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Numerical Solution of 2D Inhomogeneous Helmholtz Equation using the Meshless Radial Basis Function Method

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Abstract

The meshless method based on the radial basis function (RBF) is used in this study to solve the inhomogeneous elliptic two-dimensional Helmholtz equation. The radial basis function and the method of approximate particular solution (MAPS) approach are used to solve the inhomogeneous Helmholtz problem. The technique converts the problem into a set of equations using an RBF-MAPS, which is then solved using a system solver. Two numerical examples are used to evaluate the method's effectiveness and utility. The numerical outcomes closely match the exact results found in the literature.

Keywords: *MAPS, Multi Quadratic, Helmholtz equation, Radial Basis Function, Shape parameter.*

1. Introduction

The well-known inhomogeneous two-dimensional Helmholtz's equation is of the form

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \lambda u = -f(x,y) \qquad (1.1)$$

Numerous problems with steady state oscillations in magnetic theory, mechanical, acoustic, and thermal lead to the two-dimensional Helmholtz equation. The mass transfer process with volume chemical reaction of the first order for $\lambda<0$ is described by equation (1.1). Additionally, the Helmholtz equation can be obtained from any elliptic equation with a constant coefficient[1]. There are numerous applications of the Helmholtz PDE, and there are numerous boundary geometries and boundary conditions for which there are solutions to be found[2]. One of three classifications-Direct, Eigenvalue, or Inverse-can be applied to Helmholtz boundary value problems. While eigenvalue problems involve determining the system's natural frequencies, direct problems call for the solution of the Helmholtz PDE for predetermined boundary conditions. In contrast, the inverse problem's solution is provided; the task is to find appropriate boundaries that satisfy it[2]. In this paper, the proposed method is used to solve the Direct problem with Dirichlet boundary conditions. The Helmholtz problem has been tackled analytically and numerically by numerous scholars in the past. Analytical solutions are restricted to domains having a specific shape, like a circle or rectangle[3]. Additionally, only a relatively small class of differential equations with forcing terms have analytical solutions available. The approximate solutions must therefore be obtained using numerical techniques. The Helmholtz equation can be solved numerically in several ways, some of which are listed below. Finite Volume method[4], Differential Transform method[5], Finite difference method[6], High-order Nystrom Discretization[7]. Most of these methods require the generation of meshes.

Many Researchers have tried to use meshless methods to solve Helmholtz problem. To note a few Semi-Analytical Meshless Method [8], Meshless thin plate spline methods[9], The method of approximate particular solutions for solving elliptic problems[10], Meshless RBF for waveguide problem.

Many researchers have lately used the Kansa approach, one of the meshless methods established by Kansa [12, 13] in 1990, to approximate the solutions of Nonlinear partial differential equations. It is obtained by collocating the RBFs, notably the multiquadric RBFs, for the numerical approximation of the solution (MQ). RBF collocation methods are mathematically straightforward and meshless in comparison to conventional mesh-based procedures. In this paper, a numerical approach using the method of Approximate Particular solutions and the radial basis function is used for solving the Helmholtz problem.

The structure of the paper is as follows: The radial basis functions approximation method is presented in Section 2. In Section 3, the approach is applied to the Helmholtz equation using the method of approximate particular solutions. In Section 4, the numerical experiments are provided. Finally, Section 5 summarizes a quick conclusion.

2. Radial Basis Function approximation

A Radial Basis Function (RBF) is a real-valued function whose value equals the distance between a Centre & an input, therefore $\emptyset$ (x) $= \widehat{\emptyset}(\|x\|)$. If c is a fixed point, often known as a Centre, then $\emptyset$ (x) $= \widehat{\emptyset}(\|x-c\|)$ [15].

The distance is normally measured in Euclidean distance, however alternative metrics are occasionally employed. They are frequently utilized as a collocation $\{\varphi_k\}$ that serves as the foundation for a function space of interest. RBF sums are frequently used to approximate a function. RBFs that are regularly utilized [16] are shown in Table 1.

