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Fractional calculus approach to investigation of creep behavior of Wengé Wood (*Millettia Laurentii*)

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Abstract

In this research work aiming at describing the time-dependent mechanical property of the wood species Wengé (***Millettia Laurentii***) during the creep, a new method of building creep model based on variable-order fractional derivatives is proposed. The order of the fractional derivative is allowed to be a function of the independent variable (time), rather than a constant of arbitrary order. Through the segmentation treatment, according to different creep stages of the experimental results, it is found that the improved creep model based on variable-order fractional derivatives agrees well with the experimental data. In addition, the fact is verified that variable order of fractional derivatives can be regarded as a step function, which is reasonable and reliable. In addition, through further piecewise fitting, the parameters in the model are determined on the basis of existing experimental results. All predicted results reveal that the theoretical model proposed in the manuscript properly depicts the creep properties, providing an excellent agreement with the experimental data.

Keywords: *Wood, Millettia Laurentii, Creep, Fractional derivatives, Spring-pot, Fractional rheological model.*

1. Introduction

During the last few years, much attention has been paid to the application of wood in various engineering works in the world. Due to the expensiveness of traditional materials like steel, iron, concrete, etc., wood has become a world-wide alternative **[1- 4]**. Meanwhile creep is among the fundamental factors limiting its long-term application as excessive deformation or reduced stiffness occurs over an extended period of time. For material design related to the load-bearing capacity of products, the evaluation of creep behavior is indispensable **[5-6]**. Determining how to describe the creep process for wood remains a challenging problem, while it is essential to research the wood creep property in wood engineering. As a result, during several decades much effort has been directed toward the study of creep behavior of wood, most of which was devoted to modeling the wood creep. As a matter of fact, various creep constitutive models of wood have been proposed [**7-14**].

In addition, the fractional derivative has long history dependence or the so-called memory effects. It has been found that fractional calculus is a powerful tool for modeling the viscoelastic behaviors and particularly suited for building the time-dependent constitutive model. Moreover, an increasing effort has been devoted to the application of fractional calculus to viscoelastic and viscoplastic constitutive models. Authors [**15-16**] realized remarkable contributions and they established a solid foundation in fractional derivative models. The use of fractional-derivative-based constitutive models is motivated, to some extent, by the fact that fewer parameters are required to represent material creep behavior than in the classical component models [**17**]. In fact, the property of material is time-dependent under loading, which means that the order of fractional derivative should be variable. Recently the constitutive model based on variable-order fractional derivatives has been getting attention [**18**]. In a sense, the variable-order fractional calculus is the extension of fractional calculus. Theoretically, the change of the order can exhibit the evolution of mechanical properties of materials. This manuscript proposes a new model based on variable order fractional derivatives in order to precisely describe the nonlinear creep behavior for wood.

2. Materials and methods

2.1 Riemann-Liouville fractional order integration

There are many fractional order calculus definitions, among which the creep characteristics of wood are defined and described by Riemann-Liouville theory. This is how the Riemann-Liouville fractional integration with the order p of function f(t) is defined on the interval [0, + ∞[, for t > 0, $p \in [0, +\infty[$ and Re(p) > 0 [**19**]

$$ {}_{0}^{R}I_{t}^{p} f\left(t\right) = \frac{1}{\Gamma\left(p\right)} \int_{0}^{t} \left(t-\tau\right)^{p-1} f\left(\tau\right) d\tau,\ t \succ 0,\ p \succ 0 \tag{1} $$

Where Γ stands for the Gamma function defined as

$$ \Gamma\left(p\right) = \int_{0}^{+\infty} t^{p-1} e^{-t} dt,\ Re(p) \succ 0 \tag{2} $$

2.2 Riemann-Liouville fractional order derivative

The Riemann-Liouville fractional order differential form is defined as

$$ {}_{0}^{R}D_{t}^{p} f(t) = \frac{1}{\Gamma\left(n-p\right)} \frac{d^{n}}{dt^{n}} \int_{0}^{t} \left(t-\tau\right)^{n-p-1} f\left(\tau\right) d\tau,\ p \succ 0,\ n-1 \leq p \prec n \tag{3} $$

where p is the fractional order, n is a positive integer larger than p and d is a differential operator.

