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Derivation of Spatial Evolution of Quantum System in the Interaction Picture within the Framework of Generalized Special Relativity

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Abstract

Using generalized special relativity a useful expression of the perturbed momentum is found. This expression is used to describe the behavior of the quantum system in the interaction picture. The spatial evolution of the Schrodinger Equation in the interaction picture is similar to that of time evaluation, where the time differential is replaced by the space one and the Hamiltonian by the momentum operator. The same holds for the unitary operator, where the time integral is replaced by the space one and the Hamiltonian with the momentum operator.

Key Words: momentum perturbation, interaction picture, spatial evolution unitary operator

Introduction

Quantum laws are used to describe the behavior of atoms and elementary particles. According to the time evolution there are three versions. The first one is the Schrodinger picture in which the time evolution is described by the wave function. The second one is the Heisenberg picture in which the time evolution is described by the operators. The third representation is the so called interaction representation in which the time evolution of the system is described by the wave vector and the operator which is the interaction Hamiltonian instead of the total Hamiltonian [1,2,3]These versions succeeded in describing the time evolution but says nothing about the spatial evolution of the quantum system. This motivated some authors to propose some models to cure this defect [4, 5, 6]. In one of them the ordinary Schrodinger equation is developed to describe the behavior of the system using the momentum operator [7].In another approach the generalized special relativity is used to describe the spatial evolution of the system by using a perturbed momentum [8,9]. Different attempts were also be made to make quantum laws more flexible in describing the quantum system [10, 11]. This encourages constructing a new model to help in describing the spatial evolution of the quantum system.

Time Evolution of Quantum System within the Frame Work of Generalized Special Relativity

The energy in generalized special relativity is given by

$$E = mc^2 = \frac{g_{00} m_0 c^2}{\sqrt{g_{00} - {v^2}/{c^2}}} = \frac{g_{00} E_0}{\sqrt{g_{00} - {v^2}/{c^2}}} \qquad (2.1)$$

For very small velocity compared to the speed of light

$$v \ll c$$

Thus

$$E = g_{00}^{\frac{1}{2}} E_0 = g_{00}^{\frac{1}{2}} E_0 \qquad (2.2)$$

Using the fact that $(v < E, m\varphi < c^2 m, \varphi < c^2)$

$$g_{00}^{\frac{1}{2}} = \left(1 + 2\,{\varphi}/{c^2}\right)^{\frac{1}{2}}$$

$$E = \left(1 + \frac{\varphi}{c^2}\right) E_0 = \left(1 + \frac{m_0 \varphi}{m_0 c^2}\right) E_0$$

$$E = E_0 + \frac{V E_0}{E_0} = E_0 + V \qquad (2.3)$$

Thus the corresponding Hamiltonian is given by

$$\hat{H} = \hat{H}_0 + \hat{H}_i \qquad (2.4)$$

Where $\hat{H}_0$ standing for the unperturbed Hamiltonian, while $\widehat{H_\iota}$ represents the interaction Hamiltonian which causes perturbation.

To simplify treatment, it is convenient to modify Schrodinger equation. This modification requires the time evolution of the wave equation to be in terms of the interaction Hamiltonian instead of the total Hamiltonian. This requires

$$|\psi\rangle = e^{-\frac{i}{\hbar}\hat{H}_0 t} |\psi\rangle_I$$

$$|\psi\rangle_I = e^{\frac{i}{\hbar}\hat{H}_0 t} |\psi\rangle$$

$$|\psi\rangle = |\psi\rangle_I e^{\frac{iH_0 t}{\hbar}} \qquad (2.5)$$

$$i\hbar \frac{d|\psi\rangle}{dt} = \hat{H} |\psi\rangle \qquad (2.6)$$

$$\hat{H} = \hat{H}_0 + \hat{H}_i \qquad (2.7)$$

$$i\hbar \frac{d}{dt}|\psi\rangle = \hat{H}|\psi\rangle$$

$$i\hbar \frac{d}{dt}\left[e^{-\frac{iH_0 t}{\hbar}}|\psi\rangle_I\right] = \hat{H}|\psi\rangle \qquad (2.8)$$

$$i\hbar\left[e^{-\frac{iH_0 t}{\hbar}}\left(-\frac{i}{\hbar}\hat{H}_0\right)|\psi\rangle_I + \frac{d|\psi\rangle_I}{dt}\right]$$

