ON FUZZY ALMOST P-SPACES

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ABSTRACT

In this paper we discuss several characterizations of fuzzy almost P-spaces and the conditions under which fuzzy topological spaces become fuzzy almost P-spaces, are investigated. Some results concerning functions that preserve fuzzy almost P-spaces in the context of images and preimages are obtained.

KEY WORDS : Fuzzy $G_\delta$-set, fuzzy $F_\sigma$-set, fuzzy dense set, fuzzy nowhere dense set, fuzzy submaximal space, fuzzyD-Baire space, fuzzy strongly irresolvable space, somewhat fuzzy continuous function, somewhat fuzzy open function.

1. INTRODUCTION

In order to deal with uncertainties, the concept of fuzzy sets and fuzzy set operations were first introduced by L.A.ZADEH [23] in his classical paper in the year 1965, describing fuzziness mathematically for the first time. This inspired mathematicians to fuzzify mathematical structures. The concepts of fuzzy topology was defined by C.L.CHANG [3] in the year 1968. The paper of Chang paved the way for the subsequent tremendous growth of the numerous fuzzy topological concepts. Since then much attention has been paid to
generalize the basic concepts of general topology in fuzzy setting and thus a modern theory of fuzzy topology has been developed.

A.K.MISHRA [8] introduced the concepts of P-spaces as a generalization of $\omega_\alpha$-additive spaces of SIKORSKI [10] and COHEN, L.W. and C. GOFFMAN [5]. The concept of P-spaces in fuzzy setting was introduced by G. BALASUBRAMANIAN in [13]. Almost P-spaces in classical topology was introduced by A.I. Veksler [21] and was also studied further by R. Levy [7]. Chang IL Kim [4] studied several characterizations almost P-spaces. The concept of almost P-spaces in fuzzy setting was introduced by the authors in [16]. In this paper we discuss several characterizations of fuzzy almost P-spaces and the conditions under which fuzzy topological spaces become fuzzy almost P-spaces, are investigated. Some results concerning functions that preserve fuzzy almost P-spaces in the context of images and preimages are obtained.

2. PRELIMINARIES

Now we introduce some basic notions and results used in the sequel.

In this work by $(X,T)$ or simply by $X$, we will denote a fuzzy topological space due to CHANG (1968).

**Definition 2.1**: Let $\lambda$ and $\mu$ be any two fuzzy sets in a fuzzy topological space $(X,T)$. Then we define:

(i). $\lambda \lor \mu : X \rightarrow [0,1]$ as follows : $(\lambda \lor \mu)(x) = \text{Max} \{ \lambda(x), \mu(x) \}$.

(ii). $\lambda \land \mu : X \rightarrow [0,1]$ as follows : $(\lambda \land \mu)(x) = \text{Min} \{ \lambda(x), \mu(x) \}$.

(iii). $\mu = \lambda^c \iff \mu(x) = 1 - \lambda(x)$. 
More generally, for a family \( \{ \lambda_i / i \in I \} \) of fuzzy sets in \((X,T)\), \( \psi = \bigvee_i (\lambda_i) \) and \( \delta = \bigwedge_i (\lambda_i) \), are defined respectively as \( \psi(x) = \sup_i \{ \lambda_i(x), x \in X \} \) and \( \delta(x) = \inf_i \{ \lambda_i(x), x \in X \} \).

**Definition 2.2**: Let \((X,T)\) be a fuzzy topological space and \( \lambda \) be any fuzzy set in \((X,T)\). We define the interior and the closure of \( \lambda \) respectively as follows:

(i) \( \text{int} (\lambda) = \bigvee \{ \mu / \mu \leq \lambda, \mu \in T \} \)

(ii) \( \text{cl} (\lambda) = \bigwedge \{ \mu / \lambda \leq \mu, 1-\mu \in T \} \).

**Lemma 2.1 [1]**: For a fuzzy set \( \lambda \) of a fuzzy topological space \( X \),

(i) \( 1 - \text{int} (\lambda) = \text{cl} (1 - \lambda) \),

(ii) \( 1 - \text{cl} (\lambda) = \text{int} (1 - \lambda) \).

**Definition 2.3 [14]**: A fuzzy set \( \lambda \) in a fuzzy topological space \((X,T)\) is called fuzzy dense if there exists no fuzzy closed set \( \mu \) in \((X,T)\) such that \( \lambda \leq \mu < 1 \). That is, \( \text{cl}(\lambda) = 1 \).

**Definition 2.4 [14]**: A fuzzy set \( \lambda \) in a fuzzy topological space \((X,T)\) is called fuzzy nowhere dense if there exists no non-zero fuzzy open set \( \mu \) in \((X,T)\) such that \( \mu < \text{cl}(\lambda) \). That is, \( \text{int cl}(\lambda) = 0 \).

**Definition 2.5 [2]**: A fuzzy set \( \lambda \) in a fuzzy topological space \((X,T)\) is called a fuzzy \( F_\alpha \)-set in \((X,T)\) if \( \lambda = \bigvee_{i=1}^\infty (\lambda_i) \) where \( 1 - \lambda_i \in T \) for \( i \in I \).