Remark

We should bear in mind that in mathematical modeling, the use of Riemann-Liouville fractional derivatives leads to initial conditions containing limit values of fractional derivatives at the inferior born t = 0. A solution of this problem has been proposed by **Caputo M.** [**20**]. Let $p \geq 0$ (with $n-1 \leq p < n$ and $n \in N^{*}$), f(t) is a continuum function n times differentiable. The Caputo fractional derivative is defined as:

$$ {}_{0}^{C}D_{t}^{p} f\left(t\right) = \frac{1}{\Gamma\left(n-p\right)} \int_{0}^{t} \left(t-\tau\right)^{n-p-1} f^{(n)}\left(\tau\right) d\tau,\ t \succ 0 \tag{4} $$

where p is the order of the derivative.

2.3 Establishment of the fractional order creep model

The ideal solid stress-strain relationship follows Hooke's law:

$$ \sigma\left(t\right) = E\varepsilon\left(t\right) \tag{5} $$

The ideal fluid stress-strain relationship follows Newton's viscous law:

$$ \sigma\left(t\right) = \eta \frac{d\varepsilon\left(t\right)}{dt} \tag{6} $$

where σ(t) is a function of stress versus time, ε(t) is a function of strain versus time, E is the elastic modulus and t is time.

Ideal solids and ideal fluids are idealized models, which can only be used for approximate calculations in practical engineering. Wood is a material between an ideal solid and an ideal fluid, and a software element can be used to describe an intermediate material between a pure elastomer and a Newtonian fluid. Therefore, the wood rheological properties can be described by combining a model with the spring-pot (Able dashpot) instead of the Newton dashpot. The Newton dashpot and the spring-pot are shown in figure 1.

Figure 1 : (a) Newton dashpot ; **(b)** Spring-pot or Able dashpot

The constitutive equation based on the fractional derivative is established as:

$$\sigma(t)=\eta^{p}\frac{d^{p}\varepsilon(t)}{dt^{p}},\ 0\leq \mathrm{p}\leq 1 \tag{7}$$

where the stress σ remains constant; that is, σ = cst. This software element can describe the creep deformation. Fractional order integrals are performed on both sides of equation (7), by using the Riemann-Liouville fractional order integral definition; we can get:

$$\varepsilon(t)=\frac{\sigma}{\eta^{p}}\frac{t^{p}}{\Gamma(1+p)},\ 0\leq p\leq 1 \tag{8}$$

where σ denotes the constant stress level.

Figure 2 depicts the common fractional derivative Maxwell composed by the Hooke body and the spring-pot. Suppose the strain of the Hooke body is ε_1, that of the spring-pot is ε_2.

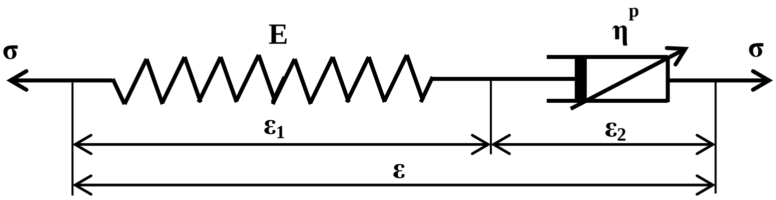

Figure 2 : Fractional derivative Maxwell model

The stress-strain relationship of the Hooke body is:

$$\varepsilon_1=\frac{\sigma}{E} \tag{9}$$

where E stands for elastic modulus of the Hooke body. The stress-strain relationship of the spring-pot reads

$$\varepsilon_2(t)=\frac{\sigma}{\eta^{p}}\frac{t^{p}}{\Gamma(1+p)},\ 0\leq p\leq 1 \tag{10}$$

In this way, the order of the fractional derivative can be regarded as a function of time i.e. p = α(t), $0 \leq \alpha(t) \leq 1$. α(t) is given as follows, then:

$$\sigma(t)=\eta^{\alpha(t)}\ {}_{t_{k-1}}^{\ C}D_{t}^{\alpha(t)}\varepsilon_2(t),\ 0\leq\alpha(t)\leq 1,\ t_{k-1}\leq t\prec t_k \tag{11}$$

Where α(t) representing the fractional derivative order is a function of time and varies according to table 1 and $\eta^{\alpha(t)}$ is the corresponding viscosity coefficient.

