Minimal Quasi-\(\Gamma\)-absorbent in \(\Gamma\)-Groupoid-lattice

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Abstract

Otto. Steinfeld and Rédei [1] generalize concepts from groups, rings and semigroups to groupoid lattices. In our paper [5][6], we have introduced the notion of semiprime absorbents in groupoid lattices and \(\Gamma\)-absorbents in groupoid lattices. Here in this paper we will discuss some properties of minimal quasi- \(\Gamma\)-absorbent using semiprime \(\Gamma\)-absorbent in \(\Gamma\)-groupoid lattices.

Keywords: \(\Gamma\)-Groupoid lattice, semiprime \(\Gamma\)-absorbent, minimal \(\Gamma\)-absorbent, quasi-\(\Gamma\)-absorbent.

1. Introduction

A binary operation \(\Gamma\) on \(C\) is defined as a mapping from \(C \times C \rightarrow C\) that \(\Gamma\) assigns to each pair \((a, b) \in C \times C\) exactly one element \(\Gamma(a, b) \in C\). Instead of \(\Gamma\) one mostly writes \(\cdot\). Let \(C\) and \(\Gamma\) be two non-empty sets. A mapping from \(C \times \Gamma \times C \rightarrow C\) will be called a \(\Gamma\)-multiplication in \(C\) and is denoted by \(\triangleleft\). The result of the \(\Gamma\)-multiplication for \((a, b) \in C\) and \(\gamma \in \Gamma\) is denoted by \(a \triangleleft b\). Let \(C = \{x, y, z, \ldots\}\) and \(\Gamma = \{a, b, y, \ldots\}\) be two non-empty sets. Then \(C\) is called a \(\Gamma\)-groupoid if it satisfies \(xay \in C \ \forall \ x, y \in C\) and \(a \in \Gamma\). A \(\Gamma\)-semigroup is a \(\Gamma\)-groupoid such that the operation \((.)\) is associative.

Definition 1.1: A partially ordered \(\Gamma\)-groupoid is a non-empty set \(C\) satisfying the following properties:

i) \(C\) is a \(\Gamma\)-groupoid w.r.t. multiplication \((\cdot)_\Gamma\);

ii) \(C\) is a partially ordered set w.r.t. a partial ordering \(\leq\);

iii) If \(a \leq b\) then \(cay \leq cyb\) and \(ayc \leq byc\ \forall a, b, c \in C\), and \(y \in \Gamma\).

We know that a lattice "\(V\)" is called a complete if every subset of "\(V\)" has a least upper bound (join "\(\lor\)") and greatest lower bound (meet "\(\land\") in "\(V\)". The greatest element is denoted by "\(\top\)" and the least element is denoted by "\(\bot\)."

Definition 1.2: A \(\Gamma\)-groupoid lattice is a partially ordered groupoid \((V, (\cdot)_\Gamma, \leq)\) such that \(V\) is a complete lattice with respect to the partial ordering \(\leq\) and it has the following properties:

iv) \(ay \leq a\ \forall a \in V\) and \(y \in \Gamma\)

v) \(0 \gamma e = \gamma 0 = 0, \forall y \in \Gamma\)

for the greatest element \(\top\) and least element \(0\) of \(V\), where \(V\) denotes a \(\Gamma\)-groupoid lattice. Condition iii and v implies,

vi) \(0 \gamma a = a \gamma 0 = 0\ \forall a \in V, y \in \Gamma\)

Definition 1.3: An element \(b\) of \(V\) is called \(\Gamma\)-absorbent of the element \(a\) of \(V\) if

vii) \(b \leq a\)

viii) and \(\gamma ay \leq b,\)

ix) \(bya \leq b, \forall a, b \in V\) and \(y \in \Gamma\).

holds, \(b\) is a left- \(\Gamma\)-absorbent of \(a\) if vii and viii holds and right- \(\Gamma\)-absorbent if vii and ix holds.

Definition 1.4: An element \(k\) of \(V\) is called quasi- \(\Gamma\)-absorbent of \(a \in V\) if \(k \leq a\) and \(kya \land yk \leq k\) for all \(y \in \Gamma\).

Definition 1.5: By a bi- \(\Gamma\)-absorbent of \(a \in V\) we mean an element \(b \in V\) such that \(b \leq a\) and \((bya)ya \land \gamma (ayb) \leq b\) for all \(y \in \Gamma\).

An element \(x\) of \(\Gamma\)-groupoid lattice \(V\) is called idempotent if the identity \(x = x\gamma x\) holds for all \(y \in \Gamma\).

