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Analytical Approximate Solutions For Oscillators With Fractional Order Restoring Force And Relativistic Oscillators

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Abstract
Oscillators with fractional order restoring forces and relativistic oscillators are investigated in this paper by using the Equivalent Linearization method based on a weighted averaging. Averaging value is calculated in a new way called the weighted averaging value by introducing a weighted coefficient function. The amplitude-frequency relationships are presented by considering three nonlinear oscillators. The obtained solutions have been compared with approximate analytical solutions, exact solutions and numerical solutions. Comparisons show the reliability of this method.

Keywords: *equivalent linearization method, weighted averaging, fractional order, relativistic oscillator.*

1. Introduction

Nonlinear oscillation problem is very important in the physical science, mechanical structures and other kind of mathematical sciences. Most of real systems are modeled by nonlinear differential equations which are important issues in mechanical structures, mathematical physics and engineering. In most cases, it is difficult to solve such equations, especially analytically; and in addition, the most important information such as the natural circular frequency of a nonlinear oscillation which depends on the initial conditions will be lost during the procedure of numerical simulation.

Oscillators with fractional order restoring forces and relativistic oscillators are common problems in engineering and have attracted the attention of many authors. The exact expression for the frequency of oscillators with general positive power non-linearity in the force equation and the corresponding direct first order Harnomic Balance approximate result utilizing the first Fourier coefficient are presented by Gottlied [7]. Mickens used the Harnomic Balance method (HBM) to analyse oscillations in an $x^{4/3}$ potential [9]. He's modified Lindstedt-Poincaré method (MLPM) is applied by Yildirim to analyse nonlinear oscillatiors with fractional powers [10]. Cveticanin et al. [11] used Hamiltonian Approach (HA) to the generalized nonlinear oscillator with fractional power. A new analytical method for solving the differential equations which describe the motion of the oscillator with fraction order elastic force is introduced by Cveticanin [12]. Using the first integral of motion, the exact period of vibration in the form of the Euler Beta function is obtained. By applying He's Modified Lindstedt-Poincaré method (MLPM), Yildirim [14] has determined the periodic solutions to nonlinear oscillator equations for case of the elastic restoring forces are nonpolynomial functions of the displacement. Mickens [8] has used the Harnomic Balance method (HBM) to determine Periodic solutions of the relativistic harmonic oscillator. Younesian et al. [13] used He's Energy Balance method (EBM) to present analytical approximate solutions for the generalized nonlinear oscillator. Azami et al. [15] applied He's Min-Max approach (MMA) for analyzing of the relativistic oscillator. The relativistic harmonic oscillator equation is investigated by Biazar et al. [16] using Homotopy perturbation method (HPM). And approximate analytical solutions for oscillation of a mass attached to a stretched elastic wire are presented by Sun et al. [17]. The approximation is based on combining Newton's method with the Harmonic Balance method.

The Equivalent Linearization method (EML) is one of the common approaches to approximate analysis of dynamical systems. The original linearization for deterministic systems was proposed by Krylov and Bugoliubov [1]. Then Caughey [2] expanded the method for stochastic systems. To date, there have been some extended versions of the Equivalent Linearization method [3, 4]. It has been shown that the Equivalent Linearization method is presently the simplest tool widely used for analyzing nonlinear stochastic problems. Nevertheless, the accuracy of the Equivalent Linearization method with conventional averaging normally reduces for middle or strong nonlinear systems. A reason is that some terms will vanish in the averaging process, for example, the averaging values of the functions *sin(t)* and *cos(t)* over one period will be equal to zero. Recently, Anh [5] proposed a new way for determining averaging values, instead of using conventional averaging process author introduced weighted coefficient functions, thus the averaging value was given in a new way called the weighted averaging value. And the proposed method have been applied effectively in analyzing of some strongly nonlinear oscillators such as the nonlinear Duffing oscillator with third, fifth, and seventh powers of the amplitude, the strongly nonlinear oscillators in forms $(1+\varepsilon u^2)\ddot{u}+u=0$ and $\ddot{u}+\dfrac{u^3}{1+u^2}=0$, and the cubic Duffing with discontinuity [6].

