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Sandwich Theorems for Higher-Order Derivatives of Multivalent Analytic Functions Defined by Convolution Structure with Linear Operator

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Abstract

The purpose of the present paper is to derive some applications of first order differential subordination and superordination results involving Hadamard product for multivalent analytic functions with linear operator defined in the open unit disk. These results are applied to obtain sandwich results.

Keywords: Analytic functions, Differential subordination, Differential superordination, Higher-order derivatives, Hadamard product, Linear operator.

1. Introduction

Let $\mathcal{H}$ be the class of analytic functions in U and let $\mathcal{H}[a,p]$ be the subclass of $\mathcal{H}$ consisting of functions of the form:

$$f(z) = a + a_p z^p + a_{p+1} z^{p+1} + \cdots ,$$
$$(a \in \mathbb{C}, p \in \mathbb{N} = \{1,2,\ldots\}).$$

Let $\mathcal{A}_p$ denote the class of functions of the form:

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} z^{n+p}, \quad p \in \mathbb{N}, \qquad (1.1)$$

which are analytic and multivalent in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. For simplicity, let $\mathcal{A}_1 = \mathcal{A}$.

Upon differentiating both sides of (1.1) j-times with respect to z, we obtain

$$f^{(j)}(z) = \delta(p,j) z^{p-j} + \sum_{n=1}^{\infty} \delta(p+n,j) a_{n+p} z^{n+p-j}$$
$$(p \in \mathbb{N}, j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, p > j),$$

where

$$\delta(p,j) = \frac{p!}{(p-j)!} = \begin{cases} 1 & (j = 0) \\ p(p-1)\ldots(p-j+1) & (j \neq 0) \end{cases}.$$

Let $f, g \in \mathcal{H}$. The function f is said to be subordinate to g, or g is said to be superordinate to f, if there exists a Schwarz function w analytic in U with $w(0) = 0$ and $|w(z)| < 1 \; (z \in U)$ such that $f(z) = g(w(z))$. This subordination is denoted by $f \prec g$ or $f(z) \prec g(z) (z \in U)$. It is well known that, if the function g is univalent in U, we have the following equivalence (see [15]):

$$f \prec g \; (z \in U) \iff f(0) = g(0), \; f(U) \subset g(U).$$

Let $k, h \in \mathcal{H}$ and $\psi(r,s,t;z): \mathbb{C}^3 \times U \longrightarrow \mathbb{C}$. If k and $\psi(k(z), zk'(z), z^2 k''(z); z)$ are univalent functions in U and if k satisfies the second-order differential superordination:

$$h(z) \prec \psi(k(z), zk'(z), z^2 k''(z); z), \qquad (1.2)$$

then k is called a solution of the differential superordination (1.2). (If f is subordinate to g, then g is superordinate to f). An analytic function q is called a subordinant of (1.2), if $q \prec k$ for all k satisfying (1.2). An univalent subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all the subordinants q of (1.2) is called the best subordinant.

For the functions $f \in \mathcal{A}_p$ given by (1.1) and $g \in \mathcal{A}_p$ defined by

$$g(z) = z^p + \sum_{n=1}^{\infty} b_{n+p} z^{n+p}, \quad p \in \mathbb{N},$$

we define the Hadamard product (or convolution) $f * g$ of the functions f and g (as usual) by

$$(f * g)(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} b_{n+p} z^{n+p} = (g * f)(z).$$

For $a \in \mathbb{R}, c \in \mathbb{R} \setminus \mathbb{Z}_0^-$, *where* $\mathbb{Z}_0^- = \{0,-1,-2,\ldots\}$, $0 \leq \lambda < 1$, $p \in \mathbb{N}$, $\alpha > -p$, $\mu, \nu \in \mathbb{R}$ with $\mu - \nu - p < 1$ and $f \in \mathcal{A}_p$. The linear operator $\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha}(a,c): \mathcal{A}_p \longrightarrow \mathcal{A}_p$ (see [7]) is defined by

$$\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha}(a,c) f(z) = z^p + \sum_{n=1}^{\infty} \frac{(c)_n (p+1-\mu)_n (p+1-\lambda+\nu)_n (\alpha+p)_n}{(a)_n (p+1)_n (p+1-\mu+\nu)_n n!} a_{n+p} z^{n+p}. \qquad (1.3)$$