**Definition 2.6 [2]**: A fuzzy set \( \lambda \) in a fuzzy topological space \((X,T)\) is called a fuzzy \( G_\delta \)-set in \((X,T)\) if \( \lambda = \bigwedge_{i=1}^\infty (\lambda_i) \) where \( \lambda_i \in T \) for \( i \in I \).

**Definition 2.7 [1]**: A fuzzy set \( \lambda \) in a fuzzy topological space \((X,T)\) is called
(i). a fuzzy regular open set in \((X,T)\) if \(\text{int} \, cl(\lambda) = \lambda\),

(ii). a fuzzy regular closed set in \((X,T)\) if \(\text{cl} \, \text{int}(\lambda) = \lambda\),

(iii). a fuzzy semi-open set in \((X,T)\) if \(\lambda \leq \text{cl} \, \text{int}(\lambda)\),

(iv). a fuzzy semi-closed set in \((X,T)\) if \(\text{int} \, \text{cl}(\lambda) \leq \lambda\).

**Lemma 2.2** [1]: For a family \(\mathcal{A}\) of \(\{\lambda_\alpha\}\) of fuzzy sets of a fuzzy topological space \((X,T)\), \(\bigvee \, \text{cl}(\lambda_\alpha) \leq \text{cl}(\bigvee \lambda_\alpha)\). In case \(\mathcal{A}\) is a finite set, \(\bigvee \, \text{cl}(\lambda_\alpha) = \text{cl}(\bigvee \lambda_\alpha)\). Also \(\bigvee \, \text{int}(\lambda_\alpha) \leq \text{int}(\bigvee \lambda_\alpha)\) in \((X,T)\).

**Definition 2.9** [14]: A fuzzy set \(\lambda\) in a fuzzy topological space \((X,T)\) is called a fuzzy first category set if \(\lambda = \bigvee_{i=1}^\infty (\lambda_i)\), where \((\lambda_i)\)'s are fuzzy nowhere dense sets in \((X,T)\). Any other fuzzy set in \((X,T)\) is said to be of fuzzy second category.

**Definition 2.10** [14]: A fuzzy topological space \((X,T)\) is called fuzzy first category if \(\bigvee_{i=1}^\infty (\lambda_i) = 1_X\), where \((\lambda_i)\)'s are fuzzy nowhere dense sets in \((X,T)\). A topological space which is not of fuzzy first category, is said to be of fuzzy second category.

**Definition 2.11** [3]: Let \((X,T)\) and \((Y,S)\) be any two fuzzy topological spaces. Let \(f\) be a function from the fuzzy topological space \((X,T)\) to the fuzzy topological space \((Y,S)\). Let \(\lambda\) be a fuzzy set in \((Y,S)\). The inverse image of \(\lambda\) under \(f\) written as \(f^{-1}(\lambda)\) is the fuzzy set in \((X,T)\) defined by \(f^{-1}(\lambda)(x) = \lambda(f(x))\) for all \(x \in X\). Also the image of \(\lambda\) in \((X,T)\) under \(f\) written as \(f(\lambda)\) is the fuzzy set in \((Y,S)\) defined by
Lemma 2.3[3]: Let \( f : (X, T) \to (Y, S) \) be a mapping. For fuzzy sets \( \lambda \) and \( \mu \) of \( (X, T) \) and \( (Y, S) \) respectively, the following statements hold:

1. \( f^{-1}(\mu) \leq \mu \);
2. \( f^{-1}f(\lambda) \geq \lambda \);
3. \( f(1 - \lambda) \geq 1 - f(\lambda) \);
4. \( f^{-1}(1 - \mu) = 1 - f^{-1}(\mu) \);
5. If \( f \) is one-to-one, then \( f^{-1}(\lambda) = \lambda \);
6. If \( f \) is onto, then \( f f^{-1}(\mu) = \mu \);
7. If \( f \) is one-to-one and onto, then \( f(1 - \lambda) = 1 - f(\lambda) \).

Lemma 2.4 [1]: Let \( f : (X, T) \to (Y, S) \) be a mapping and \( \{\lambda_\alpha\} \) be a family of fuzzy sets of \( Y \). Then

(a) \( f^{-1}(\bigcup \lambda_\alpha) = \bigcup f^{-1}(\lambda_\alpha) \).

(b) \( f^{-1}(\bigcap \lambda_\alpha) = \bigcap f^{-1}(\lambda_\alpha) \).

Lemma 2.5 [6]: Let \( f : (X, T) \to (Y, S) \) be a mapping and \( \{A_j\}, j \in J \) be a family of fuzzy sets in \( X \). Then

(a) \( f(\bigcup_{j \in J} A_j) = \bigcup_{j \in J} f(A_j) \).

(b) \( f(\bigcap_{j \in J} A_j) \leq \bigcap_{j \in J} f(A_j) \).

3. FUZZY ALMOST P-SPACES

Almost P-spaces in classical topology were introduced by A.I. Veksler [21] and was also studied further by R.Levy [7]. Chang IL Kim [4] studied several characterizations of almost P-
spaces. Motivated by these ideas, the concept of almost P-spaces in fuzzy setting is introduced in [16].

**Definition 3.1 [16]:** A fuzzy topological space \((X,T)\) is called a fuzzy almost P-space if for every non-zero fuzzy \(G_\delta\)-set \(\lambda\) in \((X,T)\), \(\text{int}(\lambda) \neq 0\) in \((X,T)\).