Table 1: Value of α(t) versus time

Order	Time period
α_1	$0 \leq t < t_1$
α_2	$t_1 \leq t < t_2$
.	.
α_n	$t_{n-1} \leq t < t_n$

Let the stress be constant, we can obtain

$$\begin{cases} \varepsilon_2(t) = \sum_{k=1}^{n} \frac{\sigma}{\eta^{\alpha_k}} \frac{(t-t_{k-1})^{\alpha_k}}{\Gamma(1+\alpha_k)} \\ 0 \le \alpha(t) = \alpha_k \le 1,\ t_{k-1} \le t \prec t_k \end{cases} \tag{12}$$

Where σ stands for constant stress, α_k denotes the value of α(t) at a given moment.

Considering the two parts of strain, and combining (9) and (12), the constitutive equation of variable-order fractional derivatives creep model can be represented as:

$$\begin{cases} \varepsilon(t) = \varepsilon_1 + \varepsilon_2(t) = \frac{\sigma}{E} + \sum_{k=1}^{n} \frac{\sigma}{\eta^{\alpha_k}} \frac{(t-t_{k-1})^{\alpha_k}}{\Gamma(1+\alpha_k)} \\ 0 \le \alpha(t) = \alpha_k \le 1,\ t_{k-1} \le t \prec t_k \end{cases} \tag{13}$$

Equation (13) is the nonlinear creep constitutive model of wood, based on fractional calculus theory.

2.4 Experimental setup

Creep experiment is the foundation of creep property investigation on wood, determining the parameters in creep constitutive models. The current experiments were conducted at Dschang University (Cameroon) using a four points flexural test machine (figure 4) coupled with a strain-bridge possessing a high accuracy. The indoor temperature was 23°C and the relative humidity was 65% during all the process. All the wood samples were extracted from the same billet of *Millettia Laurentii* wood, originating from Kyé-Ossi natural forest in Cameroon south region. The specimens were prepared with a required dimension of 20mm×20mm×360mm.

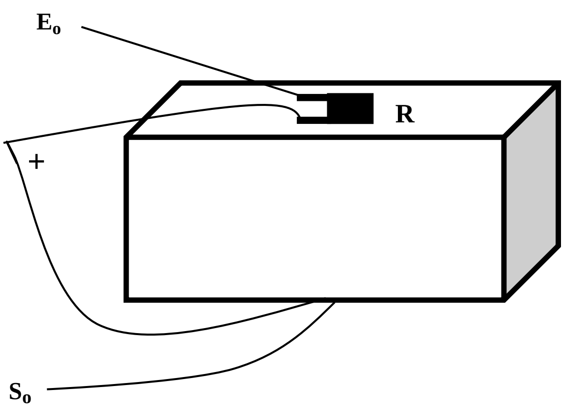

Figure 3 : Wood tube carrying two symmetrical gauges

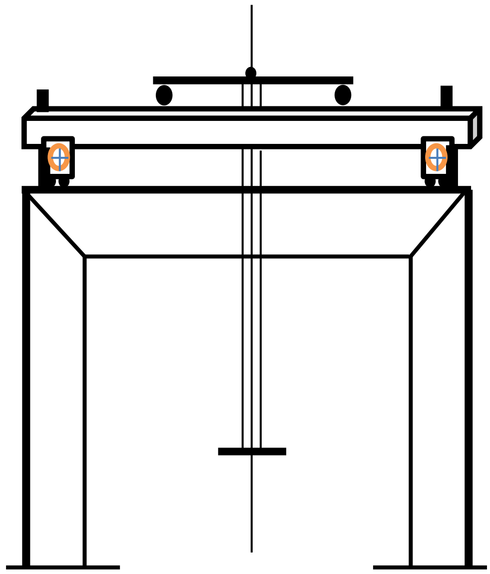

Figure 4: Experiment set-up for creep test

In figure 3, E_o and S_o represent the gauge electrodes that are directly connected to the strain bridge and R stands for the gauge electrical resistance. During the test the sample is lain on the test machine in such a way that one gauge is on the top and another one symmetrically on the opposite face of the sample. The wood specimens were tested under four points flexural loading following the French Norm NF B 51-003 that labels general requirements for physical and mechanical tests.