$$= \hat{H}e^{-\frac{iH_0 t}{\hbar}}|\psi\rangle_I + e^{-\frac{iH_0 t}{\hbar}}\hat{H}_0|\psi\rangle_I + i\hbar e^{-\frac{iH_0 t}{\hbar}}\frac{d|\psi\rangle_I}{dt}$$

$$= (\hat{H}_0 + \hat{H}_i)e^{-\frac{iH_0 t}{\hbar}}|\psi\rangle_I$$

$$= \hat{H}_0 e^{-\frac{iH_0 t}{\hbar}}|\psi\rangle_I + i\hbar e^{-\frac{iH_0 t}{\hbar}}\frac{d|\psi\rangle_I}{dt} = \hat{H}_0 e^{-\frac{iH_0 t}{\hbar}}|\psi\rangle_I + \hat{H}_i e^{-\frac{i\hat{H}_0}{\hbar}}|\psi\rangle_I \qquad (2.9)$$

Cancelling similar terms and multiplying both sides by $e^{\frac{iH_0 t}{\hbar}}$ gives

$$i\hbar\frac{d|\psi\rangle_I}{dt} = e^{\frac{iH_0 t}{\hbar}}\hat{H}_i e^{-\frac{i\hat{H}_0 t}{\hbar}}|\psi\rangle_I \qquad (2.10)$$

To simplify this equation it is convenient to define operator

$$\hat{H}_I = e^{\frac{iH_0 t}{\hbar}}\hat{H}_i e^{-\frac{i\hat{H}_0 t}{\hbar}} \qquad (2.11)$$

Inserting equation (11) in equation (12) yields

$$i\hbar\frac{d|\psi\rangle_I}{dt} = \hat{H}_I|\psi\rangle_I \qquad (2.12)$$

This is the ordinary Schrodinger equation in the interaction representation.

This equation can also be derived by bearing in mind that the expect value is the same in Schrodinger and interaction picture, i.e.

$$\langle\psi|\hat{H}|\psi\rangle = |\psi\rangle_I H_I \langle\psi|_I \qquad (2.13)$$

In view of equations (5), (7) and (11) one gets

$$\langle\psi|\hat{H}|\psi\rangle = |\psi\rangle_I e^{-\frac{i\hat{H}_{0_0} t}{\hbar}}(\hat{H}_0 + \hat{H}_i)e^{-\frac{i\hat{H}_0 t}{\hbar}}\langle\psi|_I$$

$$= |\psi\rangle_I e^{-\frac{i\hat{H}_0 t}{\hbar}}\hat{H}_0 e^{-\frac{i\hat{H}_0 t}{\hbar}}\langle\psi|_I + \langle\psi|_I e^{-\frac{i\hat{H}_0 t}{\hbar}}\hat{H}_i e^{-\frac{i\hat{H}_0 t}{\hbar}}\langle\psi|_I$$

$$= \langle\psi|_I\hat{H}_0|\psi\rangle_I + |\psi\rangle_I H_I\langle\psi|_I \qquad (2.14)$$

This means that for equations (14) and (13) to be typical to each other, the expectation value in Schrodinger picture. This can be satisfied only when

$$\hat{H}_0\langle\psi|_I = 0$$

Thus

$$\langle\psi|_I\hat{H}_0|\psi\rangle_I = 0 \qquad (2.15)$$

This is consistent with the fact that in the interaction picture the original Hamiltonian is absorbed in the wave vector and disappear as an energy operator according to this transformation.

$$|\psi\rangle \to |\psi\rangle_I = e^{\frac{i\hat{H}_0 t}{\hbar}}|\psi\rangle$$

$$\hat{H} = \hat{H}_0 + \hat{H}_i \to H_I = e^{\frac{i\hat{H}_0 t}{\hbar}} H_i e^{-\frac{i\hat{H}_0 t}{\hbar}} \qquad (2.16)$$

This equivalent to make

$$\hat{H}_0 \to 0 \qquad (2.17)$$

This expression of energy operator. Thus it is quite obvious to have

$$\hat{H}_0 \to 0 \Rightarrow \hat{H}_0|\psi\rangle_I = 0 \qquad (2.18)$$

Momentum Perturbation Equation in the Interaction Picture

The momentum operator is related to the spatial differential change according to the relation

$$\hat{P} = \frac{\hbar}{i}\vec{\nabla} \qquad (3.1)$$

In one dimension

$$\hat{P} = \frac{\hbar}{i}\frac{\partial}{\partial x} \qquad (3.2)$$

To see how the momentum operator look like in a curved space time, one uses the expression for x and t in a curved space time for velocity, i.e.

