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Minimal Quasi-Γ-absorbent in Γ-Groupoid-lattice

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Abstract

Otto. Steinfeld and Rédei [1] generalize concepts from groups, rings and semigroups to groupoid lattices. In our paper [5][6], we have introduced the notion of semiprime absorbents in groupoid lattices and Γ-absorbents in groupoid lattices. Here in this paper we will discuss some properties of minimal quasi- Γ-absorbent using semiprime Γ-absorbent in Γ-groupoid lattices.

Keywords: *Γ -Groupoid lattice, semiprime Γ -absorbent, minimal Γ-absorbent, quasi-Γ-absorbent.*

1. Introduction

A binary operation Γ on C is defined as a mapping from $C \times C \to C$ that is Γ assigns to each pair $(a, b) \in C \times C$ exactly one element $\Gamma(a, b) \in C$. Instead of $\Gamma(a, b)$ one mostly writes $a\Gamma b$. Let C and Γ be two non-empty sets. A mapping from $C \times \Gamma \times C \to C$ will be called a Γ -multiplication in C and is denoted by $(\cdot)_\Gamma$. The result of the Γ-multiplication for $(a, b) \in C$ and $\gamma \in \Gamma$ is denoted by $a\gamma b$. Let $C = \{x, y, z, \ldots\}$ and $\Gamma = \{\alpha, \beta, \gamma, \ldots\}$ be two non empty sets. Then C is called a Γ-groupoid if it satisfies $x\alpha y \in C \;\; \forall\, x, y \in C$ and $\alpha \in \Gamma$. A Γ-semigroup is a Γ-groupoid such that the operation $(\cdot)_\Gamma$ is associative.

Definition 1.1: A partially ordered Γ-groupoid is a non-empty set C satisfying the following properties:

i) C is a Γ-groupoid w.r.t. multiplication "$(\cdot)_\Gamma$ ";
ii) C is a partially ordered set w.r.t. a partial ordering " $\leq$ ";
iii) If $a \leq b$ then $c\gamma a \leq c\gamma b$ and $a\gamma c \leq b\gamma c \;\; \forall a, b, c \in C$, and $\gamma \in \Gamma$.

We know that a lattice "V" is called a complete if every subset of "V" has aleast upper bound (join " $\vee$ ") and greatest lower bound (meet " $\wedge$ ") in "V". The greatest element is denoted bt "e" and the least element is denoted by "0".

Definition 1.2: A Γ-groupoid lattice is a partially ordered groupoid $\langle V, (\cdot)_\Gamma, \leq \rangle$ such that V is a complete lattice with respect to the partial ordering $\leq$ and it has the following properties:

iv) $a\gamma a \leq a \quad \forall\, a \in V$ and $\gamma \in \Gamma$
v) $0\gamma e = e\gamma 0 = 0, \forall\, \gamma \in \Gamma$

for the greatest element e and least element 0 of V, where V denotes a Γ-groupoid lattice. Condition iii and v implies,

vi) $0\gamma a = a\gamma 0 = 0 \quad \forall\, a \in V, \gamma \in \Gamma$

Definition 1.3: An element b of V is called Γ-absorbent of the element a of V if

vii) $b \leq a$
viii) and $a\gamma b \leq b,$
ix) $b\gamma a \leq b, \;\; \forall a, b \in V$ and $\gamma \in \Gamma$.

holds, b is a left- Γ-absorbent of a if vii and viii holds and right- Γ-absorbent if vii and ix holds.

Definition 1.4: An element k of V is called quasi- Γ-absorbent of $a \in V$ if $k \leq a$ and $k\gamma a \wedge a\gamma k \leq k$ for all $\gamma \in \Gamma$.

Definition 1.5: By a bi- Γ-absorbent of $a \in V$ we mean an element $b \in V$ such that $b \leq a$ and $(b\gamma a)\gamma a \wedge b\gamma(a\gamma b) \leq b$ for all $\gamma \in \Gamma$.

An element x of Γ-groupoid lattice V is called idempotent if the identity $x = x\gamma x$ holds for all $\gamma \in \Gamma$.