In this paper, Anh's method will be applied to analyse oscillators with fractional order restoring force and relativistic oscillators.

2. Overview of the study method

To introduce an overview of the Equivalent Linearization method, we consider the oscillation described by following nonlinear differential equation:

$$\ddot{X}+g(X)=0, \quad X(0)=A, \; \dot{X}(0)=0 \tag{1}$$

where *g(X)* is a nonlinear function of *X* and *A* is the initial amplitude.

The idea of the Equivalent Linearization method is to replace the nonlinear term *g(X)* in Eq. (1) by the linear term as follows:

$$g(X)\rightarrow \alpha X \tag{2}$$

By this manner, the linearized equation of Eq. (1) is given by:

$$\ddot{X}+\alpha X=0 \tag{3}$$

where the cofficient α of the linear term is determined by using the mean-square criterion:

$$e(X)=g(X)-\alpha X \rightarrow \underset{\alpha}{Min} \tag{4}$$

Thus, from:

$$\frac{\partial\left\langle e^2(X)\right\rangle}{\partial\alpha}=0$$

yields:

$$\alpha=\frac{\left\langle g(X)X\right\rangle}{\left\langle X^2\right\rangle} \tag{5}$$

In Eq. (5), the symbol $\langle\square\rangle$ donotes the time-averaging operator in classical meaning [1]:

$$\left\langle f(t)\right\rangle=\lim_{T\to+\infty}\frac{1}{T}\int_0^T f(t)dt \tag{6}$$

For a ω-frequency function $f(\omega t)$, the averaging process is taken during one period *T*, i.e.

$$\left\langle f(\omega t)\right\rangle=\frac{1}{T}\int_0^T f(\omega t)dt=\frac{1}{2\pi}\int_0^{2\pi} f(\tau)d\tau, \quad \tau=\omega t \tag{7}$$

The everaging values in Eqs. (6) and (7) are called the classical or conventional averaging values. They often give incorrect results, especially for some periodic functions such as *sine* or *cosine* ones.

In this paper, the weighted averaging value proposed by Anh [5] is used to calculate averaging values in Eq. (5) instead of the conventional averaging values in Eq. (6) or (7). The idea of the proposed method as follows [5]: replacing the constant coefficient *1/T* in Eq. (6) by a weighted coefficient function *h(t)*. Thus we get so-called a weighted average value:

$$\left\langle x(t)\right\rangle=\int_0^T h(t)x(t)dt \tag{8}$$

where *h(t)* satisfies the condition:

$$\int_0^T h(t)dt=1$$

In the Ref. [5], Anh has proposed a weighted function as follows:

$$h(t)=s^2\omega^2 te^{-s\omega t}, s>0 \tag{9}$$

where *s* is constant.

It is seen that the weighted coefficient (9), obtained as a product of the optimistic weighted coefficient *t* and the pessimistic weighted coefficient $e^{-s\omega t}$, has one maximal value at $t_{\max}=1/(\omega s)$, and then decreases to zero as $t\rightarrow\infty$. If one requires that the time t_{max} is equal to $T/n=2\pi/(n\omega)$ where *n* is a natural number or zero, we get $s=n/(2\pi)$. So the meaning of *s* can be specified as follows: for $n = 1$, $s=1/(2\pi)$ the weighted coefficient (9) has maximal value after one period, and for $n=4$, $s=4/(2\pi)$ the weighted coefficient (9) has maximal value after quarter period, and for $n=0$, $s=0$ the weighted coefficient (9) has maximal value at infinity, this case corresponds to the conventional averaging value. The detailed properties of the weighted function *h(t)* in Eq. (9) can be viewed in Refs. [5, 6].

With the priodic solution of linearized equation (3), the averaging values in Eq. (5) can be calculated by using Eq. (8):

$$\left\langle x(\omega t)\right\rangle = \int_0^{+\infty} s^2\omega^2 te^{-s\omega t}x(\omega t)dt \qquad (10)$$

As ω – periodic function $x(\omega t)$ can be expanded into Fourier series, hence we can easy calulate (10) by using Laplace tranformation.

In the next section, the proposed method will be applied to analyse some oscillations with fraction order restoring force and relativistic oscillations.