**Proposition 3.1:** If \(\lambda\) is a fuzzy \(F_\sigma\) -set in a fuzzy almost P-space \((X,T)\), then \(\text{cl}(\lambda) \neq 1\).

**Proof:** Let \(\lambda\) be a fuzzy \(F_\sigma\) -set in a fuzzy almost P-space \((X,T)\). Then, \((1 - \lambda)\) is a fuzzy \(G_\delta\)-set in \((X,T)\). Since \((X,T)\) is a fuzzy almost P-space, for the fuzzy \(G_\delta\)-set \((1 - \lambda)\), we have \(\text{int}(1 - \lambda) \neq 0\). This implies that \(1 - \text{cl}(\lambda) \neq 0\) and hence we have \(\text{cl}(\lambda) \neq 1\).

**Remark 3.1:** In view of the proposition 3.1, we have the following result: “If \(\lambda\) is a fuzzy \(G_\delta\)-set in a fuzzy almost P-space \((X,T)\), then \(\text{cl}(1 - \lambda) \neq 1\).”

The following propositions give the conditions for a fuzzy topological space to be a fuzzy almost P-space.

**Proposition 3.2:** If each non-zero fuzzy \(G_\delta\)-set is a fuzzy regular closed set in a fuzzy topological space \((X,T)\), then \((X,T)\) is a fuzzy almost P-space.

**Proof:** Let \(\lambda\) be a non-zero fuzzy \(G_\delta\)-set in \((X,T)\) such that \(\text{cl} \text{ int}(\lambda) = \lambda\). We claim that \(\text{int}(\lambda) \neq 0\). Assume the contrary. Then \(\text{int}(\lambda) = 0\), will imply that \(\text{cl} \text{ int}(\lambda) = \text{cl}(0) = 0\) and hence we will have \(\lambda = 0\), a contradiction to \(\lambda\) being a non-zero fuzzy \(G_\delta\)-set in \((X,T)\). Hence we must have \(\text{int}(\lambda) \neq 0\), for a fuzzy \(G_\delta\)-set \(\lambda\) in \((X,T)\) and therefore \((X,T)\) is a fuzzy almost P-space.
Proposition 3.3: If each non-zero fuzzy $G_\delta$-set is a fuzzy semi-open set in a fuzzy topological space $(X,T)$, then $(X,T)$ is a fuzzy almost $P$-space.

Proof: Let $\lambda$ be a non-zero fuzzy $G_\delta$-set in $(X,T)$ such that $\lambda \leq \text{cl}(\text{int}(\lambda))$. We claim that $\text{int}(\lambda) \neq 0$. Assume the contrary. Then $\text{int}(\lambda) = 0$, will imply that $\text{cl}(\text{int}(\lambda)) = \text{cl}(0) = 0$ and hence we will have $\lambda = 0$, a contradiction to $\lambda$ being a non-zero fuzzy $G_\delta$-set in $(X,T)$. Hence we must have $\text{int}(\lambda) \neq 0$, for a fuzzy $G_\delta$-set $\lambda$ in $(X,T)$ and therefore $(X,T)$ is a fuzzy almost $P$-space.

The following proposition shows that a fuzzy first category set, is not a fuzzy dense set in a fuzzy almost $P$-space.

Proposition 3.4: If $\lambda$ is a fuzzy first category set in a fuzzy almost $P$-space $(X,T)$, then $\text{cl}(\lambda) \neq 1$, in $(X,T)$.

Proof: Let $\lambda$ be a fuzzy first category set in $(X,T)$, then $\lambda = \bigvee_{i=1}^{\infty}(\lambda_i)$, where $(\lambda_i)$'s are fuzzy nowhere dense sets in $(X,T)$. Now $\lambda_i \leq \text{cl}(\lambda_i)$, implies that $\bigvee_{i=1}^{\infty}(\lambda_i) \leq \bigvee_{i=1}^{\infty}\text{cl}(\lambda_i)$ and hence we have $\text{cl}(\bigvee_{i=1}^{\infty}(\lambda_i)) \leq \text{cl}(\text{cl}(\bigvee_{i=1}^{\infty}(\lambda_i)))$. Then $\text{cl}(\lambda) \leq \text{cl}(\bigvee_{i=1}^{\infty}\text{cl}(\lambda_i))$ ...(1). Now $\bigvee_{i=1}^{\infty}\text{cl}(\lambda_i)$ is a fuzzy $F_\sigma$-set in $(X,T)$. Since $(X,T)$ is a fuzzy almost $P$-space, by proposition 3.1, $\text{cl}(\bigvee_{i=1}^{\infty}\text{cl}(\lambda_i)) \neq 1$. ...(2). Now we claim that $\lambda$ is not a fuzzy dense set in $(X,T)$. Assume the contrary. Suppose that $\lambda$ is a fuzzy dense set, then $\text{cl}(\lambda) = 1$, implies from (1), that $1 \leq \text{cl}(\bigvee_{i=1}^{\infty}\text{cl}(\lambda_i))$. That is, $\text{cl}(\bigvee_{i=1}^{\infty}\text{cl}(\lambda_i)) = 1$, a contradiction to (2). Hence we must have $\text{cl}(\lambda) \neq 1$, in $(X,T)$.

