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An Optimal Control Approach to Maintenance Modeling

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Abstract
The paper presents an approach for finding the optimal maintenance period of a machine by considering together its resale value, constant production rate and preventive maintenance rate as utility function's maximization problem. In general, a declination of single machine's resale value over time is an important input to the optimal maintenance modeling tasks. In this paper, an optimal control theory is applied on the purpose of finding the optimal maintenance period which leads to highest present value of the machine.
Keywords: optimal maintenance; utility function; machine resale value; dynamic optimization.

1. Introduction

Consider a single machine whose resale value gradually declines over some time. Recognizing the importance of optimal maintenance and the following impact on the overall machine's present value cannot be understated. Some other researchers highlighted the importance of optimal maintenance in their papers and studies [9,10,11,12,13,14,15]. One can conclude that the machine's optimal maintenance period estimation is a very important part of system's present value. It takes into account the resale value of machine, constant production rate and preventive maintenance rate during the lifetime of the object. The main goal of the proposed study is to find an approach how to estimate the optimal maintenance period by considering the machine's resale value as utility function's maximization problem.

2. Theoretical Background

Nowadays, the applied mathematical modelling (e.g. applied optimization, optimal control) is widely used in many research areas, see papers [1,2,3,4,5,6,7,8,17]. An optimal control theory (dynamic optimization) will be further applied to solve for this task.

Suppose an optimal control task in discrete time with the following periods: *0,1,2...T* [5]. The general consideration is that the state variable x_t is measured at the beginning of each period t and the control variable u_t is applied during this period t. Figure 1 shows this problem statement:

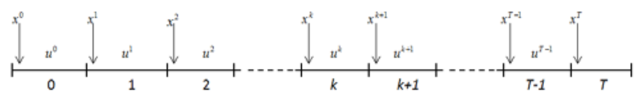

Fig.1. Discrete time optimal control problem

With some continuously differentiable functions:

$$f : E^n \times E^m \times \Theta \to E^n, F : E^n \times E^m \times \Theta \to E^1,$$

$$g : E^m \times \Theta \to E^s, S : E^m \times \Theta \cup \{T\} \to E^1$$

The model dealing with the optimal maintenance task is with the following notation:

- *T*- the lifetime of the considered object.
- *x(t)*- the resale value (e.g. in euro/dollars) of the machine at time *t*; where: *x(0)*= x_0 .
- *u(t)*- the preventive maintenance rate at time *t.*
- *g(t)*- the maintenance effectiveness function at time *t.*
- *d(t)*- the obsolescence function at time *t.*
- π - the constant production rate per unit time per unit resale value.

It is assumed that *g(t)* is a non-increasing function of time and *d(t)* is non-decreasing function of time, and that for all *t:*

$$u(t) \in \Omega = [0;U] \quad (1)$$

where: U is a positive constant.

In this case we consider that the present value of the machine is a sum of two terms and can be modeled in the following way [5]:

$$Max\left\{ J = \int_0^T [\pi x(t) - u(t)]e^{-rt}dt + x(T)e^{-rT} \right\} \quad (2)$$

The state variable x is affected by the obsolescence factor, the amount of preventive maintenance and the maintenance effectiveness function [5]. Thus,

$$\dot{x}(t) = -d(t) + g(t)u(t), \quad x(0) = x_0 \quad (3)$$

One can also highlight that on the purpose of realism:

$$-d(t) + g(t)u(t) \le 0, \ t \ge 0. \quad (4)$$

The assumption given by Eq.(4) implies that preventive maintenance is not so effective as to enhance the resale value of the machine over its previous values; rather it can at most slow down the decline of the resale value, even when preventive maintenance is performed at the maximum rate U [5]. The optimal control problem is to maximize the Eq.(2) subject to constraints expressed by Eq.(1) and Eq.(3).