If $a_\mu \; (\mu \in \Lambda)$ are elements of the Γ-groupoid lattice V, then from iii, we have

x) $b\gamma\left(\wedge_{\mu\in\Lambda} a_\mu\right) \leq \wedge_{\mu\in\Lambda} b\gamma a_\mu, \quad \left(\wedge_{\mu\in\Lambda} a_\mu\right)\gamma\, b \leq \wedge_{\mu\in\Lambda} a_\mu \gamma\, b \quad \forall\, b \in V, \gamma \in \Gamma$

xi) $b\gamma\left(\vee_{\mu\in\Lambda} a_\mu\right) \geq \vee_{\mu\in\Lambda} b\gamma a_\mu, \quad \left(\vee_{\mu\in\Lambda} a_\mu\right)\gamma b \geq \vee_{\mu\in\Lambda} a_\mu \gamma\, b \quad \forall\, b \in V, \gamma \in \Gamma$

Definition 1.6: The quasi absorbent $k \neq 0$ of an element a of a $\gamma \in \Gamma$-groupoid lattice V is said to be minimal if a has no non-zero quasi absorbent k_1 such that $k_1 < k$. A minimal left, right or bi-absorbent of a can be defined analogously.

Proposition 1.1[6]: Let $b_\lambda(\lambda \in \Lambda)$ are the Γ-absorbent (left, right, quasi, bi-Γ-absorbent) of a groupoid lattice $a \in V$. Then the meet $\wedge_{\lambda \in \Lambda} b_\lambda$ is a Γ-absorbent (left, right, quasi, bi- Γ-absorbent) of the element a.

Proposition 1.2[6]:If r and l are right- and left- Γ-absorbent of $a \in V$, respectively, then $r\gamma l \leq r \wedge l$, rl is a bi-absorbent and $r \wedge l$ is a quasi absorbent of a for all $\gamma \in \Gamma$.

2. Minimal-Γ-absorbent in groupoid lattices

The right- Γ-absorbent $R \neq 0$ of an element a of a Γ-groupoid lattice V is said to be minimal if a has no nonzero right-Γ-absorbents s such that $s < R$. A minimal Γ-absorbent, left-, quasi-, or bi-Γ-absorbent of a can be define analogously.

Proposition 2.1: If a is an element a of a Γ-groupoid lattice$V$, if r and l are minimal right and left-Γ-absorbents of a, then $k = r \wedge l$ is a quasi-Γ-absorbent of a.

Proof: Let $r \wedge l = k \neq 0$. By proposition 1.2, $k = r \wedge l$ is a quasi-Γ-absorbent of a. Assume that k is not a minimal quasi-Γ-absorbent of a. Then there exists a quasi-Γ-absorbent k of a such that $0 < k' < k$. Since $k' < l$ imply that $a\gamma k'$ is a left-Γ-absorbent of a such that $0 < a\gamma k' < l$ for all $\gamma \in \Gamma$. If $a\gamma k' = 0$ then the element k' is a left-Γ-absorbent of a with $0 < k' < k \leq l$, which is impossible. Thus $a\gamma k' = l$. Similarly, one can show that $k'\gamma a = r$. Hence $k = r \wedge l = a\gamma k' \wedge k'\gamma a < k'$, which contradicts the assumption $k' < k$. So we conclude that k is a minimal quasi-Γ-absorbent of a, indeed.

Theorem 2.1: A quasi- Γ-absorbent k of an element a of V is minimal if and only if any two of its nonzero elements generates the same left-Γ-absorbent and same right-Γ-absorbent of a Γ-groupoid lattice.

Proof: Suppose that k is a minimal quasi- Γ-absorbent in Γ-groupoid lattice V. By the minimality of k implies that $a\gamma b \wedge k = k\ \forall a, b \in V, \gamma \in \Gamma$. Thus $k \leq a\gamma b$. Let $0 \neq c \in k$, then

$c \leq a\gamma b$

$\Rightarrow a\gamma c \leq (a\gamma a)\gamma b$

$\Rightarrow a\gamma c \leq a\gamma b \quad \because a\gamma a \leq a$

Similarly, $a\gamma b \leq a\gamma c$ and therefore $a\gamma b = a\gamma c$. Thus any two nonzero elements of a Γ-groupoid lattice generates the same left-Γ-absorbent and same right-Γ-absorbent.