Proposition 3.5: If each non-zero fuzzy first category set is a fuzzy dense set in a fuzzy topological space $(X,T)$, then $(X,T)$ is not a fuzzy almost $P$-space.
Proof: Let $\lambda$ be a fuzzy first category set in $(X,T)$ such that $\text{cl}(\lambda) = 1$. Then $\lambda = V_{i=1}^{\infty}(\lambda_i)$, where $(\lambda_i)$'s are fuzzy nowhere dense sets in $(X,T)$. Now $1 - \text{cl}(\lambda_i)$ is a fuzzy open set in $(X,T)$. Let $\mu = \Lambda_{i=1}^{\infty} [1 - \text{cl}(\lambda_i)]$. Then $\mu$ is a fuzzy $G_{\delta}$-set in $(X,T)$. Now $\mu = \Lambda_{i=1}^{\infty} [1 - \text{cl}(\lambda_i)] = 1 - \sum_{i=1}^{\infty} \text{cl}(\lambda_i) \leq 1 - \sum_{i=1}^{\infty} (\lambda_i) = 1 - \lambda$. That is, $\mu \leq 1 - \lambda$.

Then $\text{int}(\mu) \leq \text{int}(1 - \lambda)$ and hence $\text{int}(\mu) \leq 1 - \text{cl}(\lambda) = 1 - 1 = 0$. That is, $\text{int}(\mu) = 0$.

Hence, for the fuzzy $G_{\delta}$-set $\mu$ in $(X,T), \text{int}(\mu) = 0$. Therefore $(X,T)$ is not a fuzzy almost $P$-space.

Proposition 3.6: If $\gamma$ is a fuzzy residual set in a fuzzy almost $P$-space $(X,T)$, then $\text{int}(\gamma) \neq 0$, in $(X,T)$.

Proof: Let $\gamma$ be a fuzzy residual set in $(X,T)$. Then, $(1 - \gamma)$ is a fuzzy first category set in $(X,T)$ and hence by proposition 3.4, $\text{cl}(1 - \gamma) \neq 1$ in $(X,T)$. Therefore $1 - \text{int}(\gamma) = \text{cl}(1 - \gamma) \neq 1$. Therefore $\text{int}(\gamma) \neq 0$, in $(X,T)$.

Proposition 3.7: If $\gamma$ is a fuzzy residual set in a fuzzy almost $P$-space $(X,T)$, then there exists a fuzzy $G_{\delta}$-set $\mu$ in $(X,T)$ such that $\mu \leq \gamma$ in $(X,T)$.

Proof: Let $\gamma$ be a fuzzy residual set in $(X,T)$. Then, $(1 - \gamma)$ is a fuzzy first category set in $(X,T)$. Let $\lambda = 1 - \gamma$. Then $\lambda = V_{i=1}^{\infty}(\lambda_i)$, where $(\lambda_i)$'s are fuzzy nowhere dense sets in $(X,T)$. Now $[1 - \text{cl}(\lambda_i)]$ is a fuzzy open set in $(X,T)$. Let $\mu = \Lambda_{i=1}^{\infty} [1 - \text{cl}(\lambda_i)]$. Then $\mu$ is a fuzzy $G_{\delta}$-set in $(X,T)$. Since $(X,T)$ is a fuzzy almost $P$-space, $\text{int}(\mu) \neq 0$ in $(X,T)$. Let $\text{int}(\mu) = \delta$, where $\delta \in T$. Now $\mu = \Lambda_{i=1}^{\infty} [1 - \text{cl}(\lambda_i)] = 1 - \sum_{i=1}^{\infty} \text{cl}(\lambda_i) \leq 1 - \sum_{i=1}^{\infty} (\lambda_i) = 1 - \lambda$. That is, $\mu \leq \gamma$. Thus, if $\gamma$ is a fuzzy residual set in a fuzzy almost $P$-space $(X,T)$, then there exists a fuzzy $G_{\delta}$-set $\mu$ in $(X,T)$ such that $\mu \leq \gamma$ in $(X,T)$.  

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Proposition 3.8: If μ is a fuzzy residual set in a fuzzy almost P-space (X,T), then μ is not a fuzzy nowhere dense set in (X,T).

Proof: Let μ be a fuzzy residual set in a fuzzy almost P-space (X,T). Then, by proposition 3.6, int (μ) ≠ 0, in (X,T). We claim that int cl (μ) ≠ 0. Assume the contrary. Then int cl (μ) = 0 and int (μ) ≤ int cl (μ), will imply that int (μ) = 0, a contradiction. Hence μ is not a fuzzy nowhere dense set in (X,T).

Remark 3.1: If λ is a fuzzy semi-open set in a fuzzy topological space (X,T), then cl (λ) is a fuzzy regular closed set in (X,T). For, if λ is a fuzzy semi-open set in (X,T), then λ ≤ cl [int (λ)] and hence cl (λ) ≤ cl (cl [int (λ)]) = cl [int (λ)] ≤ cl [cl (λ)] …….(1). Also we have cl [cl (λ)] ≤ cl (cl (λ)) = cl (λ) ……(2). From (1) and (2), cl [int (cl (λ))] = cl (λ) and hence cl (λ) is a fuzzy regular closed set in (X,T).