Table 1: Frequently used Radial Basis Functions

Type of RBF	$\emptyset(r)$
Multiquadratic	$\sqrt{1+\epsilon^2 r^2}$
Inverse Quadratic	$1/(1+\epsilon^2 r^2)$
Inverse Multiquadratic	$1/\sqrt{1+\epsilon^2 r^2}$
Gaussian	$e^{-\epsilon^2 r^2}$
Spline (Polyharmonic)	$r^k, k = 1,3,5,----$
	$r^k \log(r), k = 2,4,6,----$
Spline (Thin plate)	$r^2 \log(r)$

Where $r = \|x - x_j\|$ *and* ϵ is a shape parameter for scaling the radial kernel's input. As a result, RBFs are frequently employed to create function approximations of the form

$$u(x) = \sum_{i=1}^{n} w_i \, \emptyset(\|x - x_i\|) \tag{2.1}$$

where the total of n RBFs, each with a different Center x_i and weighted by an estimated coefficient w_i, is used to represent the approximation function u(x). The matrix method of linear least squares can be used to find the weights w_i.

3. Meshless method for the Helmholtz equation

To solve Helmholtz equation

$$\Delta u + \lambda^2 u = -f(x,y); (x,y) \in \Gamma \tag{3.1}$$

$$\boldsymbol{B}u = g(x,y); \ (x,y) \in \partial\Gamma \tag{3.2}$$

Where $\boldsymbol{B}$ is a boundary differential operator.

Rewriting equation (3.1)

$$\Delta u = h(x,y,u);\ (x,y) \in \Gamma \tag{3.3}$$

where $h(x,y,u) = -\lambda^2 u - f(x,y)$ (3.4)

The RBFs approximation of h can be written as

$$h(x,y,u) = \sum_{i=1}^{n} w_i \varphi(r_i) \tag{3.5}$$

where $r_i = \|(x,y) - (x_i, y_i)\|$.

An RBF approximate particular solution of equation (3.1) can be written as

$$\breve{u}_p(x) = \sum_{i=1}^{n} w_i \boldsymbol{\Phi}(r_i) \tag{3.6}$$

where $\boldsymbol{\Phi}$ is obtained by solving analytically $\Delta\boldsymbol{\Phi} = \varphi$ (3.7)

$\boldsymbol{\Phi}$ in equation (3.7) can be obtained by repeated integration of φ [14]

The commonly used RBFs φ and their corresponding particular solutions $\boldsymbol{\Phi}$ [10] are shown in Table 2.

Table 2: Radial Basis Functions and their corresponding particular solutions

RBF Type	$\varphi(r)$	$\Phi(r)$
Multi Quadratic	$\sqrt{r^2+c^2}$	$\frac{1}{9}(4c^2+r^2)\sqrt{r^2+c^2} - \frac{c^3}{3}\ln\left(c + \sqrt{r^2+c^2}\right)$
Inverse Multi Quadratic	$\frac{1}{\sqrt{r^2+c^2}}$	$\sqrt{r^2+c^2} - c\ln\left(c+\sqrt{r^2+c^2}\right) - c\ ln\ 2c$
Polyharmonic Spline	$r^{2m}\ln(r)$	$\frac{r^{2m+2}\ln(r)}{4(m+1)^2} - \frac{r^{2m+2}}{4(m+1)^3}$
Polyharmonic Spline	r^{2m-1}	$\frac{r^{2m+1}}{(2m+1)^2}$

Thus, the solutions of equations (3.1) and (3.2) can be approximated as

$$u(x,y) = \breve{u}(x,y) = \sum_{i=1}^{n} w_i \boldsymbol{\Phi}(r_i) \tag{3.8}$$

$$\therefore \Delta u(x,y) = \Delta\breve{u}(x,y) = \sum_{i=1}^{n} w_i \Delta\boldsymbol{\Phi}(r_i)$$

i.e. $\Delta u(x,y) = \Delta\breve{u}(x,y) = \sum_{i=1}^{n} w_i \varphi(r_i)$ (3.9)

using equation (3.7).

From equations (3.5) & (3.9) we have

$$\sum_{i=1}^{n} w_i \varphi(r_i) = -\lambda^2 \breve{u} - f(x,y);\ for\ (x,y) \in \Gamma \tag{3.10}$$

Using (3.8), equation (3.10) can be rewritten as

$$\sum_{i=1}^{n} w_i \left(\varphi(r_i) + \lambda^2 \boldsymbol{\Phi}(r_i)\right) = -f(x,y);\ \text{for}\ (x,y) \in \Gamma. \tag{3.11}$$

Now, boundary conditions (3.2) can be written as

$$\sum_{i=1}^{n} w_i \boldsymbol{B\Phi}(r_i) = g(x,y);\ (x,y) \in \partial\Gamma \tag{3.12}$$

We select two sets of collocation points for numerical implementation as shown in the figure 1.