$$v = \frac{dx_c}{dt_c} = \frac{\sqrt{g_{xx}}dx}{\sqrt{g_{00}}dt} \qquad (3.3)$$

Where

$$dx_c = \sqrt{g_{xx}}dx \qquad dt_c = \sqrt{g_{00}}dt \qquad (3.4)$$

But in solution relative to Special Relativity

$$g_{xx} = g_{00}^{-1} \qquad (3.5)$$

Where

$$dt = \sqrt{g_{00}}dt_0 \qquad dx = \sqrt{g_{xx}}dx_0$$

$$g_{xx} = \gamma^2 = \left(1 - {v^2}/{c^2}\right) \qquad (3.6)$$

Thus equation (3) and (5) gives

$$v = \frac{1}{g_{00}}\frac{dx}{dt} = \frac{1}{g_{00}}v_0 \qquad (3.7)$$

This is the expression for the velocity in a curved space-time.
The momentum in a curved space-time takes the form

$P = mv \qquad (3.8)$

Where the mass is given by

$m = \frac{g_{00}m_0}{\sqrt{g_{00} - v_i^2/c^2}} \qquad (3.9)$

Inserting equations (7) and (9) in (8) yields

$$P = \frac{g_{00}m_0v_0}{\sqrt{g_{00} - v_i^2/c^2}} = P_0\left(g_{00} - v_i^2/c^2\right)^{-\frac{1}{2}} \qquad (3.10)$$

Where the momentum in Euclidean free space is given by

$P_0 = m_0v_0 \qquad (3.11)$

Bearing in mind that for weak field

$$g_{00} = \left(1 + 2\varphi/c^2 - v_i^2/c^2\right)$$

The momentum is given by

$$P = P_0\left(1 + \frac{2\varphi}{c^2} - \frac{v_i^2}{c^2}\right)^{-\frac{1}{2}}$$

This expression relates momentum in a curved space-time to that in Euclidean space.

Since the potential is less than the total energy

$$V_0 < E_0$$
$$m_0\varphi < m_0c^2$$

Therefore

$\frac{\varphi}{c^2} < 1 \qquad (3.12)$

Similarly, the kinetic energy is also less than the total energy. Hence

$$\frac{1}{2}m_0v_i^2 < m_0c^2$$

$v_i^2 < c^2 \qquad (3.13)$

As a result

$$\left(1+\frac{2\varphi}{c^2}-\frac{v_i^2}{c^2}\right)^{-\frac{1}{2}}=\left(1-\frac{2\varphi}{c^2}-\frac{1}{2}\frac{v_i^2}{c^2}\right)$$
$$=\left(1-\frac{\left(m_0\varphi-\frac{1}{2}m_0v_i^2\right)}{m_oc^2}\right)$$
$$=\left(1+\frac{(T_0-V_0)}{E_0}\right)=\left(1+\frac{L_0}{E_0}\right) \qquad (3.14)$$

Where the free space Lagrargian is defined to be

$$L_0=T_0-V_0 \qquad (3.15)$$

Hence, the curved space operator can be written as sum of perturbed and non-perturbed momentum in the form

$$\hat{P}=\hat{P}_0\left(1+\frac{L_0}{E_0}\right)=\hat{P}_0+\hat{P}_i \qquad (3.16)$$

Where the perturbed momentum is given by

$$P_i=\frac{P_0L_0}{E_0} \qquad (3.17)$$

The Schrodinger Hamiltonian is related to the interaction one according to the relation

$$|\psi\rangle=e^{\frac{iP_0x}{\hbar}}|\psi\rangle_I$$
$$\langle\psi|=\langle\psi|_I e^{-\frac{iP_0x}{\hbar}} \qquad (3.18)$$