3. Some Examples and Discussions

3.1 Case 1

The oscillator equation with single-term positive-power non-linearity [10]:

$$\ddot{X} + sign(X)\left|X\right|^p = 0,\quad X(0) = A,\quad \dot{X}(0) = 0 \qquad (11)$$

where p is a constant ($p<1$) and $sign(X)$ is the sign function.

For this oscillator, the nonlinear term is:

$$g(X) = sign(X)\left|X\right|^p \qquad (12)$$

From Eq. (5), the approximate frequency ω is determined as follows:

$$\omega = \sqrt{\frac{\left\langle sign(X)X\left|X\right|^p\right\rangle}{\left\langle X^2\right\rangle}} \qquad (13)$$

The priodic solution of the linearized equation is:

$$X(t) = A\cos(\omega t) \qquad (14)$$

With priodic solution (14), we calculate averaging operators in Eq. (13) by using Eq. (8):

$$\left\langle X^2\right\rangle = \left\langle A^2\cos^2\omega t\right\rangle = \int_0^{+\infty} A^2s^2\omega^2te^{-s\omega t}\cos^2(\omega t)\,dt$$
$$= \int_0^{+\infty} A^2s^2\tau e^{-s\tau}\cos^2(\tau)\,d\tau = A^2\frac{s^4+2s^2+8}{(s^2+4)^2} \qquad (15)$$

$$\left\langle sign(X)X\left|X\right|^p\right\rangle = \left\langle A^{p+1}sign[A\cos(\omega t)]\cos(\omega t)\left|\cos(\omega t)\right|^p\right\rangle$$
$$= \int_0^{+\infty} A^{p+1}s^2\omega^2te^{-s\omega t}sign[\cos(\omega t)]\cos(\omega t)\left|\cos(\omega t)\right|^p\,dt$$
$$= \int_0^{+\infty} A^{p+1}s^2\tau e^{-s\tau}sign[\cos(\tau)]\cos(\tau)\left|\cos(\tau)\right|^p\,d\tau \qquad (16)$$

It requires the evaluation of the first Fourier (cosine) series of $sign[A\cos(\tau)]\left|\cos(\tau)\right|^p$, as follows [10]:

$$sign[\cos(\tau)]\left|\cos(\tau)\right|^p = $$
$$= \frac{4\Gamma(1+p/2)}{\sqrt{\pi}(p+1)\Gamma(p/2+1/2)}\cos(\tau) \qquad (17)$$
$$+\text{higher order harmonics}$$

Substituting Eq. (17) into Eq. (16), we get:

$$\left\langle sign(X)X\left|X\right|^p\right\rangle =$$
$$= A^{p+1}\frac{4\Gamma(1+p/2)}{\sqrt{\pi}(p+1)\Gamma(p/2+1/2)}\frac{s^4+2s^2+8}{(s^2+4)^2} \qquad (18)$$

Substituting Eqs. (15) and (18) into Eq. (13), we get the approximate frequency:

$$\omega = A^{(p-1)/2}\sqrt{\frac{4\Gamma(1+p/2)}{\sqrt{\pi}(p+1)\Gamma(p/2+1/2)}} \qquad (19)$$

and from Eq. (14) we have the approximate solution:

$$X(t) = A\cos\left[A^{(p-1)/2}\sqrt{\frac{4\Gamma(1+p/2)}{\sqrt{\pi}(p+1)\Gamma(p/2+1/2)}}\,t\right] \qquad (20)$$

The approximate frequency in Eq. (19) is the same as the frequency obtained by Gottlied et al. [7] as well as Yildirim [10]. Table 1 compares frequency commensurate for different values of p, obtained from the proposed method and the Harnomic Balance method (HBM) (Mickens [9]). The exact values for some cases are also reported. It is seen that the present method accuracy is higher than the HBM.

The exact frequency is given (using Gamma functions, where $p<1$) [7]:

$$\omega_{ex} = \frac{\sqrt{\pi}}{2\sqrt{2}}\frac{(1-p)}{\sqrt{1+p}}\frac{\Gamma\left(\frac{1-p}{2(1+p)}\right)}{\Gamma\left(\frac{1}{1+p}\right)}A^{\frac{p-1}{2}} \qquad (21)$$

The graph of relative error based on the parameter p is shown in Fig. 1, it shows that when p increases, the relative error decreases.