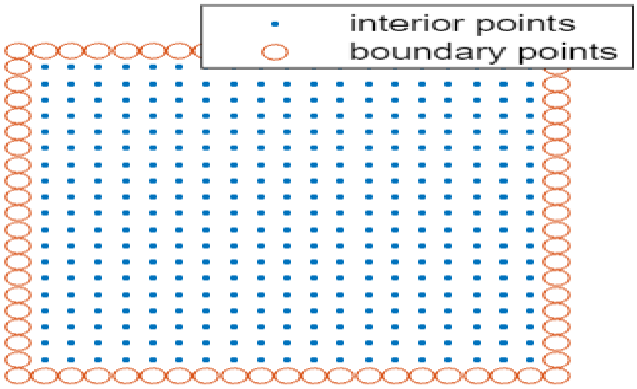

Fig 1: Nodes distribution in the domain

Let n_i be the number of interior points $(\cdot)$, $\{(x_i,\ y_i)\}_1^{n_i}$ and n_b the number of boundaries points (o), $\{(x_i,\ y_i)\}_{n_i+1}^{n_i+n_b}$.

Let $n = n_i + n_b$

By collocation method, from equations (3.11) & (3.12), we have

$$\sum_{i=1}^{n} w_i \psi(r_{ij}) = f(x_j,\ y_j); 1 \le j \le n_i \tag{3.13}$$

$$\sum_{i=1}^{n} w_i \boldsymbol{B\Phi}(r_{ij}) = g(x_j,\ y_j); n_{i+1} \le j \le n \tag{3.14}$$

where

$$\psi(r_{ij}) = \varphi(r_{ij}) + \lambda^2 \boldsymbol{\Phi}(r_{ij}) \tag{3.15}$$

and

$$r_{ij} = \|(x_j,\ y_j) - (x_i,\ y_i)\|.$$

Any standard matrix solver can be used to solve the above system. After determining $\{w_i\}_1^n$, the approximate particular solution in equation (3.8) becomes the approximate solution $\breve{u}$ of equations (3.1) & (3.2)

i.e. $\breve{u}(x,y) = \sum_{i=1}^{n} w_i \boldsymbol{\Phi}(r_i)$ (3.16)

4. Numerical Examples and comparisons

The numerical results of the suggested approach of solving the Helmholtz equation are shown in this section. We use two different problems to demonstrate the correctness and adaptability of the suggested technique. We use numerical errors, as stated by

$$\mathrm{RMSE} = \sqrt{\frac{1}{n_t}\sum_{j=1}^{n_t}(\breve{u}_j - u_j)^2} \tag{4.1}$$

Where $\tilde{u}$ is approximate solution of u and u the exact.

Example 1: We test the proposed method by solving the 2-D non-homogeneous Helmholtz equation with Dirichlet Boundary conditions

$$\Delta u(x,y) - \lambda^2 u(x,y) = (\sinh(x) + \cosh(y))(1-\lambda^2); (x,y) \in \Gamma$$

$u(x,y) = \sinh(x) + \cosh(y)\,; \in \partial\Gamma$

The analytical solution is $u(x,y) = \sinh(x) + \cosh(y)$

where $\cup\ \partial\Gamma$: Unit square

Solving the equation numerically by the proposed method, the various errors for different values of shape parameter are shown in the Table 3 proving the proposed method's excellent accuracy. The solution profile is as shown in figure 2.

Table 3: RMSE Vs Shape Parameter Example 1

Interior points	Boundary points	Shape parameter	RMSE_in	RMSE_edge
361	80	0.49	3.26E-07	7.43E-07
289	68	0.59	3.07E-07	8.49E-07
196	40	0.69	4.38E-07	2.04E-06