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3. Results and discussion

From the creep experiment lasting for 10 hours that is 600 minutes, we obtained the experimental creep curve of wood shown in figure 5. By further processing, the creep rate curve of wood is acquired as illustrated in figure 6.

As shown in the above two figures, the creep curve exhibits only deceleration creep and steady-state creep, without an apparent accelerating creep stage. Therefore, the creep curve can be divided into two segments, and the segment point of time is comprehensively determined in accordance with the creep curve and creep rate curve, that is **t_1 = 142 minutes**. From equation (13), the curve form of the first stage can be expressed as follows:

$$\begin{cases} \varepsilon(t) = \dfrac{\sigma}{E} + \dfrac{\sigma}{\eta^{\alpha_1}} \dfrac{(t-t_0)^{\alpha_1}}{\Gamma(1+\alpha_1)} = \dfrac{\sigma}{E} + \dfrac{\sigma}{\eta^{\alpha_1}} \dfrac{t^{\alpha_1}}{\Gamma(1+\alpha_1)} \\ 0 = t_0 \le t \prec t_1 \end{cases} \tag{14}$$

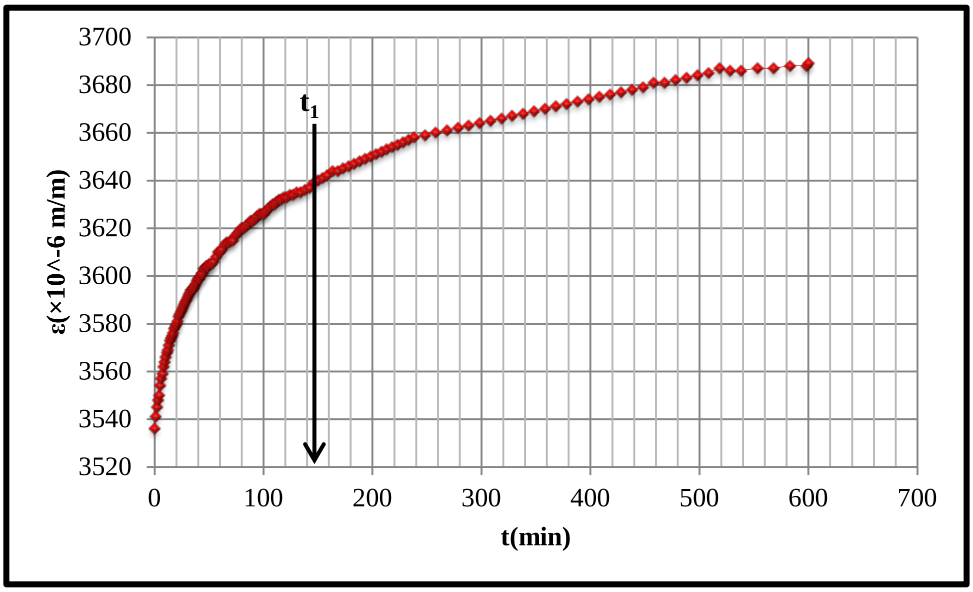

Figure 5: Creep curve of wood under flexural load of **σ = 33,1MPa**

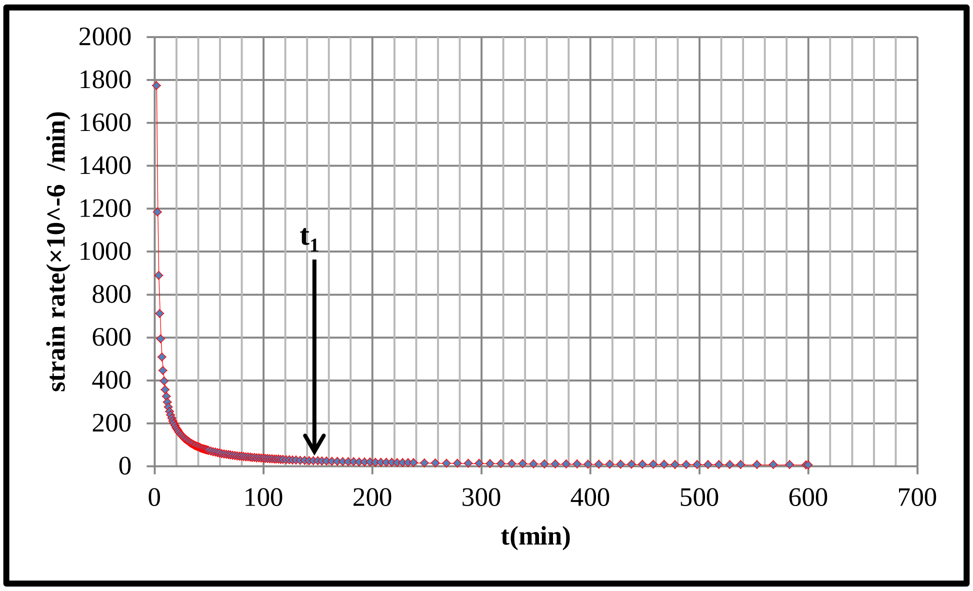

Figure 6: Creep rate curve of wood under flexural load of σ = 33,1MPa

