Analytic Mean Labeling of Cycle Related Graphs

P. Lawrence Rozario Raj¹ and K. Vivek²

¹ Assistant Professor, P.G. and Research Department of Mathematics, St. Joseph’s College, Tiruchirappalli – 620 002, Tamil Nadu, India.

² M.Phil Scholar, P.G. and Research Department of Mathematics, St. Joseph’s College, Tiruchirappalli – 620 002, Tamil Nadu, India.

Abstract
In this paper, we investigate an analytic mean labeling for some graphs obtained by duplication of graph elements, \( S'(K_{1,n}) \) and tadpole.

Keywords: Analytic Mean Labeling, Analytic Mean Graph, Duplication of a vertex, Duplication of an edge.

1. Introduction

By a graph, we mean a finite, undirected graph without loops and multiple edges, for terms not defined here, we refer to Harary [2]. For standard terminology and notations related to graph labeling, we refer to Gallian [1]. In [4], Tharmaraj et al. introduce the concept of an analytic mean labeling of graph. Analytic mean labeling of various types of graphs are presented in [5]. The brief summaries of definition which are necessary for the present investigation are provided below.

2. Definitions

Definition :2.1
A graph labeling is the assignment of unique identifiers to the edges and vertices of a graph.

Definition :2.2 [3]
For a graph \( G \), the splitting graph \( S'(G) \) of a graph \( G \) is obtained by adding a new vertex \( v' \) corresponding to each vertex \( v \) of \( G \) such that \( N(v) = N(v') \).

Definition :2.3 [7]
Duplication of a vertex \( v_k \) by a new edge \( e = v_i v_{i+1} \) in a graph \( G \) produces a new graph \( G' \) such that \( N(v_k) = \{v_i, v_{i+1}\} \).

Definition :2.4 [7]
Duplication of an edge \( e = v_i v_{i+1} \) by a vertex \( v_k \) in a graph \( G \) produces a new graph \( G' \) such that \( N(v_k) = \{v_i, v_{i+1}\} \).

Definition :2.5 [9]
Duplication of a vertex \( v_k \) of a graph \( G \) produces a new graph \( G' \) by adding a new vertex \( v'_k \) in such that \( N(v_k) = N(v'_k) \).

Definition :2.6 [8]
Duplication of an edge \( e_k = v_k v_{k+1} \) of a graph \( G \) produces a new graph \( G' \) by adding an edge \( e'_k = v'_k v'_{k+1} \) such that \( N(v'_k) = N(v_k) \cup \{v'_{k+1}\} \) and \( N(v'_{k+1}) = N(v_k) \).

Definition :2.7 [6]
The tadpole graph is formed by joining the end point of a path \( P_m \) to a cycle \( C_n \). It is denoted by \( C_n @ P_m \).

Definition :2.8 [4]
A \((p,q)\) graph \( G(V,E) \) is said to be an analytic mean graph if it is possible to label the vertices \( v \) in \( V \) with distinct elements from \( 0,1,2,\ldots, p-1 \) in such a way that when each edge \( e = uv \) is labeled with \( f^*(e) = |f(u) - f(v)|^2 \) if \( |f(u)|^2 - |f(v)|^2 \) is even and if \( |f(u)|^2 - |f(v)|^2 \) is odd and the edge labels are distinct. In this case, \( f^* \) is called an analytic mean labeling of \( G \). A graph with an analytic mean labeling is called an analytic mean graph.
3. Main Results

Theorem 3.1

$S'(K_{1,n})$ is an analytic mean graph.

Proof:

Let $v_1, v_2, v_3, \ldots, v_n$ be the pendant vertices and $v$ be the apex vertex of $K_{1,n}$ and $u, u_1, u_2, u_3, \ldots, u_n$ are added vertices corresponding to $v, v_1, v_2, v_3, \ldots, v_n$ to obtain $S'(K_{1,n})$.

Let $G$ be the graph $S'(K_{1,n})$

Then $|V(G)| = 2n + 2$ and $|E(G)| = 3n$.

Define $f: V(G) \rightarrow \{0, 1, 2, \ldots, 2n+1\}$ by

$f(v) = 0$,

$f(u) = 1$,

$f(v_i) = 2i + 1$, for $1 \leq i \leq n$;

$f(u_i) = 2i$, for $1 \leq i \leq n$;

Let $f^*$ be the induced edge labeling of $f$.

Then the induced edge labels are

$\{2, 4, 5, \ldots, \frac{2(2n+1)^2 + 4n + 2}{2}\}$.

Therefore, $S'(K_{1,n})$ is an analytic mean graph.

Example 3.1

Analytic mean labeling of $S'(K_{1,4})$ is given in figure 3.1.