If \(a_\mu (\mu \in \Lambda)\) are elements of the \(\Gamma\)-groupoid lattice \(V\), then from iii, we have

x) \(\gamma (\land_{\mu \in \Lambda} a_\mu) \leq \land_{\mu \in \Lambda} a_\mu, (\land_{\mu \in \Lambda} a_\mu)b \leq \land_{\mu \in \Lambda} a_\mu b, \forall b \in V, y \in \Gamma\)

xi) \(\gamma (\lor_{\mu \in \Lambda} a_\mu) \geq \lor_{\mu \in \Lambda} a_\mu, (\lor_{\mu \in \Lambda} a_\mu)b \geq \lor_{\mu \in \Lambda} a_\mu b, \forall b \in V, y \in \Gamma\)
Definition 1.6: The quasi absorbent $k \neq 0$ of an element $a$ of a $\Gamma$-groupoid lattice $V$ is said to be minimal if $a$ has no non-zero quasi absorbent $k_1$ such that $k_1 < k$. A minimal left, right or bi-absorbent of $a$ can be defined analogously.

Proposition 1.1[6]: Let $b_2(\lambda \in \Lambda)$ are the $\Gamma$-absorbent (left, right, quasi, bi-$\Gamma$-absorbent) of a groupoid lattice $a \in V$. Then the meet $\bigwedge_{\lambda \in \Lambda} b_2$ is a $\Gamma$-absorbent (left, right, quasi, bi-$\Gamma$-absorbent) of the element $a$.

Proposition 1.2[6]: If $r$ and $l$ are right- and left-$\Gamma$-absorbent of $a \in V$, respectively, then $ryl \leq r \wedge l$, $rl$ is a bi-absorbent and $r \wedge l$ is a quasi absorbent of $a$ for all $y \in \Gamma$.

2. Minimal-$\Gamma$-absorbent in groupoid lattices

The right-$\Gamma$-absorbent $R \neq 0$ of an element $a$ of a $\Gamma$-groupoid lattice $V$ is said to be minimal if $a$ has no nonzero right-$\Gamma$-absorbers $s$ such that $s < R$. A minimal $\Gamma$-absorbent, left-, quasi-, or bi-$\Gamma$-absorbent of $a$ can be defined analogously.

Proposition 2.1: If $a$ is an element $a$ of a $\Gamma$-groupoid lattice $V$, if $r$ and $l$ are minimal left and right-$\Gamma$-absorbers of $a$, then $k = r \wedge l$ is a quasi-$\Gamma$-absorbent of $a$.

Proof: Let $r \wedge l = k \neq 0$. By proposition 1.2, $k = r \wedge l$ is a quasi-$\Gamma$-absorbent of $a$. Assume that $k$ is a minimal quasi-$\Gamma$-absorbent of $a$. Then there exists a quasi-$\Gamma$-absorbent $k$ of $a$ such that $0 < k' < k$. Since $k' < l$ imply that $a y k'$ is a left-$\Gamma$-absorbent of $a$ such that $0 < a y k' < l$ for all $y \in \Gamma$. If $a y k' = 0$ then the element $k'$ is a left-$\Gamma$-absorbent of $a$ with $0 < k' < k \leq l$, which is impossible. Thus $a y k' = l$. Similarly, one can show that $k' y a = r$. Hence $k = r \wedge l = a y k' \wedge k' y a < k'$, which contradicts the assumption $k' < k$. So we conclude that $k$ is a minimal quasi-$\Gamma$-absorbent of $a$, indeed.

Theorem 2.1: A quasi-$\Gamma$-absorbent $k$ of an element $a$ of $V$ is minimal if and only if any two of its nonzero elements generates the same left-$\Gamma$-absorbent and same right-$\Gamma$-absorbent of a $\Gamma$-groupoid lattice.

Proof: Suppose that $k$ is a minimal quasi-$\Gamma$-absorbent in $\Gamma$-groupoid lattice $V$. By the minimality of $k$ implies that $a y k = k \forall a, b \in V, y \in \Gamma$. Thus $k \leq a y b$. Let $0 \neq c \in k$, then $c \leq a y b$.

\[ a y c \leq (a y a) y b \]

Similarly, $a y b \leq a y c$ and therefore $a y b = a y c$. Thus any two nonzero elements of a $\Gamma$-groupoid lattice generates the same left-$\Gamma$-absorbent and same right-$\Gamma$-absorbent.

Conversely, suppose that any two nonzero elements of the quasi-$\Gamma$-absorbent $k$ generate the same left and right-$\Gamma$-absorber of $a \in V$. Let $k$ be a non-zero quasi-$\Gamma$-absorbent of $a \in V$, then $k \leq k$. First we assume that $a y k \wedge k \neq \{0\} \forall y \in \Gamma$ and $k y a \wedge k \neq \{0\} \forall y \in \Gamma$. Let $0 \neq l \in a y k \wedge k$ and $0 \neq r \in k y a \wedge k$. Then for $0 \neq b \in k, b \in a y b = a y l \leq a y k$ and $b \in a y b = r y a \leq k y a$. Thus $b \in a y k \wedge k y a \leq k$ and so $k = k$. Now assume that $k y a \wedge k = \{0\}$. Let $0 \neq c \in k$. Then for $0 \neq b \in k, b = x y c, \forall x, b, c \in V, y \in \Gamma$. Obviously, $b \in x y c$. Thus $b = (n + a_1) y c$ for some $a_1 \in a \in V$. It follows that $a y c = b - n y c \in a y k \wedge k = \{0\}$. So $b = n y c \in k$ and $k = k$. Thus $k$ is minimal.