It is easily verified from (1.3) that

$$z\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha}(a,c) f(z)\right)'_z = (\alpha+p) \mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha+1}(a,c) f(z) - \alpha \mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha}(a,c) f(z). \qquad (1.4)$$

Differentiating (1.4) j-times with respect to z, we get

$$z\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha}(a,c) f(z)\right)^{(j+1)} = (\alpha+p)\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha+1}(a,c) f(z)\right)^{(j)} - (\alpha+j)\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha}(a,c) f(z)\right)^{(j)}. \qquad (1.5)$$

Note that the linear operator $\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha}(a,c)$ unifies many other operators considered earlier. In particular:

(1) $\mathcal{L}_{0,\nu}^{0,p,\alpha}(a,c) \equiv \mathcal{J}_p^{\alpha}(a,c)$ (see Cho et.. al. [2])

(2) $\mathcal{L}_{0,\nu}^{0,p,\alpha}(a,a) \equiv D^{\alpha+p-1}$ (see Goel and Sohi [3])

(3) $\mathcal{L}_{0,\nu}^{0,p,1}(p+1-\lambda,1) \equiv \Omega_z^{(\lambda,p)}$ (see Srivastava and Aouf [14])

(4) $\mathcal{L}_{0,\nu}^{0,1,\alpha-1}(a,c) \equiv \mathcal{J}_c^{a,\alpha}$ (see Hohlov [5])

(5) $\mathcal{L}_{0,\nu}^{0,1-\alpha,\alpha}(a,c) \equiv \mathcal{L}_p(a,c)$ (see Saitoh [13])

(6) $\mathcal{L}_{0,\nu}^{0,p,1}(p+\alpha,1) \equiv \mathcal{J}_{\alpha,p}\,, \alpha \in \mathbb{Z}, \alpha > -p$ (see Liu and Noor [8])

Recently several authors, Shanmugam et al. [13], Goyal et al. [4], Murugusundaramoorthy and Magesh [11, 12], Magesh et al. [9], Ibrahim and Darus [6], Wanas [16,17], Wanas and Joudah [18], Wanas and Majeed [20] and Wanas and Lupas [19] have obtained sandwich results for certain classes of analytic functions.

The main object of the present investigation is to find sufficient condition for certain normalized analytic functions f in U such that $(f * \Psi)(z) \neq 0$ and f to satisfy

$$q_1(z) \prec \left(\frac{\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha+1}(a,c)(f*\Phi)(z)\right)^{(j)}}{\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha}(a,c)(f*\Psi)(z)\right)^{(j)}} \right)^{\gamma} \prec q_2(z),$$

where q_1 and q_2 are given univalent functions in U with $q_1(0) = q_2(0) = 1$ and $\Phi(z) = z^p + \sum_{n=1}^{\infty} r_{n+p} z^{n+p}$, $\Psi(z) = z^p + \sum_{n=1}^{\infty} e_{n+p} z^{n+p}$ are analytic functions in U with $r_{n+p} \geq 0, e_{n+p} \geq 0$.

To establish our main results, we need the following definition and lemmas.

Definition 1.1 [10]. Denote by Q the set of all functions f that are analytic and injective on $\overline{U} \setminus E(f)$, where

$$E(f) = \left\{ \zeta \in \partial U : \lim_{z \to \zeta} f(z) = \infty \right\}$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(f)$.

Lemma 1.1 [10]. Let q be univalent in the unit disk U and let θ and ϕ be analytic in a domain D containing $q(U)$ with $\phi(w) \neq 0$ when $w \in q(U)$. Set $Q(z) = zq'(z)\phi(q(z))$ and $h(z) = \theta(q(z)) + Q(z)$. Suppose that

(1) $Q(z)$ is starlike univalent in U,

(2) $Re\left\{\frac{zh'(z)}{Q(z)}\right\} > 0$ for $z \in U$.