The spatial evaluation of the system is related to momentum operator according to the relation

$$\frac{\hbar}{i}\frac{d}{dx}|\psi\rangle=\hat{P}|\psi\rangle \qquad (3.19)$$

In view of equation (3.18) and equation (3.19)

$$\frac{\hbar}{i}\frac{d}{dx}e^{\frac{i\hat{P}_0x}{\hbar}}|\psi\rangle_I=\frac{\hbar}{i}\left[\frac{i}{\hbar}\hat{P}_0e^{\frac{iP_0x}{\hbar}}|\psi\rangle_I+e^{\frac{iP_0x}{\hbar}}\frac{d}{dx}|\psi\rangle_I\right]$$
$$=\left(\hat{P}_0+\hat{P}_i\right)e^{\frac{iP_0x}{\hbar}}|\psi\rangle_I\hat{P}_0e^{\frac{i\hat{P}_0x}{\hbar}}|\psi\rangle_I$$
$$+\frac{\hbar}{i}e^{\frac{i\hat{P}_0x}{\hbar}}\frac{d}{dx}|\psi\rangle_I$$
$$=\hat{P}_0e^{\frac{i\hat{P}_0x}{\hbar}}|\psi\rangle_I+\hat{P}_ie^{\frac{i\hat{P}_0x}{\hbar}}|\psi\rangle_I$$

Multiply both sides by $e^{\frac{i\hat{P}_0x}{\hbar}}$, one gets

$$\frac{\hbar}{i}\frac{\partial|\psi\rangle_I}{\partial x}=e^{-\frac{i\hat{P}_0x}{\hbar}}P_ie^{\frac{i\hat{P}_0x}{\hbar}}|\psi\rangle_I$$
$$\frac{\hbar}{i}\frac{d|\psi\rangle_I}{dx}=\hat{P}_I|\psi\rangle_I \qquad (3.20)$$

Where

$P_I = e^{-\frac{i\hat{P}_0 x}{\hbar}} P_i e^{\frac{i\hat{P}_0 x}{\hbar}}$ (3.21)

The mathematical form of equation (21) can also be found using the fact that the expectation values are the same in all representations.

Thus

$\langle\psi|\hat{P}|\psi\rangle_I = \langle\psi|_I \hat{P}_I |\psi\rangle_I$ (3.22)

With the aid of equations (3.16), (3.21) and (3.22), one gets

$$\langle\psi|_I e^{-\frac{i\hat{P}_0 x}{\hbar}}(\hat{P}_0 + \hat{P}_i)e^{\frac{i\hat{P}_0 x}{\hbar}}|\psi\rangle_I$$
$$= \langle\psi|_I e^{-\frac{i\hat{P}_0 x}{\hbar}}\hat{P}_0 e^{\frac{i\hat{P}_0 x}{\hbar}}|\psi\rangle_I$$
$$+ \langle\psi|_I e^{-\frac{i\hat{P}_0 x}{\hbar}} P_i e^{\frac{i\hat{P}_0 x}{\hbar}}|\psi\rangle_I$$

$= \langle\psi|_I \hat{P}_0|\psi\rangle_I + \langle\psi|_I \hat{P}_I|\psi\rangle_I$ (3.23)

Equation (3.23) should be typical to (3.22). This requires

$\langle\psi|_I \hat{P}_0|\psi\rangle_I = 0$ (3.24)

One can prove this by bearing that in the interaction picture

$$|\psi\rangle \rightarrow |\psi\rangle_I = e^{-\frac{i\hat{P}_0 x}{\hbar}}|\psi\rangle$$

$\hat{P} = \hat{P}_0 + \hat{P}_i \rightarrow P_I = e^{-\frac{i\hat{P}_0 x}{\hbar}} P_i e^{\frac{i\hat{P}_0 x}{\hbar}}$ (3.25)

In view of equations (3.20) and (3.16) it is clear that $\hat{P}_0$ gives no contribution to the equation of motion. Thus as if

$\hat{P}_0 \rightarrow 0$ (3.27)

Thus

$\hat{P}_0|\psi\rangle_I = 0|\psi\rangle_I = 0$ (3.28)

Hence equations (3.23) becomes

$\langle\psi|\hat{P}|\psi\rangle = \langle\psi|_I \hat{P}_I|\psi\rangle_I$ (3.29)

Which is typical to equation (3.22)

Spatial Evolution of Unitary Operator

The spatial evolution of the wave function in the wave vector space takes the form

$\frac{\hbar}{i}\frac{d}{dx}|\psi\rangle_I = \hat{P}_I|\psi\rangle_I$ (4.1)

The unitary operator $\mathcal{U}$ can be defined to be

$|\psi\rangle_I = \hat{\mathcal{U}}|\psi\rangle_0$ (4.2)

Where the stationary wave vector is defined to satisfy

$$x = x_0 = 0$$

$P_I = 0$ (4.3)

$|\psi(x)\rangle_I = |\psi\rangle_I = |\psi(x=0)\rangle_I = |\psi\rangle_0$ (4.4)