Conversely, suppose that any two nonzero elements of the quasi- Γ-absorbent k generate the same left and right-Γ-absorbent of $a \in V$. Let $\bar{k}$ be a non-zero quasi- Γ-absorbent of $a \in V$, then $\bar{k} \leq k$. First we assume that $a\gamma\bar{k} \wedge k \neq \{0\}\ \forall\gamma \in \Gamma$ and $\bar{k}\gamma a \wedge k \neq \{0\}\ \forall\gamma \in \Gamma$. Let $0 \neq l \in a\gamma\bar{k} \wedge k$ and $0 \neq r \in \bar{k}\gamma a \wedge k$. Then for any $0 \neq b \in k, b \in a\gamma b = a\gamma l \leq a\gamma\bar{k}$ and $b \in a\gamma b = r\gamma a \leq \bar{k}\gamma a$. Thus $b \in a\gamma\bar{k} \wedge \bar{k}\gamma a \leq \bar{k}$ and so $k = \bar{k}$. Now assume that $\bar{k}\gamma a \wedge k = \{0\}$. Let $0 \neq c \in \bar{k}$. Then for $0 \neq b \in k, x\gamma b = x\gamma c,\ \forall x, b, c \in V, \gamma \in \Gamma$. Obviously, $b \in x\gamma c$. Thus $b = (n + a_1)\gamma c$ for some $a_1 \in a$ of V. It follows that $a\gamma c = b - n\gamma c \in a\gamma\bar{k} \wedge k = \{0\}$. So $b = n\gamma c \in \bar{k}$ and $k = \bar{k}$. Thus k is minimal.

Proposition 2.2: Let m be a two-sided Γ-absorbent of a Γ-groupoid lattice $a \in V$. For all $\gamma \in \Gamma$ the following are equivalent.

i) If b is a (two-sided) Γ-absorbent of a such that $b\gamma b \leq m$, then $b \leq m$;

ii) If l is a left-Γ-absorbent of a such that $l\gamma l \leq m$, then $l \leq m$;

iii) If r is a right-Γ-absorbent of a such that $r\gamma r \leq m$, then $r \leq m$.

Proof: i) $\Rightarrow$ ii) Let l be left-Γ-absorbent of $a \in V$ satisfying $l\gamma l \leq m$, then $l \vee l\gamma a$ is the two sided Γ-absorbent of a generated by l. Hence $(l \vee l\gamma a)^2 = l\gamma l \vee l\gamma l\gamma a \vee l\gamma a\gamma l \vee l\gamma a\gamma l\gamma a \leq m$. Similarly one can show i) $\Rightarrow$ iii). The implications ii) $\Rightarrow$ i) and iii) $\Rightarrow$ i) are evident.

The Γ-absorbent m of $a \in V$ is called semiprime Γ-groupoid lattice if one of the conditions i)−iii) is satisfied for m.

Theorem 2.2. Every minimal quasi-Γ-absorbent k of a semiprime Γ-groupoid lattice is the meet of a minimal left Γ-absorbent l and minimal right- Γ-absorbent r of a.

Proof: Since $a\gamma k \wedge k\gamma a$ is a quasi-Γ-absorbent of $a \in V$ for all $\gamma \in \Gamma$. Therefore the minimality of k implies that $a\gamma k \wedge k\gamma a = 0$ or $a\gamma k \wedge k\gamma a = k$. Assume that $a\gamma k \wedge k\gamma a = 0$. Then either $a\gamma k = 0$ or $a\gamma k \neq 0$. The case $a\gamma k = 0$ is impossible, since then k would be a nonzero left-Γ-absorbent of a satisfying $a\gamma k \neq 0$. If

$a\gamma k \neq 0$, then $k\gamma a\gamma k \leq a\gamma k \wedge k\gamma a = 0$ implies $(a\gamma k)\gamma(k\gamma a) = 0$ which contradicts the condition that 0 is a semiprime Γ-absorbent of a. So we have $a\gamma k \wedge k\gamma a = k$. We shall show that $a\gamma k$ is a minimal left-Γ-absorbent and $k\gamma a$ is a minimal right-Γ-absorbnet of a. Assume that $a\gamma k$ is not a minimal left-Γ-absorbent of a. Then there exists a non-zero left-Γ-absorbent l of a such that $l \leq a\gamma k$. Since $a\gamma k \wedge k\gamma a$ is a quasi- Γ -absorbent of a such that $a\gamma l \wedge k\gamma a \leq a\gamma k \wedge k\gamma a = k$, the minimality of the quasi-Γ -absorbent k implies that $a\gamma l \wedge k\gamma a = k$ implies $k \leq a\gamma l \leq l$, whence $a\gamma k \leq a\gamma l \leq$. This excludes the existence of a nonzero left- Γ -absorbent l such that $l \leq a\gamma k$. Thus $a\gamma k$ is a minimal left-Γ-absorbent of a, indeed. Similarly, one can show that $k\gamma a$ is a minimal right-Γ-absorbnet of a.

Proposition 2.2: Every minimal quasi- Γ-absorbent k of a semiprime Γ -groupoid lattice $a \in V$ has the form $k = e\gamma a \wedge a\gamma f = e\gamma a\gamma f \;\; \forall\, e\gamma e = e, f\gamma f = f$ and $\gamma \in \Gamma$, where $e\gamma a, a\gamma f$ are minimal right and minimal left Γ-absorbents of a, respectively.