If k is analytic in U, with $k(0) = q(0)$, $k(U) \subset D$ and

$$\theta(k(z)) + zk'(z)\phi(k(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)), \qquad (1.6)$$

then $k \prec q$ and q is the best dominant of (1.6).

Lemma 1.2 [1]. Let q be convex univalent in the unit disk U and let θ and ϕ be analytic in a domain D containing $q(U)$. Suppose that

(1) $Re\left\{\frac{\theta'(q(z))}{\phi(q(z))}\right\} > 0$ for $z \in U$,

(2) $Q(z) = zq'(z)\phi(q(z))$ is starlike univalent in U.

If $k \in H[q(0),1] \cap Q$, with $k(U) \subset D, \theta(k(z)) + zk'(z)\phi(k(z))$ is univalent in U and

$$\theta(q(z)) + zq'(z)\phi(q(z)) \prec \theta(k(z)) + zk'(z)\phi(k(z)), \qquad (1.7)$$

then $q \prec k$ and q is the best subordinant of (1.7).

2. Coefficient Inequalities

Theorem 2.1. Let $\Phi, \Psi \in \mathcal{A}_p$, $\beta_1, \beta_2, \tau \in \mathbb{C}$, $\eta, \gamma \in \mathbb{C} \setminus \{0\}$ and let q be convex univalent in U with $q(0) = 1$ and assume that q satisfies:

$$Re\left\{1 + \frac{\beta_1 \tau}{\eta} + \frac{\beta_2(\tau+1)}{\eta} q(z) + (\tau - 1)\frac{zq'(z)}{q(z)} + \frac{zq''(z)}{q'(z)}\right\} > 0, \qquad (2.1)$$

Suppose that $z(q(z))^{\tau-1} q'(z)$ is starlike univalent in U. If $f \in \mathcal{A}_p$ satisfies the differential subordination:

$$\varphi_1(f, \Phi, \Psi, \beta_1, \beta_2, \tau, \eta, \gamma, a, c, \mu, \nu, \lambda, p, \alpha, j; z) \prec (\beta_1 + \beta_2 q(z))(q(z))^{\tau} + \eta z (q(z))^{\tau-1} q'(z), \qquad (2.2)$$

where

$$\varphi_1(f, \Phi, \Psi, \beta_1, \beta_2, \tau, \eta, \gamma, a, c, \mu, \nu, \lambda, p, \alpha, j; z) = \beta_1 \left(\frac{\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha+1}(a,c)(f*\Phi)(z)\right)^{(j)}}{\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha}(a,c)(f*\Psi)(z)\right)^{(j)}} \right)^{\gamma\tau} + \beta_2 \left(\frac{\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha+1}(a,c)(f*\Phi)(z)\right)^{(j)}}{\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha}(a,c)(f*\Psi)(z)\right)^{(j)}} \right)^{\gamma(\tau+1)}$$
$$+ \eta\gamma \left(\frac{\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha+1}(a,c)(f*\Phi)(z)\right)^{(j)}}{\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha}(a,c)(f*\Psi)(z)\right)^{(j)}} \right)^{\gamma\tau} \left((\alpha+p+1) \frac{\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha+2}(a,c)(f*\Phi)(z)\right)^{(j)}}{\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha+1}(a,c)(f*\Phi)(z)\right)^{(j)}} - (\alpha+p) \frac{\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha+1}(a,c)(f*\Psi)(z)\right)^{(j)}}{\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha}(a,c)(f*\Psi)(z)\right)^{(j)}} - 1 \right), \qquad (2.3)$$

then

$$\left(\frac{\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha+1}(a,c)(f*\Phi)(z)\right)^{(j)}}{\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha}(a,c)(f*\Psi)(z)\right)^{(j)}} \right)^{\gamma} \prec q(z)$$

and q is the best dominant of (2.2).