where $\varepsilon(t_0) = \frac{\sigma}{E}$ thus

$$E = \frac{\sigma}{\varepsilon(t_0)} \tag{15}$$

Taking the logarithmic operation on both sides of equation (14), we can obtain

$$\log\left[\varepsilon(t) - \frac{\sigma}{E}\right] = \log\left[\frac{\sigma}{\eta^{\alpha_1}} \frac{t^{\alpha_1}}{\Gamma(1+\alpha_1)}\right], \; 0 \le \alpha_1 \le 1, \; 0 = t_0 \le t_1 \tag{16}$$

$$\log\left[\varepsilon(t) - \varepsilon(t_0)\right] = \alpha_1 \log(t) - \log\left[\frac{\eta^{\alpha_1}\Gamma(1+\alpha_1)}{\sigma}\right]; \; 0 \le \alpha_1 \le 1, \; 0 \le t \prec t_1 \tag{17}$$

Let us suppose that

$$\begin{cases} x = \log(t) \\ y = \log\left[\varepsilon(t) - \varepsilon(t_0)\right] \end{cases} \tag{18}$$

Then equation (17) becomes a linear equation about ***x*** and ***y***.

$$y = a_1 x + b_1 \tag{19}$$

By dealing with the creep experimental data of the first stage, we can calculate to get the data set about ***x*** and ***y***. Hence, whether x and y is a linear correlation can be determined by linear fitting analysis of the data set about x and y. If x and y are linearly related, one can further get α_1 and η^{α_1}, derived from the fitting coefficients a_1 and b_1.

$$\begin{cases} \alpha_1 = a_1 \\ \eta^{\alpha_1} = \dfrac{\sigma}{\Gamma(1+\alpha_1)} 10^{-b_1} \end{cases} \tag{20}$$