In this case we consider the problem in a discrete time since (due to some feasibility reasons) it is impossible to have records in continuous time [16]. In the case of annual/monthly/weekly/daily information, the time series of preventive maintenance rate can be expressed as:

$$\vec{u} = \{ u_0, u_1, u_2, ..., u_T \} \quad (5)$$

Suppose the total time period in our case study T=36 months (i.e. the considered machine lifetime is 3 years). Therefore, for discrete case [17], the resource value can be described as:

$$V(x_0, \vec{u}) = \sum_{t=0}^{T} \beta^t C(u_t) \quad (6)$$

where: $C(u_t)$ - cash flow for ***t**-th* month;

$\beta = 1/(1+r)$ - discounting factor

r - the interest rate.

The fundamental concept proposed by the interest rate theory is the net present value (see Eq.(6)) of the cash flow over time [17]. It should be noted that the arbitrage absence supposes that the value of obligation (agreement, contract) should be the net present value of the cash flow.

Further, we can consider a cash flow, i.e. the discrete series of periodic payments $C(u_t)$, $t = 0,1,2,...,T$. In this case the interest rate r is given in a discrete complexity and it is applied on the payment periods (see Fig.1). Then the net present value (NPV) is defined by the following expression [16]:

$$V(x_0, \vec{u}) = \sum_{t=0}^{T} \frac{C(u_t)}{(1+r)^t} \quad (7)$$

From the theoretical background point of view, the cash flow in our case study can be modelled as:

$$V(x_0, \vec{u}) = \left[\sum_{t=0}^{T} \beta^t \left(\pi x(t) - u(t) \right) \right] + x(T)\beta^T \quad (8)$$

The maintenance manager can decide to apply the time series of preventive maintenance rate in Eq.(5) which provides maximized value of the present value functional: $V(x_0, \vec{u})$ expressed by Eq.(8) with a constraint shown on Eq.(1) and Eq.(3).

The term u_t is a control variable in our research study. As well as, a numerical solution to a dynamic optimization problem requires two endpoint conditions. Suppose the initial machine's resale value is 5000 units (e.g. in euro or dollars) and the lifetime of the object is 36 months. Then the initial condition is x_0 =5000 and final one is $x_T = x_{36} \ge 0$.

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3. Practical Applications

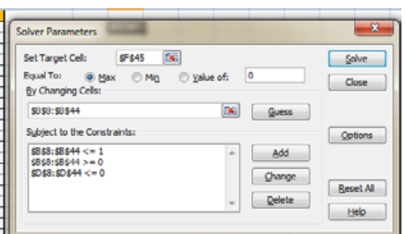
A practical application of the applied mathematical modeling by Eq.(1)-Eq.(8) is given in this chapter. This task is solved for by using the Microsoft Excel® software. Tables 1 and 2 show the construction of the dynamic optimization problem:

Table 1

	A	B	C	D	E	F
1	d	100				
2	π	0.1				
3	r	0.05				
4	β	0.95				
5						
6						
7	time, t	u	g	(-d+gU)	x	Utility function (J)
8	0	1	2	-98	5000	499
9	1	1	1.4	-98.59	4902	466
10	2	1	1.2	-98.85	4803	435
11	3	1	1.0	-99.00	4705	406
12	4	1	0.9	-99.11	4606	378
13	5	1	0.8	-99.18	4506	352
14	6	1	0.8	-99.24	4407	328
15	7	1	0.7	-99.29	4308	305
16	8	1	0.7	-99.33	4209	284
17	9	1	0.6	-99.37	4109	264
18	10	1	0.6	-99.40	4010	246
19	11	1	0.6	-99.42	3911	228
20	12	1	0.6	-99.45	3811	212
………						
40	32	1	0.3	-99.65	1820	38
41	33	1	0.3	-99.66	1720	34
42	34	1	0.3	-99.66	1621	31
43	35	1	0.3	-99.67	1521	27
44	36	1	0.3	-99.67	1421	24
45					PV	6788