Table 1. Comparison of some values of the frequency of the nonlinear oscillator with various fractional powers, Case 1

p	$\omega_{ex}A^{(1-p)/2}$ [7]	$\omega_{HBM}\ A^{(1-p)/2}$ (*R. Error %*) [9]	$\omega_{present}\ A^{(1-p)/2}$ (*R. Error %*)
1	1.00000	1.00000 (0.00%)	1.00000 (0.00%)
3/4	1.02496	-	1.02567 (0.07%)
5/7	1.02866	-	1.02961 (0.09%)
2/3	1.03365	-	1.03498 (0.13%)
3/5	1.04075	-	1.04273 (0.19%)
1/2	1.05164	-	1.05491 (0.31%)
3/7	1.05960	-	1.06405 (0.42%)
1/3	1.07045	1.04912 (1.99%)	1.07685 (0.60%)
1/4	1.08018	-	1.08868 (0.79%)
1/5	1.08613	1.04812 (3.50%)	1.09609 (0.92%)
1/6	1.09013	-	1.10170 (1.06%)
1/7	1.09302	1.04405 (4.48%)	1.10487 (1.08%)
1/8	1.09519	-	1.10768 (1.14%)
1/9	1.09689	1.04017 (5.17%)	1.10989 (1.19%)
1/10	1.09826	-	1.11168 (1.22%)
1/11	1.09938	1.03684 (5.69%)	1.11315 (1.25%)
1/∞	1.11072	1.00000 (9.97%)	1.12838 (1.59%)
0	1.11072	1.00000 (9.97%)	1.12838 (1.59%)

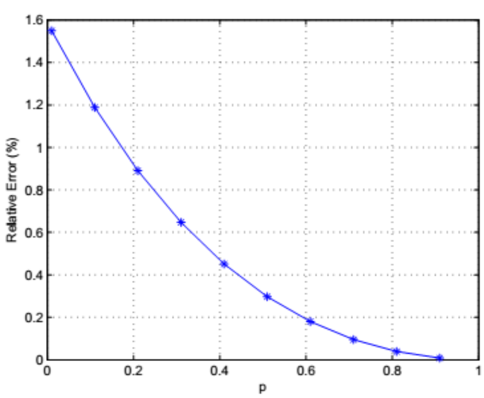

Fig. 1. The relative error for the oscillator in Case 1

Comparisons of the time history and phase plane trajectory are presented in Figure 2. It is shown that both in the time domain and in the phase plane, a very good correlation is preserved.

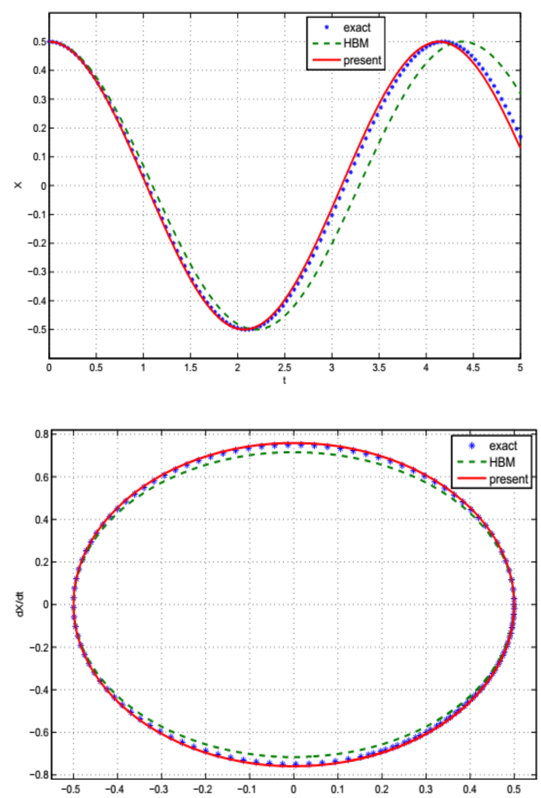

Fig. 2. Time history and phase plane of the dynamic response for *A=0.5* and *p=1/10*, Case 1

3.2 Case 2

Futhermore about the oscillator with fraction order restoring force, we consider the oscillator given by the nonlinear equation [12]:

$$\ddot{X}+C_1^2 X\left|X\right|^{\alpha-1}=0,\qquad X(0)=A,\quad \dot{X}(0)=0 \tag{22}$$

where C_1 and α are constants.