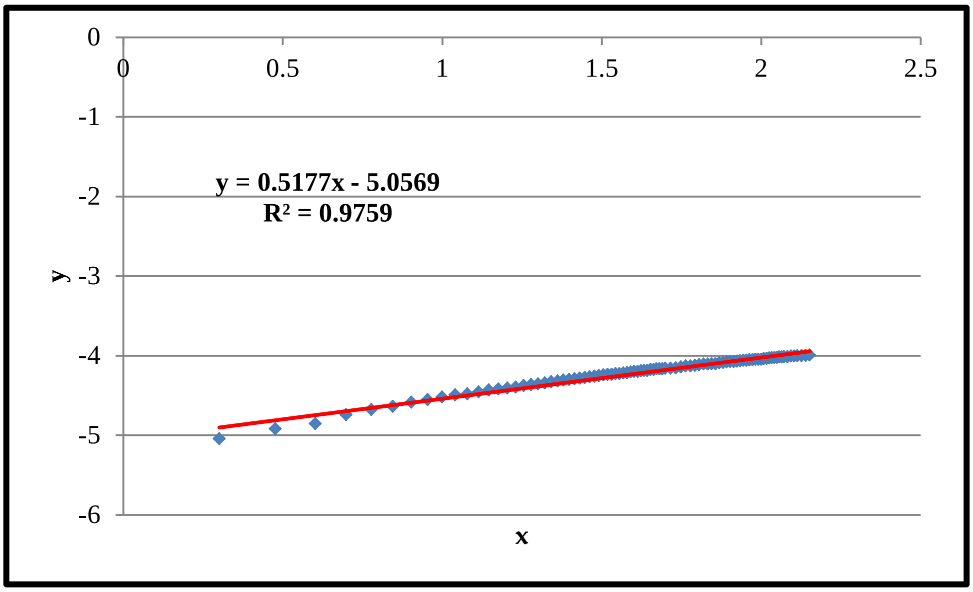

Figure 7: Deceleration creep region data with linear fitting equation after logarithmic processing

From the analysis results shown in figure 7, it is obvious that the correlation coefficient of the equation $\mathbf{R^2 > 0{,}97}$ indicating the predicted data by the fractional derivative model proposed in the paper is in good agreement with the creep experimental data of the first stage. Then the coefficients both α_1 and η^{α_1} can be obtained.

Similarly according to equation (13), the creep curve in the second stage can be given by

$$\begin{cases} \varepsilon(t) = \dfrac{\sigma}{E} + \dfrac{\sigma}{\eta^{\alpha_1}} \dfrac{t_1^{\alpha_1}}{\Gamma(1+\alpha_1)} + \dfrac{\sigma}{\eta^{\alpha_2}} \dfrac{(t-t_1)^{\alpha_2}}{\Gamma(1+\alpha_2)} \\ 0 \le \alpha_2 \le 1,\ t_1 \le t \prec t_2 \end{cases} \quad (21)$$

Taking the logarithmic operation on both sides of equation (21), then

$$\begin{cases} \log\left[\varepsilon(t) - \varepsilon(t_1)\right] = \alpha_2 \log(t-t_1) - \log\left[\dfrac{\eta^{\alpha_2}\Gamma(1+\alpha_2)}{\sigma}\right] \\ 0 \le \alpha_2 \le 1,\ t_1 \le t \prec t_2 \end{cases} \quad (22)$$

where $\varepsilon(t_1) = \frac{\sigma}{E} + \frac{\sigma}{\eta^{\alpha_1}} \frac{t_1^{\alpha_1}}{\Gamma(1+\alpha_1)}$

Similarly, we can assume

$$\begin{cases} x = \log(t-t_1) \\ y = \log\left[\varepsilon(t) - \varepsilon(t_1)\right] \end{cases} \quad (23)$$

Then equation (22) establishes another linear relationship between x and y.

$$y = a_2 x + b_2 \quad (24)$$

Analyzing the creep experimental data of the second stage, the data set about x and y can be calculated and whether x and y is a linear correlation can be confirmed by linear fitting analysis of the data set. If x and y are linearly related, one can further get α_2 and η^{α_2}, derived from the fitting coefficients a_2 and b_2.

$$\begin{cases} \alpha_2 = a_2 \\ \eta^{\alpha_2} = \dfrac{\sigma}{\Gamma(1+\alpha_2)} 10^{-b_2} \end{cases} \quad (25)$$