Proposition 2.2: Let $m$ be a two-sided-$\Gamma$-absorbent of a $\Gamma$-groupoid lattice $a \in V$. For all $y \in \Gamma$ the following are equivalent.

i) If $b$ is a (two-sided) $\Gamma$-absorbent of a such that $b y l \leq m$ then $b \leq m$;

ii) If $l$ is a left-$\Gamma$-absorbent of $a$ such that $l y l \leq m$, then $l \leq m$;

iii) If $r$ is a right-$\Gamma$-absorbent of $a$ such that $r y r \leq m$, then $r \leq m$.

Proof: i) $\Rightarrow$ ii) Let $l$ be left-$\Gamma$-absorbent of $a \in V$ satisfying $l y l \leq m$, then $l \vee l y a$ is the two-sided $\Gamma$-absorbent of $a$ generated by $l$. Hence $(l \vee l y a)^2 = l y l \vee l y a \vee l y a l \vee l y a y l a \leq m$. Similarly one can show i) $\Rightarrow$ iii). The implications ii) $\Rightarrow$ i) and iii) $\Rightarrow$ i) are evident.

The $\Gamma$-absorbent $m$ of $a \in V$ is called semiprime $\Gamma$-groupoid lattice if one of the conditions i)—iii) is satisfied for $m$.

Theorem 2.2: Every minimal quasi-$\Gamma$-absorbent $k$ of a semiprime $\Gamma$-groupoid lattice is the meet of a minimal left-$\Gamma$-absorbent $l$ and minimal right-$\Gamma$-absorbent $r$ of $a$.

Proof: Since $a y k \wedge k y a$ is a quasi-$\Gamma$-absorbent of $a \in V$ for all $y \in \Gamma$. Therefore the minimality of $k$ implies that $a y k \wedge k y k = 0$ or $a y k \wedge k y k = k$. Assume that $a y k \wedge k y k = 0$. Then either $a y k = 0$ or $a y k = 0$. The case $a y k = 0$ is impossible, since then $k$ would be a non-zero left-$\Gamma$-absorbent of $a$ satisfying $a y k \neq 0$. If
\[ ay_k \neq 0, \text{ then } k y a y_k \leq a y_k \land k y a = 0 \text{ implies } (a y_k) y (k y a) = 0 \text{ which contradicts the condition that } 0 \text{ is a semiprime } \Gamma-\text{absorbent of } a. \text{ So we have } a y_k \land k y a = k. \text{ We shall show that } a y_k \text{ is a minimal left-} \Gamma-\text{absorbent and } k y a \text{ is a minimal right-} \Gamma-\text{absorbent of } a. \text{ Assume that } k \text{ is not a minimal left-} \Gamma-\text{absorbent of } a. \text{ Then there exists a non-zero left-} \Gamma-\text{absorbent } l \text{ of } a \text{ such that } l \leq a y_k. \text{ Since } a y_k \land k y a \text{ is a quasi-} \Gamma-\text{absorbent of } a \text{ such that } a y_l \land k y a \leq a y_k \land k y a = k \text{ implies that } a y_l \land k y a = k, \text{ the minimality of the quasi-} \Gamma-\text{absorbent } k \text{ implies that } a y_l \land k y a = k \text{ implies } k \leq a y_l \leq l, \text{ whence } a y_k \leq a y_l \leq a y_l. \text{ This excludes the existence of a non-zero left-} \Gamma-\text{absorbent } l \text{ such that } l \leq a y_k. \text{ Thus } a y_k \text{ is a minimal left-} \Gamma-\text{absorbent of } a, \text{ and } a y_l \text{ is a minimal right-} \Gamma-\text{absorbent of } a. \text{ Indeed, similarly, one can show that } k y a \text{ is a minimal right-} \Gamma-\text{absorbent of } a.