Proposition 3.9: If (λ_i)'s are fuzzy regular closed sets in a fuzzy almost P-space (X,T), then cl (∪_{i=1}^{∞} (λ_i)) ≠ 1, in (X,T).

Proof: Let (λ_i)'s be fuzzy regular closed sets in (X,T). Then, (1−λ_i)'s are fuzzy regular open sets in (X,T). Since the fuzzy regular open sets are fuzzy open sets in a fuzzy topological space, (1−λ_i)'s are fuzzy open sets in (X,T) and hence \( \Lambda_{i=1}^{\infty} (1−λ_i) \) is a fuzzy Gδ-set in (X,T). Since (X, T) is a fuzzy almost P-space, int (\( \Lambda_{i=1}^{\infty} (1−λ_i) \)) ≠ 0 in (X,T). Then we have int (1−\( \bigcup_{i=1}^{\infty} (λ_i) \)) ≠ 0 and therefore cl (\( \bigcup_{i=1}^{\infty} (λ_i) \)) ≠ 1, in (X,T).

Proposition 3.10: If (λ_i)'s are fuzzy semi-open sets in a fuzzy almost P-space (X,T), then cl (∪_{i=1}^{∞} [cl (λ_i)]) ≠ 1, in (X,T).
Proof: Let \((\lambda_i)\)'s be fuzzy semi-open sets in \((X,T)\). Then, by remark 3.1, \(\text{cl} (\lambda_i)'s\) are fuzzy regular closed sets in \((X,T)\) and hence, by proposition 3.8, \(\bigvee_{i=1}^{\infty} \text{cl} (\lambda_i) \neq 1\), in \((X,T)\).

4. FUZZY ALMOST P-SPACES and OTHER FUZZY TOPOLOGICAL SPACES

Definition 4.1 [13]: A fuzzy topological space \((X,T)\) is called a fuzzy P-space if countable intersection of fuzzy open sets in \((X,T)\) is fuzzy open. That is, every non-zero fuzzy \(G_{\delta}\)-set in \((X,T)\), is fuzzy open in \((X,T)\).

Proposition 4.1: If a fuzzy topological space \((X,T)\) is a fuzzy almost P-space, then \((X,T)\) is a fuzzy second category space.

Proof: Let the fuzzy topological space \((X,T)\) be a fuzzy almost P-space. Suppose that \((X,T)\) is a fuzzy first category space. Then \(\bigvee_{i=1}^{\infty} (\lambda_i) = 1\), where \((\lambda_i)'s\) are fuzzy nowhere dense sets in \((X,T)\). Now \(\lambda_i \leq \text{cl} (\lambda_i)\), implies that \(\bigvee_{i=1}^{\infty} (\lambda_i) \leq \bigvee_{i=1}^{\infty} \text{cl} (\lambda_i) \ldots (1)\). Also \(\bigvee_{i=1}^{\infty} \text{cl} (\lambda_i)\) is a fuzzy \(F_{\sigma}\)-set in \((X,T)\). Since \((X,T)\) is a fuzzy almost P-space, by proposition 3.1, \(\text{cl} (\bigvee_{i=1}^{\infty} \text{cl} (\lambda_i)) \neq 1 \ldots (2)\). From (1), we have \(\text{cl} (\bigvee_{i=1}^{\infty} (\lambda_i)) \leq \text{cl} (\bigvee_{i=1}^{\infty} \text{cl} (\lambda_i))\). This implies that \(\text{cl} (1) \leq \text{cl} (\bigvee_{i=1}^{\infty} \text{cl} (\lambda_i))\) and hence we have \(1 \leq \text{cl} (\bigvee_{i=1}^{\infty} \lambda_i (\lambda_i))\). That is, \(\text{cl} (\bigvee_{i=1}^{\infty} \lambda (\lambda_i)) = 1\), a contradiction to (2). Hence we must have \(\bigvee_{i=1}^{\infty} (\lambda_i) \neq 1\), in \((X,T)\) and therefore \((X,T)\) is a fuzzy second category space.

Definition 4.2 [20]: A fuzzy topological space \((X,T)\) is said to be a fuzzy strongly irresolvable space if \(\text{cl int} (\lambda) = 1\), for each fuzzy dense set \(\lambda\) in \((X,T)\).
Proposition 4.2: If each non-zero fuzzy $G_\delta$-set is a fuzzy dense set in a fuzzy strongly irresolvable space, then $(X, T)$ is a fuzzy almost P-space.

Proof: Let $\lambda$ be a non-zero fuzzy $G_\delta$-set in $(X, T)$. Then, by hypothesis, $\lambda$ is a fuzzy dense set in $(X, T)$. Since $(X, T)$ is a fuzzy strongly irresolvable space, we have

$$\text{cl} \left( \text{int} \left( \lambda \right) \right) = 1.$$ 

Then, we have $\text{int} \left( \lambda \right) \neq 0$, otherwise $\text{int} \left( \lambda \right) = 0$ will imply $\text{cl} \left( \text{int} \left( \lambda \right) \right) = \text{cl} (0) = 0$, a contradiction. Hence, for every non-zero fuzzy $G_\delta$-set $\lambda$ in $(X, T)$, we have $\text{int} \left( \lambda \right) \neq 0$ in $(X, T)$. Therefore $(X, T)$ is a fuzzy almost P-space.