Proof. Let the function k be defined by

$$k(z)=\left(\frac{\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha+1}(a,c)(f*\Phi)(z)\right)^{(j)}}{\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha}(a,c)(f*\Psi)(z)\right)^{(j)}}\right)^{\gamma},\ (z\in U).\quad (2.4)$$

Then the function k is analytic in U and $k(0)=1$.
A simple computation using (2.4) gives

$$\frac{zk'(z)}{k(z)}=\gamma\left(\frac{z\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha+1}(a,c)(f*\Phi)(z)\right)^{(j+1)}}{\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha+1}(a,c)(f*\Phi)(z)\right)^{(j)}}-\frac{z\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha}(a,c)(f*\Psi)(z)\right)^{(j+1)}}{\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha}(a,c)(f*\Psi)(z)\right)^{(j)}}\right).$$

In view of (1.5), we obtain

$$\frac{zk'(z)}{k(z)}=\gamma\left((\alpha+p+1)\frac{\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha+2}(a,c)(f*\Phi)(z)\right)^{(j)}}{\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha+1}(a,c)(f*\Phi)(z)\right)^{(j)}}-(\alpha+p)\frac{\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha+1}(a,c)(f*\Psi)(z)\right)^{(j)}}{\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha}(a,c)(f*\Psi)(z)\right)^{(j)}}-1\right).$$

Also, we find that

$$\left(\beta_1+\beta_2k(z)\right)\left(k(z)\right)^{\tau}+\eta z\left(k(z)\right)^{\tau-1}k'(z)=\varphi(f,\Phi,\Psi,\beta_1,\beta_2,\tau,\eta,\gamma,a,c,\mu,\nu,\lambda,p,\alpha,j;z),\quad (2.5)$$

where $\varphi(f,\Phi,\Psi,\beta_1,\beta_2,\tau,\eta,\gamma,a,c,\mu,\nu,\lambda,p,\alpha,j;z)$ is given by (2.3).
By using (2.5) in (2.2), we have

$$\left(\beta_1+\beta_2k(z)\right)\left(k(z)\right)^{\tau}+\eta z\left(k(z)\right)^{\tau-1}k'(z)\prec\left(\beta_1+\beta_2q(z)\right)\left(q(z)\right)^{\tau}+\eta z\left(q(z)\right)^{\tau-1}q'(z).$$

By setting
$\theta(w)=(\beta_1+\beta_2w)w^{\tau}$ *and* $\phi(w)=\eta w^{\tau-1}$, $w\neq 0$,
it can be easily observed that $\theta(w)$ is analytic in $\mathbb{C}$, $\phi(w)$ is analytic in $\mathbb{C}\setminus\{0\}$ and that $\phi(w)\neq 0, w\in\mathbb{C}\setminus\{0\}$. Also, we get

$$Q(z)=zq'(z)\phi\left(q(z)\right)=\eta z\left(q(z)\right)^{\tau-1}q'(z)$$

and

$$h(z)=\theta\left(q(z)\right)+Q(z)=\left(\beta_1+\beta_2q(z)\right)\left(q(z)\right)^{\tau}+\eta z\left(q(z)\right)^{\tau-1}q'(z).$$

In light of the hypothesis of Theorem 2.1, we see that $Q(z)$ is starlike univalent in U and

$$Re\left\{\frac{zh'(z)}{Q(z)}\right\}=Re\left\{1+\frac{\beta_1\tau}{\eta}+\frac{\beta_2(\tau+1)}{\eta}q(z)+(\tau-1)\frac{zq'(z)}{q(z)}+\frac{zq''(z)}{q'(z)}\right\}>0.$$

Hence the result now follows by an application of Lemma 1.1.