Table 2

time, t	u	g	(-d+gU)	x	Utility function (J)
0	1	2	-98	5000	499
1	1	1.4	-98.59	4902	466
2	1	1.2	-98.85	4803	435
3	1	1.0	-99.00	4705	406
4	1	0.9	-99.11	4606	378
5	1	0.8	-99.18	4506	352
6	1	0.8	-99.24	4407	328
7	1	0.7	-99.29	4308	305
8	1	0.7	-99.33	4209	284
9	1	0.6	-99.37	4109	264
10	1	0.6	-99.40	4010	246
11	1	0.6	-99.42	3911	228
12	1	0.6	-99.45	3811	212
13	1	0.5	-99.47	3712	196
14	1	0.5	-99.48	3612	182
15	1	0.5	-99.50	3513	168
16	1	0.5	-99.51	3413	156
17	1	0.5	-99.53	3314	144
18	1	0.5	-99.54	3214	133
19	1	0.4	-99.55	3115	123
20	1	0.4	-99.56	3015	113
21	1	0.4	-99.57	2916	104
22	1	0.4	-99.58	2816	96
23	1	0.4	-99.59	2716	88
24	1	0.4	-99.60	2617	81
25	1	0.4	-99.61	2517	74
26	1	0.4	-99.62	2418	68
27	1	0.4	-99.62	2318	62
28	1	0.4	-99.63	2218	56
29	1	0.4	-99.63	2119	51
30	1	0.4	-99.64	2019	46
31	1	0.4	-99.65	1920	42
32	1	0.3	-99.65	1820	38
33	1	0.3	-99.66	1720	34
34	1	0.3	-99.66	1621	31
35	1	0.3	-99.67	1521	27
36	1	0.3	-99.67	1421	24
				PV	6788



The parameters considered in the study are the following: the obsolescence function *d* is assumed 100 (cell B1), the constant production rate *π* is assumed 0.1 (cell B2), the interest rate *r* is 0.05 in cell B3 (note that *π*>r or else it does not pay to produce). The maintenance effectiveness function *g(t)*=2/((1+*t*)^0.5). To compute the discounting factor, we need to use the following well-known formula: $\beta = 1/(1+r) = 1/(1+0.05) = 0.95$. In cells D8:D44- the constraint by Eq.(3) is introduced. The equation Eq.(5) is implemented in cells B8:B44 which represents the time series of the preventive maintenance rate. The initial resale value (cell E8) is x_0 = 5000 and the discounted resale values are shown in column E (cells E8:E44). An iterative algorithm via "Solver" introduced by Microsoft Excel® is applied which requires initial guess values (all guess values in the range B8:B44 were initially taken *u*=1) of the optimal time series of preventive maintenance rate. And, at 1st iteration, suppose that the preventive maintenance rate is controlled uniformly each month (*u*=1, cells B8:B44, Table 1). In the column F is located the DCF(t) which means discounted cash flow earned during month ***t***. The cell F45 (in yellow) shows the net present value $V(x_0, \vec{u})$ denoted by Eq.(8).

The net present value $V(x_0, \vec{u})$ depends on the initial resale value x_0 and on the preventive maintenance rate $\vec{u}$, i.e. $V(x_0, \vec{u})$ =6788 if x_0 =5000 and uniform series of preventive maintenance rate is applied. One may conclude in that case the uniform preventive maintenance rate is not optimal when the future cash flows are discounted. Applying the optimization algorithm in that case, we can find the optimal preventive maintenance rate series $\vec{u}$ which maximizes $V(x_0, \vec{u})$. The constraints (Eq.1 and Eq.3) can be found in the dialog box: „B8:B44>=0", „B8:B44<=1" and „D8:D44<=0" (see Table 2).

The final results are shown in Table 3- the optimal preventive maintenance rate series (*u*=1) should be applied up to month *t*=10, i.e. the result says that it is optimal to perform a preventive maintenance up to 10 months after purchase of the machine. In that case of the optimal preventive maintenance rate series is applied, then the global objective function (cell F45) increases from 6788 up to 6791. The resale value of the machine at the month *T*=36 (end of lifetime) has become 1411 which is about 30% from the initial resale value (x_0 =5000).

