n} \left| i^2(-7i + 21 - 12n) + i(-21 - 6n^2 + 24n) + (-n^3 + 6n^2 - 12n + 7) \right| \\
&= \{27, 124, 335, \ldots, (-26n^4 + 56n - 33n^2 + 7) \}
\end{align*}

InE3

\begin{align*}
f^*(u_i v_{2i}) &= U_i^{n} \left| f(u_i)^3 - f(v_{2i})^3 \right| \\
&= U_i^{n} \left| (i - 1)^3 - (n + 2i - 1)^3 \right| \\
&= U_i^{n} \left| -7i^3 + 9i^2 - 3i - n^3 - 6n^2i - 12ni^2 + 3n^2 + 12ni - 3n \right| \\
&= U_i^{n} \left| i^2(-7i + 9 - 12n) + i(-3 - 6n^2 + 12n) + (-n^3 + 3n^2 - 3n) \right| \\
&= \{64, 215, 504, \ldots, (-26n^3 + 24n^2 - 6n) \}
\end{align*}

In E4

\begin{align*}
f^*(w_i v_{2i-1}) &= U_i^{n} \left| f(w_i)^3 - f(v_{2i})^3 \right| \\
&= U_i^{n} \left| (3n + i - 1)^3 - (n + 2i - 2)^3 \right| \\
&= U_i^{n} \left| 26n^3 - 7i^3 - 21n^2 + 21n^2i - 3ni^2 + 6ni + 21i^2 - 3n - 21i + 7 \right| \\
&= U_i^{n} \left| i^2(-7i + 9 - 3n) + i(21n + 6n - 21) + (26n^3 - 21n^2 - 3n + 7) \right| \\
&= \{702, 875, 988, \ldots, 37n^3 + 6n^2 - 24n + 7 \}
\end{align*}

In E5

\begin{align*}
f^*(w_i v_{2i}) &= U_i^{n} \left| f(w_i)^3 - f(v_{2i})^3 \right| \\
&= U_i^{n} \left| (3n + i - 1)3 - (n + 2i - 1)3 \right| \\
&= U_i^{n} \left| 26n^3 + 21n^2i - 3ni^2 - 7i^3 - 24n^2 - 6ni + 9i^2 + 6n - 3i \right| \\
&= U_i^{n} \left| i^2(-7i + 9 - 3n) + i(21n - 6n - 3) + (26n^3 - 4n^2 + 6n) \right| \\
&= \{665, 784, 819, \ldots, 37n^3 - 21n^2 + 3n \}
\end{align*}

Here the edges are distinct.

Hence the C_{d}(QS_n) graph admits a square difference labeling.
Example: The graph $C_3(QS_2)$ is a cube difference labeling. ($n \geq 3$, $m \geq 1$).

Solution: If $n \geq 3$ and $m \geq 1$
ACKNOWLEDGEMENT

The author thanks to Mannonmaniam Sundaranar University for providing facilities. I wish to express my deep sense of gratitude to Dr. M. Karuppaian my husband for their immense help in my studies.

REFERENCES