By taking $q(z)=\frac{1+Az}{1+Bz}$ $(-1\leq B<A\leq 1)$ in Theorem 2.1, we obtain the following corollary:

Corollary 2.1. Let $\Phi,\Psi\in\mathcal{A}_p$, $\beta_1,\beta_2,\tau\in\mathbb{C}$, $\eta,\gamma\in\mathbb{C}\setminus\{0\}$, $-1\leq B<A\leq 1$ and assume that

$$Re\left\{\frac{\beta_1\tau}{\eta}+\frac{\beta_2(\tau+1)(1+Az)}{\eta(1+Bz)}+\frac{1+\tau(A-B)z-ABz^2}{(1+Az)(1+Bz)}\right\}>0,$$

If $f\in\mathcal{A}_p$ satisfies the differential subordination:

$$\varphi_1(f,\Phi,\Psi,\beta_1,\beta_2,\tau,\eta,\gamma,a,c,\mu,\nu,\lambda,p,\alpha,j;z)\prec\left(\beta_1+\beta_2\left(\frac{1+Az}{1+Bz}\right)\right)\left(\frac{1+Az}{1+Bz}\right)^{\tau}+\frac{\eta(A-B)(1+Az)^{\tau-1}z}{(1+Bz)^{\tau+1}},\quad (2.6)$$

where φ_1 is given by (2.3),
then

$$\left(\frac{\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha+1}(a,c)(f*\Phi)(z)\right)^{(j)}}{\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha}(a,c)(f*\Psi)(z)\right)^{(j)}}\right)^{\gamma}\prec\frac{1+Az}{1+Bz}$$

and $q(z)=\frac{1+Az}{1+Bz}$ is the best dominant of (2.6).

By fixing $\Phi(z)=\Psi(z)=\frac{z^p}{1-z}$ in Theorem 2.1, we obtain the following corollary:

Corollary 2.2. Let $\beta_1,\beta_2,\tau\in\mathbb{C}$, $\eta,\gamma\in\mathbb{C}\setminus\{0\}$ and let q be convex univalent in U with $q(0)=1$ and assume that (2.1) holds true. Suppose that $z\left(q(z)\right)^{\tau-1}q'(z)$ is starlike univalent in U. If $f\in\mathcal{A}_p$ satisfies the differential subordination:

$$\varphi_2(f,\beta_1,\beta_2,\tau,\eta,\gamma,a,c,\mu,\nu,\lambda,p,\alpha,j;z)\prec\left(\beta_1+\beta_2q(z)\right)\left(q(z)\right)^{\tau}+\eta z\left(q(z)\right)^{\tau-1}q'(z),\quad (2.7)$$

where

$$\varphi_2(f,\beta_1,\beta_2,\tau,\eta,\gamma,a,c,\mu,\nu,\lambda,p,\alpha,j;z)=\beta_1\left(\frac{\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha+1}(a,c)f(z)\right)^{(j)}}{\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha}(a,c)f(z)\right)^{(j)}}\right)^{\gamma\tau}+\beta_2\left(\frac{\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha+1}(a,c)f(z)\right)^{(j)}}{\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha}(a,c)f(z)\right)^{(j)}}\right)^{\gamma(\tau+1)}+\eta\gamma\left(\frac{\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha+1}(a,c)f(z)\right)^{(j)}}{\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha}(a,c)f(z)\right)^{(j)}}\right)^{\gamma\tau}\left((\alpha+p+1)\frac{\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha+2}(a,c)f(z)\right)^{(j)}}{\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha+1}(a,c)f(z)\right)^{(j)}}-(\alpha+p)\frac{\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha+1}(a,c)f(z)\right)^{(j)}}{\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha}(a,c)f(z)\right)^{(j)}}-1\right),\quad (2.8)$$

then

$$\left(\frac{\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha+1}(a,c)f(z)\right)^{(j)}}{\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha}(a,c)f(z)\right)^{(j)}}\right)^{\gamma} \prec q(z)$$

and q is the best dominant of (2.7).

Theorem 2.2. Let $\Phi,\Psi \in \mathcal{A}_p$, $\beta_1,\beta_2,\tau \in \mathbb{C}$, $\eta,\gamma \in \mathbb{C}\setminus\{0\}$ and let q be convex univalent in U with $q(0)=1$ and assume that q satisfies:

$$Re\left\{\frac{\beta_1\tau}{\eta}q'(z)+\frac{\beta_2(\tau+1)}{\eta}q(z)q'(z)\right\}>0. \qquad (2.9)$$