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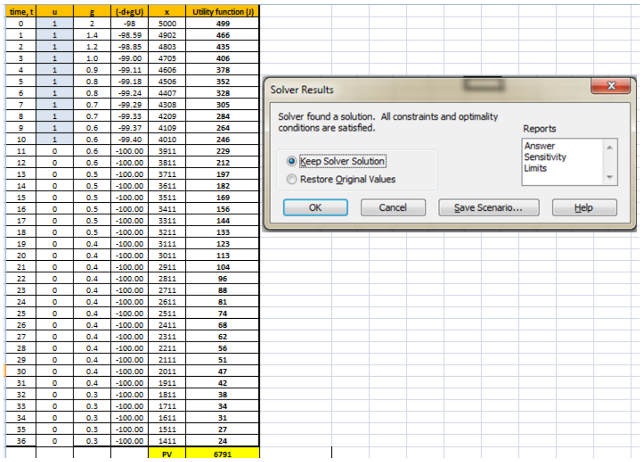
Table 3



time, t	u	g	(-d+gU)	x	Utility function (J)
0	1	2	-98	5000	499
1	1	1.4	-98.59	4902	466
2	1	1.2	-98.85	4803	435
3	1	1.0	-99.00	4705	406
4	1	0.9	-99.11	4606	378
5	1	0.8	-99.18	4506	352
6	1	0.8	-99.24	4407	328
7	1	0.7	-99.29	4308	305
8	1	0.7	-99.33	4209	284
9	1	0.6	-99.37	4109	264
10	1	0.6	-99.40	4010	246
11	0	0.6	-100.00	3911	229
12	0	0.6	-100.00	3811	212
13	0	0.5	-100.00	3711	197
14	0	0.5	-100.00	3611	182
15	0	0.5	-100.00	3511	169
16	0	0.5	-100.00	3411	156
17	0	0.5	-100.00	3311	144
18	0	0.5	-100.00	3211	133
19	0	0.4	-100.00	3111	123
20	0	0.4	-100.00	3011	113
21	0	0.4	-100.00	2911	104
22	0	0.4	-100.00	2811	96
23	0	0.4	-100.00	2711	88
24	0	0.4	-100.00	2611	81
25	0	0.4	-100.00	2511	74
26	0	0.4	-100.00	2411	68
27	0	0.4	-100.00	2311	62
28	0	0.4	-100.00	2211	56
29	0	0.4	-100.00	2111	51
30	0	0.4	-100.00	2011	47
31	0	0.4	-100.00	1911	42
32	0	0.3	-100.00	1811	38
33	0	0.3	-100.00	1711	34
34	0	0.3	-100.00	1611	31
35	0	0.3	-100.00	1511	27
36	0	0.3	-100.00	1411	24
				PV	6791

The optimal preventive maintenance rate series vs months is shown below on Fig.2:

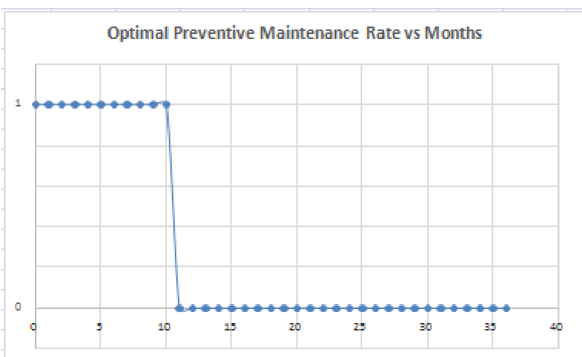


Fig.2. Optimal preventive maintenance rate vs months

4. Conclusions

The following important outcomes obtained by the proposed research study are:

1. Maximized utility function Eq.(8) requires an optimal preventive maintenance rate series $\vec{u}$ during the considered lifetime period. In case of uniform preventive maintenance the series is not optimal when the future cash flows are discounted.

2. The optimal preventive maintenance period for the considered machine was found *t=10* months which allows to perform some savings for the company in the next months.

3. The proposed modeling is very flexible since it is taking into account different important inputs like maintenance effectiveness, the obsolescence function, the constant production rate; the interest rate.

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