Theorem 3.2

The graph obtained by duplication of an arbitrary vertex by a new edge in cycle $C_n$ admits an analytic mean labeling.

Proof:

Let $v_1, v_2, v_3, \ldots, v_n$ be the vertices of cycle $C_n$ and let $G$ be the graph obtained by duplicating an arbitrary vertex of $C_n$ by a new edge.

Without loss of generality let this vertex be $v_1$ and the edge be $e = v_1' v_1''$.

Then $|V(G)| = n+2$ and $|E(G)| = n+3$.

Define $f: V(G) \rightarrow \{0, 1, 2, \ldots, n+1\}$ by

For $n = 3$

$f(v_1) = 0$, $f(v_2) = 2$, $f(v_3) = 4$, $f(v_1') = 3$ and $f(v_1'') = 1$.

Let $f^*$ be the induced edge labeling of $f$.

Then $f^*(v_1v_2) = 2$, $f^*(v_2v_3) = 6$, $f^*(v_3v_1) = 8$, $f^*(v_1'v_1) = 5$, $f^*(v_1''v_1) = 1$ and $f^*(v_1'v_1'') = 4$.

Then the induced edge labels are $\{1, 2, 4, 5, 6, 8\}$.

For $n = 4$

$f(v_1) = 0$, $f(v_2) = 2$, $f(v_3) = 3$, $f(v_4) = 4$, $f(v_1') = 5$ and $f(v_1'') = 1$.

Let $f^*$ be the induced edge labeling of $f$.

Then $f^*(v_1v_2) = 2$, $f^*(v_2v_3) = 3$, $f^*(v_3v_4) = 4$, $f^*(v_4v_1) = 8$, $f^*(v_1'v_1) = 13$, $f^*(v_1''v_1) = 1$ and $f^*(v_1'v_1'') = 12$.

Then the induced edge labels are $\{1, 2, 3, 4, 8, 12, 13\}$.

For $n \geq 5$

$f(v_i') = n$

$f(v_i'') = n+1$

$f(v_i) = i-1$, for $1 \leq i \leq n$.

Let $f^*$ be the induced edge labeling of $f$.

Then $f^*(v_1v_i) = i$, for $1 \leq i \leq n-1$;

$f^*(v_i'v_i'') = n+1$,

$f^*(v_1v_1') = \frac{(n-1)^2}{2}$ if $n$ is odd

$f^*(v_1v_1') = \frac{(n-1)^2 + 1}{2}$ if $n$ is even

$f^*(v_1''v_1) = \frac{n^2 + 1}{2}$ if $n$ is odd

$f^*(v_1''v_1) = \frac{n^2}{2}$ if $n$ is even

$f^*(v_i'v_i) = \frac{(n+1)^2}{2}$ if $n$ is odd

$f^*(v_i'v_i) = \frac{(n+1)^2 + 1}{2}$ if $n$ is even

Then the induced edge labels are

Figure 3.1
The natural text is as follows:

Therefore, the graph obtained by duplication of an arbitrary vertex by a new edge in cycle \( C_n \) is an analytic mean graph.

**Example 3.2**

Analytic mean labeling of the graph obtained by duplication of an arbitrary vertex by a new edge in cycle \( C_8 \) is given in figure 3.2.

**Theorem 3.3**

Duplication of an arbitrary edge by a new vertex in cycle \( C_n \) produces an analytic mean graph.

Proof:

Let \( v_1, v_2, \ldots, v_n \) be the vertices of cycle \( C_n \).

Let \( G \) be the graph obtained by duplication of an arbitrary edge in \( C_n \) by a new vertex.

Without loss of generality let this edge be \( e = v_n v_1 \) and the vertex be \( v' \).

Then \(|V(G)| = n+1 \) and \(|E(G)| = n+2\).

Define \( f : V(G) \rightarrow \{0,1,2,\ldots, n\} \) by

For \( n = 3 \)

\[
\begin{align*}
  f(v_1) &= 0, \\
  f(v_2) &= 1, \\
  f(v_3) &= 3 \quad \text{and} \\
  f(v') &= 2.
\end{align*}
\]

Let \( f' \) be the induced edge labeling of \( f \).

Then \( f'(v_1 v_2) = 1, f'(v_2 v_3) = 4, f'(v_3 v_1) = 5, f'(v_3 v') = 3 \) and \( f'(v_1 v') = 2 \).

Then the induced edge labels are \( \{1, 2, 3, 4, 5\} \).

Therefore, the graph obtained by duplication of an arbitrary edge by a new vertex in cycle \( C_n \) is an analytic mean graph.

**Example 3.3**

Analytic mean labeling of the graph obtained by duplication of an arbitrary edge by a new vertex in cycle \( C_8 \) is given in figure 3.3.