Proof: By Theoren 2.2, k is the meet of a minimal right- Γ-absorbent r and minimal left- Γ-absorbent l of a, that is $k = r \wedge l$. Now we shall prove the existence of nonzero idempotents e, f in a such that $r = e\gamma a$ and $l = a\gamma f$ for all $\gamma \in \Gamma$. Let m be two-sided Γ-absorbent of a such that $l \leq m$ and x be a non-zero element of l. Then the product $m\gamma x$ is a left absorbent of a. Since l is minimal, either $m\gamma x = 0$ or $m\gamma x = l$. If $m\gamma x = 0$, then the set X of all the elements x of l with $m\gamma x = 0$ is a non-zero left absorbent of a. By the minimality of l, we have $X = l$, that is $m\gamma X = m\gamma l = 0$. Since $l \leq m$, we conclude that $l\gamma l = 0$, which contradicts the condition that a is a semiprime groupoid lattice. So we have $m\gamma x = l$ for all non-zero x of l. If m is a two-sided absorbent of a such that $r \leq m$, one can show that $y\gamma m = r$ for all non-zero y of r.

Now let d be non-zer element of $k = r \wedge l$. From the above we know that

i) $a\gamma d\gamma a = l\gamma a = a\gamma r \quad \forall\, d \in k = r \wedge l.$

Since a is a semiprime groupoid lattice, $r\gamma r = r$ and $l\gamma l = l$ must hold. This and i) imply

$l = l\gamma l \leq l\gamma a = a\gamma d\gamma a$

and

$r = l\gamma l \leq a\gamma r = a\gamma d\gamma a.$

Proposition 2.3: Let e be an idempotent element of a Γ-groupoid lattice $a \in V$ and r, l are left and right Γ-absorbent of a, respectively. Then $r\gamma e$ and $e\gamma l$ are quasi-Γ -absorbents of a such that $r\gamma e = r \wedge a\gamma e$ and $e\gamma l = e\gamma a \wedge l, \;\; \forall\, \gamma \in \Gamma.$

Proof: Since $r\gamma l \leq r \wedge l$ and proposition 2.2 imply $r\gamma l \leq r \wedge a\gamma e$. Now we need only to show that $r \wedge a\gamma e \leq r\gamma e$. We know that any element of a of $r \wedge a\gamma e$ has the form $a = r_1 = s\gamma e, \;\; \forall\, r_1 \in r, s \in a$, whence $a = s\gamma e = s\gamma e\gamma e = r_1\gamma e \in r\gamma e$. Hence $r \wedge a\gamma e \leq r\gamma e$ and therefore $r\gamma e = r \wedge a\gamma e$. Similarly one can prove $e\gamma l = e\gamma a \wedge l$.

Theorem 2.3: If a quasi-Γ-absorbent k of a Γ-groupoid lattice a is a division element, then k is a minimal quasi absorbent of a.

Proof: Let k' be a quasi- Γ-absorbent of a. Such that $0 \neq k' \leq k$. Then $k\gamma k' \wedge k'\gamma k \leq a\gamma k' \wedge k'\gamma a \leq k$ implies that k' is a quasi-Γ-absorbent of a. Since k is a division element and division element has no proper quasi-Γ-absorbent, we have $k = k'$. Thus k is the minimal quasi-Γ-absorbent of a.

Proposition 2.4: Let l be a minimal left-Γ-absorbent of a Γ-groupoid lattice $a \in V$. If e is a nonzero idempotent element of l, then $e\gamma l$ is a division element of a for all $\gamma \in \Gamma$, moreover it is a minimal quasi-Γ-absorbent of a.

Proof: By proposition 2.3, $e\gamma l$ is a quasi absorbent of a. Evidently, e is a left identity of $e\gamma l$. Let $e\gamma h$ be non-zero element of $e\gamma l$. Then $l\gamma(e\gamma h)$ is a non-zero left absorbent of a such that $l\gamma(e\gamma h) \leq l$. By the minimality of l, we have $l\gamma(e\gamma h) = l$. Hence $(e\gamma l)\gamma(e\gamma h) = e\gamma l$. This implies the existence of non-zero element $e\gamma z$ of $e\gamma l$ such that $(e\gamma z)\gamma(e\gamma h) = e$. Thus $0 \neq e\gamma l$ is a division element. Hence it is minimal quasi-absorbent of a by Theorem 2.2.

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