For this oscillator, the nonlinear term is:

$$g(X)=C_1^2 X\left|X\right|^{\alpha-1} \tag{23}$$

It is similar to Case 1, the approximate frequency ω is determined as follows:

$$\omega=\sqrt{C_1^2\frac{\left\langle X^2\left|X\right|^{\alpha-1}\right\rangle}{\left\langle X^2\right\rangle}} \tag{24}$$

We calculate averaging operators in Eq. (24):

$$\left\langle X^2\right\rangle=\left\langle A^2\cos^2\omega t\right\rangle=A^2\frac{s^4+2s^2+8}{(s^2+4)^2} \tag{25}$$

$$\left\langle X^2\left|X\right|^{\alpha-1}\right\rangle=\int_0^{+\infty}A^2\left|A\right|^{\alpha-1}s^2\omega^2 te^{-s\omega t}\cos^2(\omega t)\left|\cos(\omega t)\right|^{\alpha-1}dt$$
$$=\int_0^{+\infty}A^2\left|A\right|^{\alpha-1}s^2\tau e^{-s\tau}\cos^2(\tau)\left|\cos(\tau)\right|^{\alpha-1}d\tau \tag{26}$$

With s is chosen equal to 2, from Eqs. (25), (26) and (24) we get the approximate frequency:

$$\omega=C_1A^{\frac{\alpha-1}{2}}\sqrt{\int_0^{+\infty}8\tau e^{-2\tau}\cos^2(\tau)\left|\cos(\tau)\right|^{\alpha-1}d\tau} \tag{27}$$

For the specific value of α, from Eq. (27) we can compute the corresponding value of frequency ω. Comparison of the approximate frequencies ω in Eq. (27) and the exact frequencies obtained by Cveticanin's method [12] is tabulated in Table 2. A very interesting agreement between the results is observed.

1. For the linear oscillator, when $\alpha=1$, from Eq. (24) we get

$$\omega=C_1\sqrt{\frac{\left\langle X^2\right\rangle}{\left\langle X^2\right\rangle}}=C_1 \tag{28}$$

It is well known in the linear theory of harmonic vibration.

2. For $\alpha=5/3$, from Eq. (27) we get the approximate frequency of this oscillator:

$$\omega=\sqrt{0.880296\left|A\right|^{2/3}C_1^2}=0.938241\left|A\right|^{1/3}C_1 \tag{29}$$

The exact frequency achived by Cveticanin as follows [12]:

$$\omega_{ex}=\frac{C_1\pi\sqrt{2(\alpha+1)}}{2\left|A\right|^{(1-\alpha)/2}}\frac{\Gamma\left(\frac{3+\alpha}{2(\alpha+1)}\right)}{\Gamma\left(\frac{1}{\alpha+1}\right)\Gamma\left(\frac{1}{2}\right)} \tag{30}$$

Table 2. Comparison of the approximate frequencies with the ones given by Cveticanin, Case 2

α	Cveticanin's frequency	Present frequency	R. Erorr (%)
1	C_1	C_1	0.000
5/3	$0.94081\left|A\right|^{1/3}C_1$	$0.938241\left|A\right|^{1/3}C_1$	0.273
3/2	$0.95469\left|A\right|^{1/4}C_1$	$0.952283\left|A\right|^{1/4}C_1$	0.252
4/3	$0.96915\left|A\right|^{1/6}C_1$	$0.967185\left|A\right|^{1/6}C_1$	0.203
2	$0.91468\left|A\right|^{1/2}C_1$	$0.9124\left|A\right|^{1/2}C_1$	0.249
3	$0.84721\left|A\right|C_1$	$0.848528\left|A\right|C_1$	0.156
5	$0.74683\left|A\right|^2C_1$	$0.75829\left|A\right|^2C_1$	1.534

Comparison of the time history and phase plane trajectory is presented in Figure 3 (*α=4/3*). The results are very accurate compared to the results achived by Cveticanin.

