A Note on the Hierarchical Transmuted Log–Logistic Model

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Abstract

The Transmuted Log–Logistic distribution is a flexible and simple model with applications to reliability analysis. We prove upper and lower estimates for the Hausdorff approximation of the shifted Heaviside function \( \tilde{h}_0(t) \) by a class of Transmuted Log–Logistic cumulative distribution function – (TLL–CDF). Numerical examples, illustrating our results are given.

Keywords: Transmuted Log–Logistic cumulative distribution function – (TLL–CDF), shifted Heaviside function, Hausdorff distance, upper and lower bounds.

1. Introduction

The cumulative distribution function of the (TTL) distribution is given by [1]–[2]:

\[
F(t) = \frac{e^{\mu t^\beta}}{(1 + e^{\mu t^\beta})^2} (1 + e^{\mu t^\beta} + \lambda)
\]

where \( \beta > 0, \mu \in \mathbb{R} \) and \( -1 \leq \lambda \leq 1 \).

Since the distribution was proposed to model experiments in reliability analysis. In this paper we prove upper and lower estimates for the Hausdorff approximation of the shifted Heaviside function \( \tilde{h}_0(t) \) by a class of Transmuted Log–Logistic cumulative distribution function – (TLL–CDF).

2. Preliminaries

Definition 1. The (basic) step function is:

\[
\tilde{h}_0(t) = \begin{cases} 
0, & \text{if } t < t_0, \\
[0,1], & \text{if } t = t_0, \\
1, & \text{if } t > t_0,
\end{cases}
\]

usually known as shifted Heaviside function.

Definition 2. [3], [4] The Hausdorff distance (the H–distance) [3] \( \rho(f,g) \) between two interval functions \( f, g \) on \( \Omega \subseteq \mathbb{R} \), is the distance between their completed graphs \( F(f) \) and \( F(g) \) considered as closed subsets of \( \Omega \times \mathbb{R} \).

More precisely,

\[
\rho(f,g) = \max \left\{ \sup_{A \in F(f)} \inf_{B \in F(g)} \| A - B \|, \sup_{B \in F(g)} \inf_{A \in F(f)} \| A - B \| \right\},
\]

wherein \( \| . \| \) is any norm in \( \mathbb{R}^2 \), e. g. the maximum norm \( \| (t,x) \| = \max \{|t|,|x|\} \); hence the distance between the points \( A = (t_A,x_A), B = (t_B,x_B) \) in \( \mathbb{R}^2 \) is \( \| A - B \| = \max( |t_A - t_B|, |x_A - x_B| ) \).

Let us point out that the Hausdorff distance is a natural measuring criteria for the approximation of bounded discontinuous functions [5].

3. Main Results

Let us consider the following 3-parametric sigmoid function
\[ F^*(t) = \frac{e^{\mu t^\beta}}{(1 + e^{\mu t^\beta})^2} (1 + e^{\mu t^\beta} + \lambda) \]  

(4)

**Proof.** We define the functions

\[ H(d) = F^*(t_0 + d) - 1 + d \]

(10)

\[ G(d) = a + bd. \]

(11)

From Taylor expansion

\[ H(d) - G(d) = O(d^2) \]

we see that the function \( G(d) \) approximates \( H(d) \) with \( d \to 0 \) as \( O(d^2) \) (cf. Fig. 1).

In addition \( G'(d) > 0 \) and for \( \frac{2b}{-a} > e^2 \)

\[ \left. G(d) < 0; \ G(d) \right. > 0. \]

This completes the proof of the inequalities (14).

The generated sigmoidal functions \( F^*(t) \) for

a) \( \beta = 10; \mu = 5; \lambda = 0.8; \)

b) \( \beta = 20; \mu = 9; \lambda = 0.9; \)

and c) \( \beta = 50; \mu = 15; \lambda = 0.95; \) are visualized on Fig. 2–Fig. 4.

From the Fig. 2–Fig. 4 it can be seen that the "supersaturation" is fast.

The following theorem gives upper and lower bounds for \( d \):

**Theorem 1.** Let

\[ a = - \left( 1 - \frac{e^{\mu t_0^\beta}}{(1 + e^{\mu t_0^\beta})^2} (1 + e^{\mu t_0^\beta} + \lambda) \right) \]

(7)

\[ b = 1 + e^{\mu} \left( \frac{e^{\mu t_0^{2\beta - 1} \lambda}}{(1 + e^{\mu t_0^\beta})^2} + \frac{2e^{\mu t_0^{2\beta - 1} \lambda}}{(1 + e^{\mu t_0^\beta})^3} + \frac{t_0^{\beta - 1} \beta}{(1 + e^{\mu t_0^\beta})^2} \right) (1 + e^{\mu t_0^\beta} + \lambda) \].

(8)

The \( H \)-distance \( d \) between the function \( \tilde{h}_0 \) and the sigmoidal function \( F^* \) can be expressed in terms of the parameters for \( \frac{2b}{-a} > e^2 \) as follows:

\[ d_i = \frac{1}{2b} < d < \frac{\ln \left( \frac{2b}{-a} \right)}{-a} = d_r. \]

(9)

![Fig. 1 The functions H and G](image-url)
Fig. 2 The function $F^*(t)$ for $\beta = 10; \mu = 5; \lambda = 0.8$; 
$t_0 = 0.563681$; H-distance $d = 0.108415$; 
$d_l = 0.0418; d_r = 0.132709$.

Fig. 3 The function $F^*(t)$ for $\beta = 20; \mu = 9; \lambda = 0.9$; 
$t_0 = 0.612355$; H-distance $d = 0.0665403$; 
$d_l = 0.0241137; d_r = 0.0898229$.

Fig. 4 The function $F^*(t)$ for $\beta = 50; \mu = 15; \lambda = 0.95$; 
$t_0 = 0.728395$; H-distance $d = 0.0362169$; 
$d_l = 0.0119635; d_r = 0.0529493$.

4. Conclusions

In this paper we prove upper and lower estimates for the Hausdorff approximation of the shifted Heaviside function $\hat{h}_{\beta}(t)$ by a class of Transmuted Log–Logistic cumulative distribution function – (TLL–CDF).

Numerical examples, illustrating our results are given.

We propose a software module (intellectual property) within the programming environment CAS Mathematica for the analysis of the considered family of (TLL–CDF) functions.

For other results, see [6]–[20].

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References


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