Definition 4.3 [15]: A fuzzy topological space $(X, T)$ is called a fuzzy resolvable space if there exists a fuzzy dense set $\lambda$ in $(X, T)$ such that $\text{cl} \left( 1-\lambda \right) = 1$. Otherwise $(X, T)$ is called a fuzzy irresolvable space.

Theorem 4.1 [11]: If $\lambda$ is a fuzzy dense and fuzzy $G_\delta$-set in a fuzzy topological space $(X, T)$, then $(1-\lambda)$ is a fuzzy first category set in $(X, T)$.

Proposition 4.3: If each non-zero fuzzy dense set is a fuzzy $G_\delta$-set in a fuzzy almost P-space, then $(X, T)$ is a fuzzy irresolvable space.

Proof: Let $\lambda$ be a non-zero fuzzy dense set in $(X, T)$. Then, by hypothesis, $\lambda$ is a fuzzy $G_\delta$-set and hence $\lambda$ is a fuzzy dense and fuzzy $G_\delta$-set in $(X, T)$. Then, by Theorem 4.1, $1-\lambda$ is a fuzzy first category set in $(X, T)$.

Since $(X, T)$ is a fuzzy almost P-space, by Proposition 3.4, $\text{cl} \left( 1-\lambda \right) \neq 1$, in $(X, T)$. Thus, for each fuzzy dense set $\lambda$ in $(X, T)$, we have $\text{cl} \left( 1-\lambda \right) \neq 1$, in $(X, T)$. Hence $(X, T)$ is not a fuzzy resolvable space and therefore $(X, T)$ is a fuzzy irresolvable space.

Definition 4.4 [2]: A fuzzy topological space $(X, T)$ is called a fuzzy submaximal space if for each fuzzy set in $(X, T)$ such that $\text{cl} \left( \lambda \right) = 1$, then $\lambda \in T$ in $(X, T)$.
**Proposition 4.4:** If each non-zero fuzzy $G_δ$-set is a fuzzy dense set in a fuzzy submaximal space, then $(X,T)$ is a fuzzy almost $P$-space.

**Proof:** Let $λ$ be a non-zero fuzzy $G_δ$-set in $(X,T)$ such that $\text{cl}(λ) = 1$. Since $(X,T)$ is a fuzzy submaximal space, the fuzzy dense set in $(X,T)$ is a fuzzy open set in $(X,T)$ and hence is a fuzzy semi-open set in $(X,T)$. Thus, each non-zero fuzzy $G_δ$-set is a fuzzy semi-open set in $(X,T)$. Then, by proposition 3.3, $(X,T)$ is a fuzzy almost $P$-space.

**Definition 4.5** [12]: A fuzzy topological space $(X,T)$ is called a fuzzy D-Baire space if every fuzzy first category set in $(X,T)$, is a fuzzy nowhere dense set in $(X,T)$.

The following proposition gives a condition for a fuzzy almost $P$-space to be a fuzzy D-Baire space.

**Proposition 4.5:** If $\text{int}(μ) = θ$, for each fuzzy closed set $μ$ in a fuzzy almost $P$-space, then $(X,T)$ is a fuzzy D-Baire space.

**Proof:** Let $λ$ be a fuzzy first category set in $(X,T)$. Since $(X,T)$ is a fuzzy almost $P$-space, by proposition 3.4, $\text{cl}(λ) ≠ 1$, in $(X,T)$. Let $\text{cl}(λ) = μ$, where $1− ∈ T$. Then $\text{int}[\text{cl}(λ)] = \text{int}[μ] = 0$ (by hypothesis). Thus $λ$ is a fuzzy nowhere dense set in $(X,T)$. Hence each fuzzy first category set in $(X,T)$, is a fuzzy nowhere dense set in $(X,T)$. Therefore $(X,T)$ is a fuzzy D-Baire space.

**Definition 4.6** [18]: A fuzzy topological space $(X,T)$ is called a fuzzy $GID$-space if for each fuzzy dense and fuzzy $G_δ$-set in $(X,T)$, $\text{cl} \text{int}(λ) = I$, in $(X,T)$.

**Theorem 4.2** [18]: If each fuzzy $G_δ$-set is fuzzy dense in a fuzzy $GID$ space, then $(X,T)$ is a fuzzy almost $P$-space.
Definition 4.6 [9]: A fuzzy topological space \( X \) is said to be fuzzy hyperconnected if every non-null fuzzy open subset of \( X \) is fuzzy dense in \( X \). That is, a fuzzy topological space \((X,T)\) is fuzzy hyperconnected if \( \text{cl}(\mu_i) = 1 \), for all \( \mu_i \in T \).

Proposition 4.7: If a fuzzy almost P space \((X,T)\) is a fuzzy hyperconnected space, then \((X,T)\) is a fuzzy GID space.

Proof: Let be a fuzzy dense and fuzzy \( G_\delta \)-set in \((X,T)\). Since \((X,T)\) is a fuzzy almost P – space, for the fuzzy \( G_\delta \)-set in \((X,T)\), we have \( \text{int}(\lambda) \neq 0 \) in \((X,T)\).