Suppose that $z(q(z))^{\tau-1}q'(z)$ is starlike univalent in U. Let $f \in \mathcal{A}_p$ satisfies

$$\left(\frac{\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha+1}(a,c)(f*\Phi)(z)\right)^{(j)}}{\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha}(a,c)(f*\Psi)(z)\right)^{(j)}}\right)^{\gamma} \in \mathcal{H}[q(0),1]\cap Q$$

and $\varphi_1(f,\Phi,\Psi,\beta_1,\beta_2,\tau,\eta,\gamma,a,c,\mu,\nu,\lambda,p,\alpha,j;z)$ as defined by (2.3) be univalent in U. If

$$\left(\beta_1+\beta_2 q(z)\right)\left(q(z)\right)^{\tau}+\eta z\left(q(z)\right)^{\tau-1}q'(z) \prec \varphi_1(f,\Phi,\Psi,\beta_1,\beta_2,\tau,\eta,\gamma,a,c,\mu,\nu,\lambda,p,\alpha,j;z), \qquad (2.10)$$

then

$$q(z) \prec \left(\frac{\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha+1}(a,c)(f*\Phi)(z)\right)^{(j)}}{\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha}(a,c)(f*\Psi)(z)\right)^{(j)}}\right)^{\gamma}$$

and q is the best subordinant of (2.10).

Proof. Let the function k be defined by (2.4).
In view of (1.5), the superordination (2.10) becomes

$$\left(\beta_1+\beta_2 q(z)\right)\left(q(z)\right)^{\tau}+\eta z\left(q(z)\right)^{\tau-1}q'(z) \prec \left(\beta_1+\beta_2 k(z)\right)\left(k(z)\right)^{\tau}+\eta z\left(k(z)\right)^{\tau-1}k'(z).$$

By setting $\theta(w)=\theta(w)=(\beta_1+\beta_2 w)w^{\tau}$ and $\phi(w)=\eta w^{\tau-1}$, $w\neq 0$, it is easily observed that $\theta(w)$ is analytic in $\mathbb{C}$, $\phi(w)$ is analytic in $\mathbb{C}\setminus\{0\}$ and that $\phi(w)\neq 0, w\in\mathbb{C}\setminus\{0\}$. Also, we get

$$Q(z)=zq'(z)\phi\left(q(z)\right)=\eta z\left(q(z)\right)^{\tau-1}q'(z)$$

It is clear that $Q(z)$ is starlike univalent in U and

$$Re\left\{\frac{\theta'\left(q(z)\right)}{\phi\left(q(z)\right)}\right\}=Re\left\{\frac{\beta_1\tau}{\eta}q'(z)+\frac{\beta_2(\tau+1)}{\eta}q(z)q'(z)\right\}>0.$$

Now Theorem 2.2 follows by applying Lemma 1.2.

By fixing $\Phi(z)=\Psi(z)=\frac{z^p}{1-z}$ in Theorem 2.2, we obtain the following corollary:

Corollary 2.3. Let $\beta_1,\beta_2,\tau \in \mathbb{C}$, $\eta,\gamma \in \mathbb{C}\setminus\{0\}$ and let q be convex univalent in U with $q(0)=1$ and assume that (2.9) holds true. Suppose that $z(q(z))^{\tau-1}q'(z)$ is starlike univalent in U. Let $f \in \mathcal{A}_p$ satisfies

$$\left(\frac{\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha+1}(a,c)f(z)\right)^{(j)}}{\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha}(a,c)f(z)\right)^{(j)}}\right)^{\gamma} \in \mathcal{H}[q(0),1]\cap Q$$

and $\varphi_2(f,\beta_1,\beta_2,\tau,\eta,\gamma,a,c,\mu,\nu,\lambda,p,\alpha,j;z)$ as defined by (2.8) be univalent in U. If

$$\left(\beta_1+\beta_2 q(z)\right)\left(q(z)\right)^{\tau}+\eta z\left(q(z)\right)^{\tau-1}q'(z) \prec \varphi_2(f,\beta_1,\beta_2,\tau,\eta,\gamma,a,c,\mu,\nu,\lambda,p,\alpha,j;z), \qquad (2.11)$$

then

$$q(z) \prec \left(\frac{\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha+1}(a,c)f(z)\right)^{(j)}}{\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha}(a,c)f(z)\right)^{(j)}}\right)^{\gamma}$$

and q is the best subordinant of (2.11).