Proposition 2.2: Every minimal quasi-} \Gamma-\text{absorbent } k \text{ of a semiprime } \Gamma-\text{groupoid lattice } a \in V \text{ has the form } k = e y a \land a y f = e y a f \forall e y e = e, f y f = f \text{ and } \gamma \in \Gamma, \text{ where } e y a, a y f \text{ are minimal right and minimal left } \Gamma-\text{absorbents of } a, \text{ respectively.}

Proof: By Theorem 2.2, } k \text{ is the meet of a minimal right-} \Gamma-\text{absorbent } r \text{ and minimal left-} \Gamma-\text{absorbent } l \text{ of } a, \text{ that is } k = r \land l. \text{ Now we shall prove the existence of nonzero idempotents } e, f \text{ in } a \text{ such that } r = e y a \text{ and } l = a y f \text{ for all } \gamma \in \Gamma. \text{ Let } m \text{ be two-sided } \Gamma-\text{absorbent of } a \text{ such that } l \leq m \text{ and } x \text{ be a non-zero element of } l. \text{ Then the product } m y x \text{ is a left absorbent of } a. \text{ Since } l \text{ is minimal, either } m y x = 0 \text{ or } m y x = l. \text{ If } m y x = 0, \text{ then the set } X \text{ of all the elements } x \text{ of } l \text{ with } m y x = 0 \text{ is a non-zero left absorbent of } a. \text{ By the minimality of } l, \text{ we have } X = l, \text{ that is } m y X = m y l = 0. \text{ Since } l \leq m, \text{ we conclude that } l y l = 0, \text{ which contradicts the condition that } a \text{ is a semiprime groupoid lattice. So we have } m y x = l \text{ for all non-zero } x \text{ of } l. \text{ If } m \text{ is a two-sided absorbent of } a \text{ such that } r \leq m, \text{ one can show that } y y m = r \text{ for all non-zero } y \text{ of } r.

Now let } d \text{ be non-zero element of } k = r \land l. \text{ From the above we know that}

\[ a y d y a = l y a = a y r \quad \forall d \in k = r \land l. \]

Since } a \text{ is a semiprime groupoid lattice, } r y r = r \text{ and } l y l = l \text{ must hold. This and i) imply}

\[ l = l y l \leq l y a = a y d y a \]

and

\[ r = l y l \leq a y r = a y d y a. \]

Proposition 2.3: Let } e \text{ be an idempotent element of a } \Gamma-\text{groupoid lattice } a \in V \text{ and } r, l \text{ are left and right } \Gamma-\text{absorbent of } a, \text{ respectively. Then } r y e \text{ and } e y l \text{ are quasi-} \Gamma-\text{absorbits of } a \text{ such that } r y e = r \land a y e \text{ and } e y l = e y a \land l, \forall \gamma \in \Gamma.

Proof: Since } r y l \leq r \land a y e \text{ and proposition 2.2 imply } r y l \leq r \land a y e. \text{ Now we need only to show that } r \land a y e \leq r y e. \text{ We know that any element of } a \text{ of } r \land a y e \text{ has the form } a = r_1 = s y e, \forall r_1 \in r, s \in a, \text{ whence } a = s y e = s y e y = r_1 y e \in r y e. \text{ Hence } r \land a y e \leq r y e \text{ and therefore } r y e = r \land a y e. \text{ Similarly one can prove } e y l = e y a \land l.

Theorem 2.3: If a quasi-} \Gamma-\text{absorbent } k \text{ of a } \Gamma-\text{groupoid lattice } a \text{ is a division element, then } k \text{ is a minimal quasi } \Gamma-\text{absorbent of } a.

Proof: Let } k' \text{ be a quasi-} \Gamma-\text{absorbent of } a. \text{ Such that } 0 \neq k' \leq k. \text{ Then } k y k' \land k' y k \leq a y k' \land k' y a k \leq k \text{ implies that } k' \text{ is a } \Gamma-\text{absorbent of } a. \text{ Since } k \text{ is a division element and division element has no proper quasi-} \Gamma-\text{absorbent, we have } k = k'. \text{ Thus } k \text{ is the minimal quasi-} \Gamma-\text{absorbent of } a.

Proposition 2.4: Let } l \text{ be a minimal left-} \Gamma-\text{absorbent of a } \Gamma-\text{groupoid lattice } a \in V. \text{ If } e \text{ is a nonzero idempotent element of } l, \text{ then } e y l \text{ is a division element of } \forall y \in \Gamma, \text{ moreover it is a minimal quasi-} \Gamma-\text{absorbent of } a.

Proof: By proposition 2.3, } e y l \text{ is a quasi absorbent of } a. \text{ Evidently, } e \text{ is a left identity of } e y l. \text{ Let } e y h \text{ be non-zero element of } e y l. \text{ Then } l y (e y h) \text{ is a non-zero left absorbent of } a \text{ such that } l y (e y h) \leq l. \text{ By the minimality of } l, \text{ we have } l y (e y h) = i. \text{ Hence } (e y l) y (e y h) = e y l. \text{ This implies the existence of non-zero element } e y z \text{ of } e y l \text{ such that } (e y z) y (e y h) = e. \text{ Thus } 0 \neq e y l \text{ is a division element. Hence it is minimal quasi-absorbent of } a \text{ by Theorem 2.2.}

References