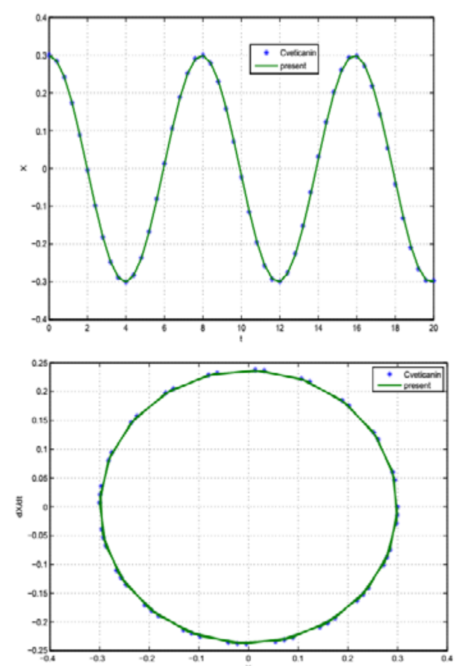

Fig. 3. Time history and phase plane of the dynamic response for *α=4/3*, Case 2

3.3 Case 3

Consider the Duffing-relativistic oscillator [13]:

$$\ddot{X} + X + X^3 - \frac{\beta X}{\sqrt{1+X^2}} = 0, \quad X(0) = A, \ \dot{X}(0) = 0 \qquad (31)$$

where β is a constant.

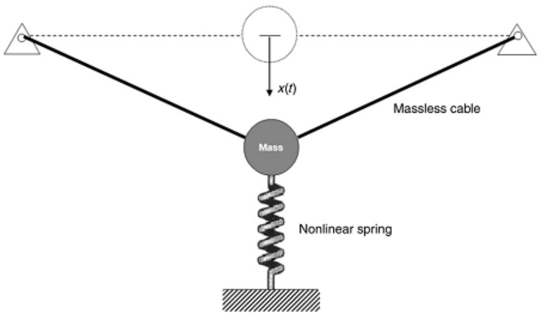

Fig. 4. Schematic of a Duffing-relativistic oscillator

This differential equation can physically govern the motion of a dynamic system consisting of an elastic cable with an attached mass connected to a nonlinear spring as shown in Figure 4.

For this oscillator, the nonlinear term is:

$$g(X) = X + X^3 - \frac{\beta X}{\sqrt{1+X^2}} \qquad (32)$$

The approximate frequency ω is determined as follows:

$$\omega = \sqrt{\frac{\langle X^2\rangle + \langle X^4\rangle - \beta\left\langle \frac{X^2}{\sqrt{1+X^2}}\right\rangle}{\langle X^2\rangle}} \qquad (33)$$

We calculate averaging operators in Eq. (33):

$$\langle X^2\rangle = \langle A^2\cos^2\omega t\rangle$$
$$= \int_0^{+\infty} A^2 s^2\omega^2 t e^{-s\omega t}\cos^2(\omega t)\,dt = A^2\frac{s^4+2s^2+8}{(s^2+4)^2} \qquad (34)$$

$$\langle X^4\rangle = \langle A^4\cos^4(\omega t)\rangle$$
$$= A^2\frac{248s^4+416s^2+1536+28s^6+s^8}{(s^2+4)^2(s^2+16)^2} \qquad (35)$$

$$\left\langle \frac{X^2}{\sqrt{1+X^2}}\right\rangle = \left\langle \frac{A^2\cos^2(\omega t)}{\sqrt{1+A^2\cos^2(\omega t)}}\right\rangle$$
$$= \int_0^{+\infty} A^2\frac{s^2\omega^2 t e^{-s\omega t}\cos^2(\omega t)}{\sqrt{1+A^2\cos^2(\omega t)}}dt \qquad (36)$$
$$= \int_0^{+\infty} A^2\frac{s^2\tau e^{-s\tau}\cos^2(\tau)}{\sqrt{1+A^2\cos^2(\tau)}}d\tau$$