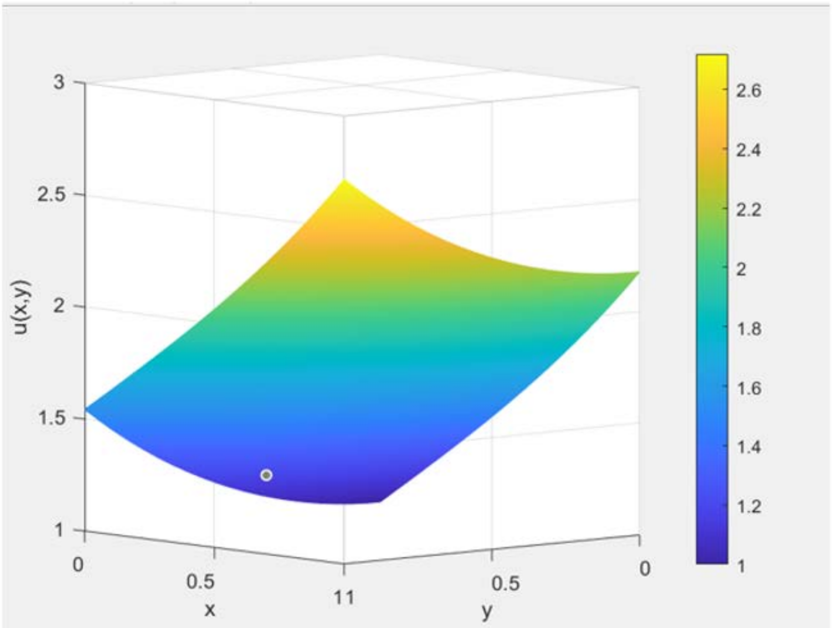

Fig 2: Exact Solution Profile Example-1

Example 2 Now, we test the proposed method by solving the 2-D non-homogeneous Helmholtz equation with Dirichlet Boundary conditions

$$(\Delta - 900)u(x,y) = -899(e^x + e^y); (x,y) \in \Gamma$$

$u(x,y) = (e^x + e^y); \in \partial\Gamma$

The analytical solution is $u(x,y) = (e^x + e^y)$

where ∪ $\partial\Gamma$: Unit square

Solving the equation numerically by the proposed method, the various errors for different values of shape parameter are shown in the Table 4 proving the proposed method's excellent accuracy. The solution profile is as shown in figure 3.

Table 4: RMSE Vs Shape Parameter Example 2

Interior points	Boundary points	Shape parameter	RMSE_in	RMSE_edge
361	80	1	1.79E-05	1.35E-05
289	68	2	1.86E-05	3.15E-05
196	40	3	5.33E-05	1.67E-05

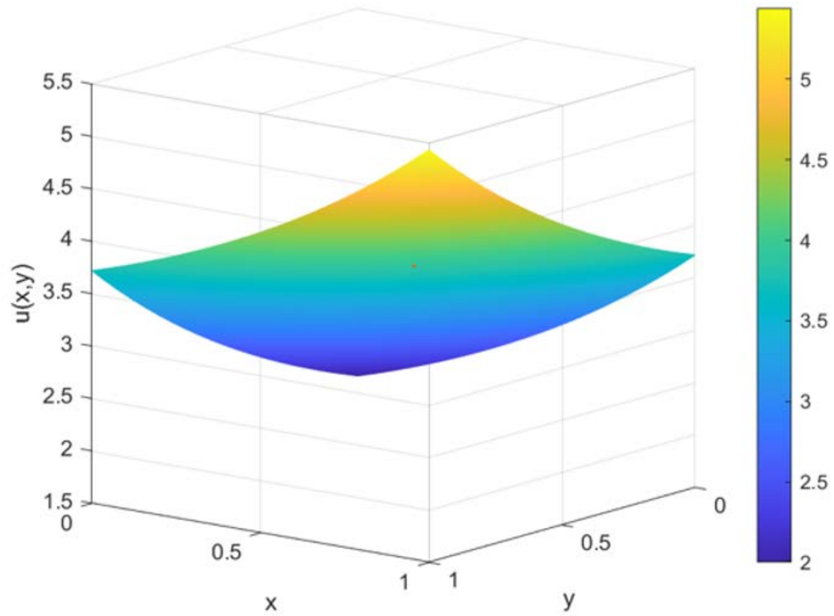

Fig 3: Exact Solution Profile Example-2

5. Conclusion

This paper discusses the approximate solution to the Helmholtz problem. The Helmholtz problem was solved using the meshless numerical approach based on the radial basis function and the method of approximate particular solutions. The multiquadric (MQ) and its Particular solution is used to convert the problem into a system of equations. To demonstrate the accuracy and adaptability of this strategy, two different problems were addressed in the numerical trials. Given its excellent performance, the suggested approach might be used to more inhomogeneous elliptical partial differential equations.

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