Concluding the results of differential subordination and superordination, we arrive at the following "sandwich results".

Theorem 2.3. Let q_1 and q_2 be convex univalent in U with $q_1(0)=q_2(0)=1$, $\beta_1,\beta_2,\tau \in \mathbb{C}$, $\eta,\gamma \in \mathbb{C}\setminus\{0\}$. Suppose q_2 satisfies (2.1) and q_1 satisfies (2.9) such that $z(q(z))^{\tau-1}q'(z)$ is starlike univalent in U. For $f,\Phi,\Psi \in \mathcal{A}_p$, let

$$\left(\frac{\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha+1}(a,c)(f*\Phi)(z)\right)^{(j)}}{\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha}(a,c)(f*\Psi)(z)\right)^{(j)}}\right)^{\gamma} \in \mathcal{H}[1,1]\cap Q$$

and $\varphi_1(f,\Phi,\Psi,\beta_1,\beta_2,\tau,\eta,\gamma,a,c,\mu,\nu,\lambda,p,\alpha,j;z)$ as defined by (2.3) be univalent in U. If

$$\left(\beta_1+\beta_2 q_1(z)\right)\left(q_1(z)\right)^{\tau}+\eta z\left(q_1(z)\right)^{\tau-1}q_1'(z) \prec \varphi_1(f,\Phi,\Psi,\beta_1,\beta_2,\tau,\eta,\gamma,a,c,\mu,\nu,\lambda,p,\alpha,j;z) \prec \left(\beta_1+\beta_2 q_2(z)\right)\left(q_2(z)\right)^{\tau}+\eta z\left(q_2(z)\right)^{\tau-1}q_2'(z),$$

then

$$q_1(z) \prec \left(\frac{\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha+1}(a,c)(f*\Phi)(z)\right)^{(j)}}{\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha}(a,c)(f*\Psi)(z)\right)^{(j)}}\right)^{\gamma} \prec q_2(z)$$

and q_1,q_2 are respectively the best subordinant and the best dominant.

By making use of Corollaries 2.2 and 2.3, we obtain the following corollary:

Corollary 2.4. Let q_1 and q_2 be convex univalent in U with $q_1(0)=q_2(0)=1$, $\beta_1,\beta_2,\tau \in \mathbb{C}$, $\eta,\gamma \in \mathbb{C}\setminus\{0\}$. Suppose q_2 satisfies (2.1) and q_1 satisfies (2.9) such that $z(q(z))^{\tau-1}q'(z)$ is starlike univalent in U. For $f \in \mathcal{A}_p$, let

$$\left(\frac{\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha+1}(a,c)f(z)\right)^{(j)}}{\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha}(a,c)f(z)\right)^{(j)}}\right)^{\gamma} \in \mathcal{H}[1,1]\cap Q$$

and $\varphi_2(f,\beta_1,\beta_2,\tau,\eta,\gamma,a,c,\mu,\nu,\lambda,p,\alpha,j;z)$ as defined by (2.8) be univalent in U. If

$$\left(\beta_1+\beta_2 q_1(z)\right)\left(q_1(z)\right)^{\tau}+\eta z\left(q_1(z)\right)^{\tau-1}q_1'(z) \prec \varphi_2(f,\beta_1,\beta_2,\tau,\eta,\gamma,a,c,\mu,\nu,\lambda,p,\alpha,j;z) \prec \left(\beta_1+\beta_2 q_2(z)\right)\left(q_2(z)\right)^{\tau}+\eta z\left(q_2(z)\right)^{\tau-1}q_2'(z),$$

then

$$q_1(z) \prec \left(\frac{\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha+1}(a,c)f(z) \right)^{(j)}}{\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha}(a,c)f(z) \right)^{(j)}} \right)^{\gamma} \prec q_2(z)$$

and q_1, q_2 are respectively the best subordinant and the best dominant.

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