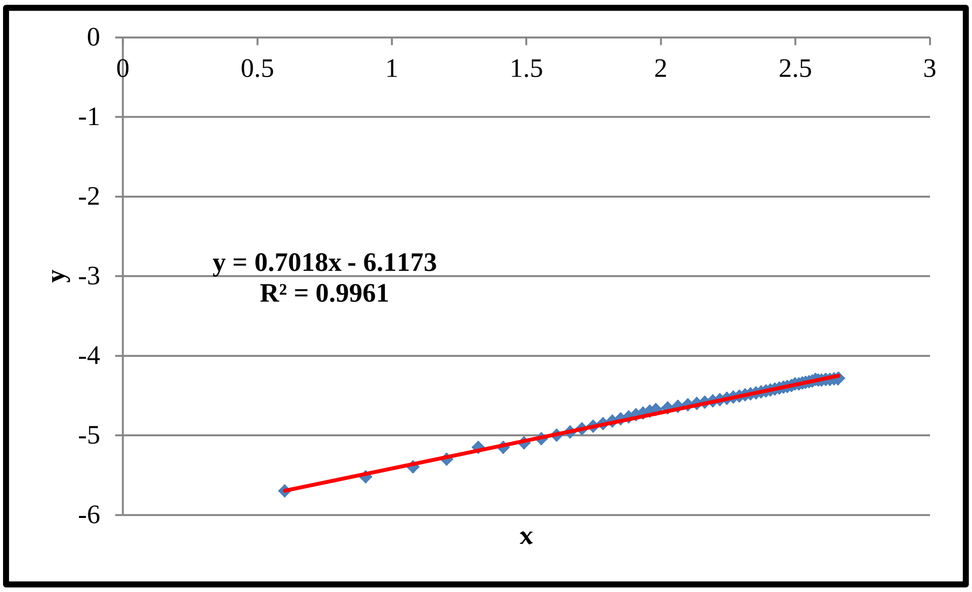

Figure 8: Steady-state creep region data with linear fitting equation after logarithmic processing

From the analysis results shown in figure 8, it can be seen that the correlation coefficient $R^2 > 0,99$ indicating the fractional derivative model proposed in the paper is in good agreement with the creep experimental data of the second stage. Eventually we can further determine the coefficients α_2 and η^{α_2}.

By segmentation treatment based on the creep experimental data, figure 9 demonstrates the fact that the variable order of fractional derivatives regarded as a step function is reasonable and reliable, and the fitting correlation coefficients of experimental data in respective stages are pretty good, which helps us to determine all the parameters (table 2).

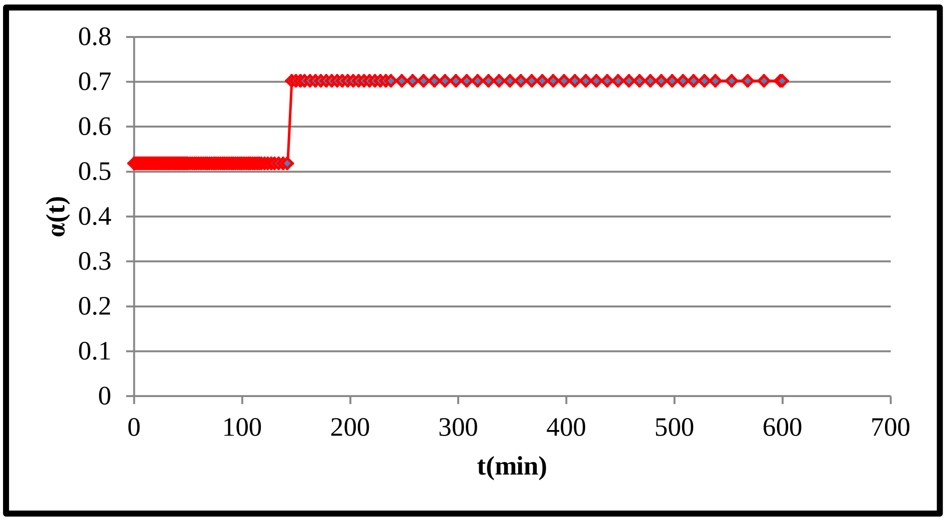

Figure 9: Values of the step function α(t)

Table 2 : Parameters determined by fitting analysis based on creep tests of wood

E(MPa)	α_1	$\eta^{\alpha_1}(MPa.min^{\alpha_1})$	α_2	$\eta^{\alpha_2}(MPa.min^{\alpha_2})$
9360,859	0,5177	$4,2544 \times 10^6$	0,7018	$4,7706 \times 10^7$

Figure 10 describes the theoretical creep curve according to the parameters given in table 2. As shown in the figure, the variable-order fractional derivative constitutive model presented in the paper has good consistency with the experimental data, which further proves that the improved creep model is reliable.