**Theorem 3.4**

The graph obtained by duplication of an arbitrary vertex of \( C_n \) admits an analytic mean labeling.

Proof:

Let \( v_1, v_2, \ldots, v_n \) be the vertices of the cycle \( C_n \).

Let \( G \) be the graph obtained by duplicating an arbitrary vertex of \( C_n \).

Without loss of generality let this vertex be \( v_1 \) and the newly added vertex be \( v'_1 \).

Then \( |V(G)| = n+1 \) and \(|E(G)| = n+2\).

Define \( f : V(G) \rightarrow \{0,1,2,\ldots, n\} \) by

For \( n = 3 \)

\[
\begin{align*}
  f(v_1) &= 0, \\
  f(v_2) &= 1, \\
  f(v_3) &= 3 \quad \text{and} \\
  f(v'_1) &= 2.
\end{align*}
\]

Let \( f' \) be the induced edge labeling of \( f \).
Then $f^*(v_1v_2) = 1$, $f^*(v_2v_3) = 4$, $f^*(v_3v_1) = 5$, $f^*(v_3v_1') = 3$ and $f^*(v_1'v_2) = 2$.

Then the induced edge labels are $\{1, 2, 3, 4, 5\}$.

For $n \geq 4$

$\begin{align*}
&\text{If } n \text{ is odd, then } f(v_i') = n, \\
&\text{If } n \text{ is even, then } f(v_i') = n+1,
\end{align*}$

Let $f^*$ be the induced edge labeling of $f$.

Then

$\begin{align*}
&f^*(v_1v_i) = \begin{cases} 
(n-1)^2 & \text{if } n \text{ is odd} \\
\frac{(n-1)^2}{2} + 1 & \text{if } n \text{ is even}
\end{cases} \\
&f^*(v_1v_i') = f(v_i) - 1
\end{align*}$

Then the induced edge labels are $\{1, 2, 3, \ldots, \frac{n^2-1}{2} \text{ or } \frac{n^2}{2}\}$.

Therefore, the graph obtained by duplication of an arbitrary vertex of $C_n$ is an analytic mean graph.

**Example 3.4**

Analytic mean labeling of the graph obtained by duplication of an arbitrary vertex in cycle $C_n$ is given in figure 3.4.

![Figure 3.4](image)

**Theorem 3.5**

The graph obtained by duplication of an arbitrary edge in $C_n$ admits an analytic mean labeling.
Example 3.5

Analytic mean labeling of the graph obtained by duplication of an arbitrary edge in cycle $C_8$ is given in figure 3.5.

![Figure 3.5](image)

Theorem 3.6

The tadpole $C_n \circ P_m$ is an analytic mean graph.

Proof:

Let $v_1, v_2, \ldots, v_m$ be the vertices of path $P_m$ and $v_m, v_{m+1}, \ldots, v_{m+n-1}$ be the vertices of cycle $C_n$.

Tadpole $G = C_n \circ P_m$ has $m+n-1$ vertices and $m+n-1$ edges.

Then $|V(G)| = m+n-1$ and $|E(G)| = m+n-1$.

Define $f : V(G) \rightarrow \{0,1,2,\ldots,m+n-2\}$ by

$$f(v_i) = i - 1, \quad \text{for } 1 \leq i \leq m+n-1$$

Let $f^*$ be the induced edge labeling of $f$.

Then

$$f^*(v_iv_{i+1}) = i, \quad \text{for } 1 \leq i \leq m+n-2$$

$$f^*(v_{m+n-1}v_1) = \begin{cases} n^2 + 2nm - 2m - 4n + 4 & \text{if } m+n \text{ is odd} \\ n^2 + 2nm - 2m - 4n + 3 & \text{if } m+n \text{ is even} \end{cases}$$

Then the induced edge labels are $\{1,2,3,\ldots,\frac{n^2 + 2nm - 2m - 4n + 4}{2} \text{ or } \frac{n^2 + 2nm - 2m - 4n + 3}{2}\}$.

Therefore, tadpole $C_n \circ P_m$ is an analytic mean graph.

Example 3.6

Analytic mean labeling tadpole graph $C_4 \circ P_5$ is given in figure 3.6.

![Figure 3.6](image)

4. Conclusions

In this paper, we prove an analytic mean labeling of $S'(K_{1,n})$, the graph obtained by duplication of an arbitrary vertex by a new edge in cycle $C_n$, the graph obtained by duplication of an arbitrary edge by a new vertex in cycle $C_n$, the graph obtained by duplication of an arbitrary vertex by of $C_n$, the graph obtained by duplication of an arbitrary edge of $C_n$ and tadpole.

References