With s is chossen equal to 2, substituting Eqs. (34) - (36) into Eq. (33), we get the approximate frequency of this oscillator:

$$\omega = \sqrt{1+0.72A^2-2\beta I} \qquad (37)$$

where

$$I = \int_0^{+\infty} 4\tau e^{-2\tau}(\cos\tau)^2/\sqrt{1+A^2(\cos\tau)^2} \qquad (38)$$

With each paticular value of initial amplitude A, the integral (38) can be calculated by using the Maple software. Therefore, the corresponding value of the approximate frequency can be obtained. Comparison of the frequency in Eq. (37) with the frequency obtained by the Energy Balance method [13] is shown in Table 3. It can be seen from Table 3 that the relative error of the present method is less than the error of the Energy Balance method. Specifically, the Energy Balance method gives a maximum relative error of 2.22% while the present method has a maximum relative error of only 0.48%.

The exact frequency of this oscillator is given by [13]:

$$\omega_{ex} = \frac{2\pi}{4\int_0^A \frac{dx}{\sqrt{(A^2-x^2)+\frac{1}{2}(A^4-x^4)-2\beta\left(\sqrt{A^2+1}-\sqrt{x^2+1}\right)}}} \qquad (39)$$

and the approximate frequency obtained by EBM [13]:

$$\omega_{EBM} = \frac{2}{A}\sqrt{\frac{A^2}{4}+\frac{3A^2}{16}-\beta\left(\sqrt{A^2+1}-\sqrt{\frac{A^2}{2}+1}\right)} \qquad (40)$$

Table 3. Frequency for different values of oscillation amplitude, Case 3

(A, β)	ω_{ex}	ω_{EBM}	R. Error (%)	$\omega_{present}$	R. Error (%)
(1, 0.1)	1.287987	1.293134	0.458019	1.281842	0.477101
(1, 0.5)	1.160519	1.170924	0.888515	1.155677	0.417227
(10, 0.1)	8.532145	8.717131	2.122095	8.543261	0.130284
(10, 0.5)	8.528856	8.714461	2.129855	8.540288	0.134039
(100, 0.1)	84.689205	86.608307	2.215840	84.858699	0.200136
(100, 0.5)	84.689171	86.608279	2.215849	84.858668	0.200140

Comparison of the time response is presented in Figure 5 for *A=100* and *β=0.1*. Again, the accuracy of this method can be observed.

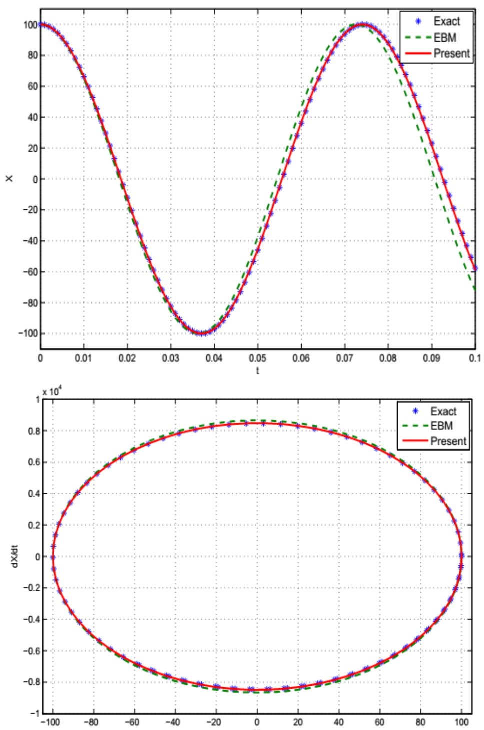

Fig. 5. Time history and phase plane of the dynamic response for *A=100* and *β=0.1*, Case 3

4. Conclusions

Analytical approximation solutions of oscillations with fractional order restoring force and Duffing-relativistic oscillator are presented by the Equivalent Linearization method based on a weighted averaging. The method is developed based on the convenience of the classical Equivalent Linearization method and the reliability of the weighted averaging value. Through three examples, the relationship between the frequencies and the initial amplitudes are given in closed forms. The approximate solutions are the harmonic oscillations, which are compared with the previous analytical results and the exact results. Comparisons show the accuracy of the present solutions. This method can be further developed for strongly nonlinear systems and multi-degree of fredom vibrations.

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