$|\psi(t=0)\rangle_I = |\psi\rangle_0 = \hat{\mathcal{U}}(0)\,|\psi_0\rangle$ (4.5)

Hence

$Ц_0 = Ц(0) = \mathrm{I}$ (4.6)

But since at

$$x = x_o = 0$$

$P_I = 0$ (4.7)

It is follow that

$\frac{\hbar}{i}\frac{d|\psi_0\rangle}{dx} = \frac{\hbar}{i}\frac{d|\psi\rangle_0}{dx} = 0|\psi\rangle_0$ (4.8)

Thus

$|\psi\rangle_0 = constant$ (4.9)

Inserting (4.2) and (4.9) in (4.1) gives

$\frac{\hbar}{i}\frac{d}{dx}\, Ц(x)|\psi_0\rangle = P_I\, Ц(x)|\psi_0\rangle$ (4.10)

Therefore

$\frac{\hbar}{i}\frac{d}{dx}\, Ц = P_I Ц$ (4.11)

This using iterated integral method and approximation, the zeroth, first, second orders of $Ц$ are given by

$$\int_{x_0}^{x_1} dЦ = \frac{i}{\hbar}\int_{x_0}^{x_1} P_I\, Ц\, dx$$

Where

$Ц(x_1) - Ц(x_0) = \frac{i}{\hbar}\int_{x_0}^{x_1} P_I(x)\, Ц(x)\, dx$ (4.12)

When

$x_1 > x_0 x_1 \approx x_0$ (4.13)

$Ц(x_1) = Ц(x_0) + \frac{i}{\hbar}\int_{x_0}^{x_1} P_I(x_0)\, Ц(x_0) dx_0 = Ц(x_0) + I$ (4.14)

Similarly

$$\int_{x_1}^{x_2} dЦ = \frac{i}{\hbar}\int_{x_1}^{x_2} P_I\, Ц\, dx \qquad (4.15)$$

$Ц(x_2) - Ц(x_1) = \frac{i}{\hbar}\int_{x_1}^{x_2} P_I(x)\, Ц(x)\, dx$ (4.16)

When

$x_2 > x_1 x_2 \approx x_1$ (4.17)

$$Ц(x_2) = Ц(x_1) + \frac{i}{\hbar}\int P_I(x_1)[Ц(x_0) + I_0]\, dx_1 = Ц(x_1)$$

$Ц(x_2) = Ц(x_0) + \mathrm{I}_0 + \frac{i}{\hbar}\int_{x_1}^{x_2} P_I(x_1)\, [Ц(x_0 + \mathrm{I}_0)] dx_1$ (4.18)

Discussion

The energy expression (2.1) in a curved space - time within the frame work of GSR is utilized to get a useful expression for the Hamiltonian (2.4). Here one assumes that the velocity is less than the speed of light. Both of Schrodinger equation and expectation Values of the Hamiltonian in Schrodinger and interaction picture are used in deriving the expression of the interaction Hamiltonian in the interaction picture (see (2.11) & (2.16)).These two expressions are typically to each other only when the unperturbed Hamiltonian gives no contribution to the energy in the interaction picture. This is in agreement with the fact that the Hamiltonian in the interaction picture is only that which causes perturbation.

Spatial evolution of the quantum system in the interaction picture is also derived using the expression of the momentum in a curved space - time within the frame work of the GSR. Here one assumes that the velocity is less than the speed of light and the potential is also less than the rest mass energy. The perturbed momentum is found to be proportional to the Lagrargian of the system thus also to the perturbation energy (see (3.3)). Fortunately this new expression resembles that of the Hamiltonian, where the time differential is replaced by the space one and the Hamiltonian is replaced by the momentum (see(3.20)). The expression of the momentum using the Schrodinger equation is typical to the one found by equating the expectation values in the interaction picture and Schrodinger picture as shown in equations (3.25)& (3.21). This requires that the unperturbed momentum to give no contribution in the interaction picture as shown in equation (3.24). Finally the spatial evolution of the unitary operator is derived using the momentum operator. It is very interesting to note that this spatial evolution resembles that of time but here one replaces time integral by spatial one, and the Hamiltonian by the momentum operator.

Conclusion

A useful expression of the spatial evolution of the quantum system in the interaction picture is derived. This expression is found to be typical to the Hamiltonian one, when one replaces the time differential with the spatial one. Another expression of the spatial evolution of the unitary operator is also found to be typical to that of the Hamiltonian one. Here one replaces the time integral with the spatial one, and the Hamiltonian with the momentum.

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