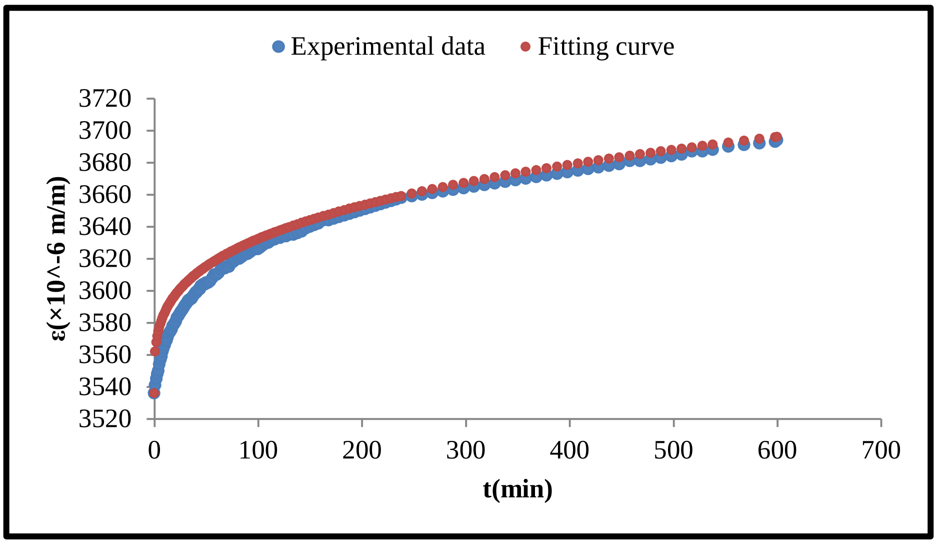

Figure 10: Comparison of experimental results and predicted data obtained from the constitutive equation

Through the segmentation treatment, according to different creep stages of the above two experimental results, it is shown that the predicted data by the new creep model in the respective stages agrees with the experimental data, and the fitting correlation coefficients are greater than 0,97. It is also verified that the variable order of fractional derivatives as a step function is reasonable and reliable.

4. Conclusion

The physics significance of the spring-pot has been a research hotspot. The commonly accepted interpretation regards the spring-pot as a constitutive element describing the material state between ideal solid and Newton fluid. In a sense, the fractional derivative seems an extension of the integer order derivative, which can reveal the mechanical properties in nature closer to viscoelastic materials properties. In order to describe the mechanical property of wood is time-dependent during the creep, a new method was presented to build the constitutive model. The Maxwell creep model for wood based on variable-order fractional derivatives was proposed and the creep constitutive equations were concluded in an explanatory manner. Through the segmentation of the experimental results according to different creep stages, it was found that the creep model based on variable-order fractional derivatives agrees well with the experimental data. It was also verified that the fact the variable order of fractional derivatives is a step function. And through piecewise fitting, the parameters in the model were determined. The theoretical curves according to the parameters were consistent well with the experimental data.

Summing up the above experimental results, it was found that in the primary creep stage, the mechanical property of wood is analog to elastic body. While in the steady creep stage, the mechanical property of wood behaves like viscous material. As we can realize, the improved Maxwell creep model proposed in this paper, based on variable-order fractional derivatives, can thorough reveal the gradual transformation process of wood creep from elastic state to viscous state. The constitutive model proposed in the manuscript not only highly agrees with the experimental results but also shows the evolution of mechanical properties of materials with the change of the order during whole process.

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Authors' contributions

Atchounga wrote the manuscript, performed the experiment of the study and was responsible for data collection; Foadieng and Talla conceived and designed the experiments. Atchounga and Talla analyzed the data. All authors read and approved the final manuscript.

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Competing interests

The authors declare that they have no competing interests.

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