Also, since \((X,T)\) is a fuzzy hyperconnected, for the fuzzy open set \( \text{int}(\lambda) \) in \((X,T)\), we have \( \text{cl}(\text{int}(\lambda)) = 1 \). Hence for a fuzzy dense and fuzzy \( G_\delta \)-set in \((X,T)\), we have \( \text{cl} \text{ int}(\lambda) = 1 \), in \((X,T)\). Therefore \((X,T)\) is a fuzzy GID space.

5. FUZZY ALMOST P-SPACES AND FUNCTIONS.

Theorem 5.1 [22]: Let \( f : (X,T) \rightarrow (Y,S) \) be a fuzzy open function from a fuzzy topological space \((X,T)\) into a fuzzy topological space \((Y,S)\). Then, for every fuzzy set \( \beta \) in \((Y,S)\), \( f^{-1}(\text{cl}(\beta)) \leq \text{cl} f^{-1}(\beta) \).

Theorem 5.2 [19]: Let \( f : (X,T) \rightarrow (Y,S) \) be a fuzzy open function from a fuzzy topological space \((X,T)\) into a fuzzy topological space \((Y,S)\). Then, for every fuzzy set \( \delta \) in \((Y,S)\), \( \text{int}(f^{-1}(\delta)) \leq f^{-1}(\text{int}(\delta)) \).

Definition 5.1 [14]: A function \( f : (X,T) \rightarrow (Y,S) \) from a fuzzy topological space \((X,T)\) into another fuzzy topological space \((Y,S)\) is called somewhat fuzzy
continuous if $\lambda \in S$ and $f^{-1}(\lambda) \neq 0$ implies that there exist a fuzzy open set $\delta$ in $(X,T)$ such that $\delta \neq 0$ and $\delta \leq f^{-1}(\lambda)$.

**Definition 5.2 [14]**: A function $f : (X,T) \rightarrow (Y,S)$ from a fuzzy topological space $(X,T)$ into another fuzzy topological space $(Y,S)$ is called somewhat fuzzy open if $\lambda \in T$ and $\lambda \neq 0$ implies that there exists a fuzzy open set $\eta$ in $(Y,S)$ such that $\eta \neq 0$ and $\eta \leq f(\lambda)$.

Let $f$ be a function from the fuzzy topological space $(X,T)$ to the fuzzy topological space $(Y,S)$. Under what conditions on “$f$” may we assert that if $(X,T)$ is a fuzzy almost $P$-space, then $(Y,S)$ is a fuzzy almost $P$-space? The following propositions establish the desired conditions.

**Proposition 5.1**: If a function $f : (X,T) \rightarrow (Y,S)$ from a fuzzy topological space $(X,T)$ into another fuzzy topological space $(Y,S)$ is fuzzy continuous function and if $\lambda$ is a fuzzy $G_\delta$-set in $(Y,S)$, then $f^{-1}(\lambda)$ is a fuzzy $G_\delta$-set in $(X,T)$.

**Proof**: Let be a fuzzy $G_\delta$-set in $(Y,S)$. Then, $= \Lambda^\infty_{=l}(\ )$, where $(\lambda_i)$’s are fuzzy open sets in $(Y,S)$. Now $f^{-1}(\ ) = f^{-1}(\Lambda^\infty_{=l}(\ )) = \Lambda^\infty_{=l} [ f^{-1}(\ ) ]$. Since the function $f$ is fuzzy continuous and $(\lambda_i)$’s are fuzzy open sets in $(Y,S)$, $[ f^{-1}(\ ) ]$’s are fuzzy open sets in $(X,T)$. Hence $f^{-1}(\lambda)$ is a fuzzy $G_\delta$-set in $(X,T)$.

**Proposition 5.2**: If a function $f : (X,T) \rightarrow (Y,S)$ from a fuzzy topological space $(X,T)$ onto another fuzzy topological space $(Y,S)$ is fuzzy continuous, and fuzzy open function and if $(X,T)$ is a fuzzy almost $P$-space, then $(Y,S)$ is a fuzzy almost $P$-space.
**Proof:** Let be a fuzzy $G_\delta$-set in $(Y,S)$. Since $f$ is a fuzzy continuous function, by proposition 5.1, $f^{-1}(\ )$ is a fuzzy $G_\delta$-set in $(X,T)$. Since $(X,T)$ is a fuzzy almost $P$-space, $\text{int}[f^{-1}(\ )] \neq 0$, in $(X,T)$. Since $f$ is a fuzzy open function from $(X,T)$ onto $(Y,S)$, for the fuzzy set in $(Y,S)$, by theorem 5.2, $\text{int}(f^{-1}(\ )) \leq f^{-1}(\text{int}(\ ))$. Let $\mu = \text{int}[f^{-1}(\ )]$. Then, $\mu$ is a fuzzy open set in $(X,T)$. Now $\mu \leq f^{-1}(\text{int}(\ ))$ implies that $f(\mu) \leq f[f^{-1}(\text{int}(\ ))]$. Since the function $f$ is onto, $f[f^{-1}(\text{int}(\ ))] = \text{int}(\ )$ and hence $f(\mu) \leq \text{int}(\ )$. Since $f$ is a fuzzy open function from $(X,T)$ onto $(Y,S)$, for the fuzzy open set $\mu$ in $(X,T)$, $f(\mu)$ is a fuzzy open set in $(Y,S)$. Hence we have $\text{int}(\ ) \neq 0$, for the fuzzy $G_\delta$-set in $(Y,S)$. Therefore $(Y,S)$ is a fuzzy almost $P$-space.

**Proposition 5.3:** If the function $f : (X,T) \rightarrow (Y,S)$ from a fuzzy topological space $(X,T)$ onto another fuzzy topological space $(Y,S)$ is a fuzzy continuous and somewhat fuzzy open function and if $(X,T)$ is a fuzzy almost $P$-space, then $(Y,S)$ is a fuzzy almost $P$-space.

**Proof:** Let be a fuzzy $G_\delta$-set in a fuzzy topological space $(Y,S)$. Then, $\Lambda^\infty = \Lambda^\infty$, where $(\lambda_i)'$s are fuzzy open sets in $(Y,S)$. Now $f^{-1}(\ ) = f^{-1}(\Lambda^\infty) = \Lambda^\infty \Lambda^\infty f^{-1}(\lambda_i)$. Since $f$ is a fuzzy continuous function from $(X,T)$ onto $(Y,S)$ and $(\lambda_i)'$s are fuzzy open sets in $(Y,S)$, $f^{-1}(\lambda_i)'$s are fuzzy open sets in $(X,T)$. Then $f^{-1}(\ )$ is a fuzzy $G_\delta$-set in $(X,T)$. Since $(X,T)$ is a fuzzy almost $P$-space, $\text{int}[f^{-1}(\ )] \neq 0$. Then there exists a fuzzy open set $\mu$ in $(X,T)$ such that $\mu \leq f^{-1}(\ )$. Then, $f(\mu) \leq f[f^{-1}(\ )]$. Since the function $f$ is onto, by lemma 2.3, $f[f^{-1}(\lambda_i)] = \lambda_i$. Hence we have $\mu \leq \lambda_i$. Since $f$ is a somewhat fuzzy open function from $(X,T)$ onto $(Y,S)$, for the fuzzy open set $\mu$ in $(X,T)$, there exists a fuzzy open set $\eta$ in $(Y,S)$ such that $\neq$.
0 and \( \eta \leq f(\mu) \). Thus we have \( \eta \leq f(\mu) \) and hence \( \eta \leq \). Hence we have \( \text{int}(\ ) \neq \emptyset \), for the fuzzy \( G_\delta \)-set in \((Y,S)\). Therefore \((Y,S)\) is a fuzzy almost P-space.

**Proposition 5.4**: If the function \( f : (X,T) \to (Y,S) \) from a fuzzy topological space \((X,T)\) onto another fuzzy topological space \((Y,S)\) is a somewhat fuzzy continuous, one-to-one and fuzzy open function and if \((Y,S)\) is a fuzzy almost P-space, then \((X,T)\) is a fuzzy almost P-space.

**Proof**: Let \( \lambda \) be fuzzy \( G_\delta \)-set in the fuzzy topological space \((X,T)\). Then 
\[
\Lambda_{i=1}^{\infty} (\lambda_i) = \Lambda_{i=1}^{\infty} (\lambda_i) = \bigvee_{i=1}^{\infty} (I - \lambda_i).
\]
Since \( f \) is one-to-one and onto, by lemma 2.3, \( f(1 - \lambda) = 1 - f(\lambda) \) and hence \( 1 - f(\lambda) = f \left[ \bigvee_{i=1}^{\infty} (I - \lambda_i) \right] = \bigvee_{i=1}^{\infty} f(I - \lambda_i) = \bigvee_{i=1}^{\infty} [I - f(\lambda_i)]. \) Then, we have \( f(\lambda) = \bigvee_{i=1}^{\infty} f(\lambda_i). \) Since \( f \) is a fuzzy open function from \((X,T)\) onto \((Y,S)\), for the fuzzy open sets \((\lambda_i)\)'s in \((X,T)\), we have \((f(\lambda_i))\)'s are fuzzy open sets in \((Y,S)\) and hence \( \Lambda_{i=1}^{\infty} f(\lambda_i) \) is a fuzzy \( G_\delta \)-set in \((Y,S)\). Then, \( f(\lambda) \) is a fuzzy \( G_\delta \)-set in \((Y,S)\). Since \((Y,S)\) is a fuzzy almost P-space, \( \text{int}[f(\lambda)] \neq \emptyset. \) Then, there exists a fuzzy open set \( \mu \) in \((Y,S)\) such that \( \mu \leq f(\lambda) \). This implies that \( f^{-1}(\mu) \leq f^{-1}(\lambda) \) (since \( f \) is one-to-one, \( f^{-1}(\lambda) = \lambda \)). Hence we have \( f^{-1}(\mu) \leq \mu \). Since \( f \) is a somewhat fuzzy continuous function, for the fuzzy open set \( \mu \) in \((Y,S)\), there exist a fuzzy open set \( \delta \) in \((X,T)\) such that \( \delta \neq 0 \) and \( \delta \leq f^{-1}(\mu) \). Then \( \delta \leq f^{-1}(\mu) \leq \mu \) and hence \( \text{int}(\ ) \neq \emptyset \) in \((X,T)\). Therefore \((X,T)\) is a fuzzy almost P-space.

**REFERENCES**

[1]. K. K. Azad., *On fuzzy semi continuity, fuzzy almost continuity and fuzzy weakly*


