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A Convolution Theorem for the Linear Canonical-Mellin Transform and Its Properties

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Abstract: The Linear canonical-Mellin transform is a mixed type of integral transform in which function is both linear canonical and Mellin transformable. A convolution theorem for linear canonical transform and Mellin transform is already known. This theorem is found to be useful in sampling of band limited signals in the linear canonical transform domain and filtering in Mellin transform domain. In this paper, we derive convolution theorem for the Linear Canonical-Mellin transform and its properties such as commutative, associative etc.

Keywords: Convolution theorem, Linear canonical transform, Mellin transform, testing function space.

1. Introduction

Linear canonical transform is a four-parameter integral transform and Fourier transform, Fractional Fourier transform and Fresnel transform are special cases of linear canonical transform. The linear canonical transform (LCT) is defined as [1]:

$$F_{(a,b,c,d)}(u) = L_{(a,b,c,d)}\big(f(t)\big)$$

$$= \begin{cases} \sqrt{\frac{1}{j2\pi b}}\, e^{j\frac{d}{2b}u^2} \int_{-\infty}^{+\infty} e^{-j\frac{1}{b}ut}\, e^{j\frac{a}{2b}t^2} f(t)dt, & b \neq 0 \\ \sqrt{d}\, e^{j\frac{cd}{2}u^2} f(du), & b = 0 \end{cases} \qquad ad - bc = 1.$$

Deng Bing et. al. [2] introduced the definition and properties of LCT briefly and analyzed the relationship between LCT and the time-frequency distributions and derived convolution theorems for the LCT according to the additivity and reversibility properties given by:

$L_{(a_2,\,b_2,\,c_2,\,d_2)}\left(L_{(a_1,\,b_1,\,c_1,\,d_1)}f(t)\right) = L_{(e,f,g,h)}(f(t))$ where $\begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}.\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$

and $L_{(d,-b,-c,a)}\left(L_{(a,b,c,d)}(f(t))\right) = f(t)$.

Deyun Wei et al. [3] introduced a new convolution structure for the LCT, which is expressed by a simple integral easy to implement in filter design and obtained a one-dimensional integral expression of the correlation for LCT based on the new convolution structure. As an application of new convolution theorem, the sampling theorem for the band limited signal in the LCT domain can be studied. Deyun Wei et al. [4] proposed that, based on generalized translation, convolution formula of two functions generalize very nicely the classical results for Fourier transform and concluded that this theorem states the powerful result that the convolution of two signals in time domain results in simple multiplication of their LCTs in the LCT domain. Navdeep Goel et al. [5] presented a modified convolution theorem in the LCT domain derived by a representation transformation in quantum mechanics and it is given by:

$$L_{(a,b,c,d)}(h(x)) = \sqrt{j2\pi b}\, e^{\frac{-j}{2}\left(\frac{d}{b}u^2\right)} F_{(a,b,c,d)}(u) G_{(a,b,c,d)}(u)$$

,

where $h(x) = \int_{-\infty}^{\infty} f(y)g(x-y)\, e^{-j\frac{a}{b}y(t-y)}dy$ is the weighted convolution.

The Mellin transform is defined as [6]:

$M[f;s] \equiv F(s) = \int_0^{\infty} f(x)\, x^{s-1}dx$

and its convolution theorem is given by $M(f * g) = M\left(\int_0^{\infty} f(y)\, g\left(\frac{x}{y}\right)\frac{dy}{y}\right) = F(s)G(s)$.

Zemanian [7] studied several integral transforms in the distributional generalized sense. The convolution structure of two-dimensional Fourier-Mellin transform is proposed by [8]. The aim this paper is to establish convolution theorem of linear canonical-Mellin transform and prove its commutative and associative property.

2. The Linear Canonical-Mellin Transform

2.1 Definition:

The conventional Linear Canonical-Mellin transform is defined as follows:

$$L_A M\{f(t,x)\} = F^A M(u,s) = \int_{-\infty}^{\infty}\int_0^{\infty} f(t,x)\, K(t,x,u,s)dtdx$$

where $K(t,x,u,s) = \sqrt{\frac{1}{2j\pi b}}\, e^{\frac{j}{2}\left[\frac{a}{b}t^2 - \left(\frac{2}{b}\right)tu + \frac{d}{b}u^2\right]} x^{s-1}$, $b \neq 0, s > 0$.

2.2 The Testing Function Space $E(R^n)$

An infinitely differentiable complex valued smooth function φ on R^n belongs to $E(R^n)$, if for each compact set $K \subset S_a$, $I \subset S_b$,

where $S_a = \{t: t \in R^n, |t| \leq a, a > 0\}$ and $S_b = \{x: x \in R^n, |x| \leq b, b > 0\}$, $K, I \in R^n$,

$$\gamma_{E,l,q} = \sup_{\substack{t\in K \\ x\in I}} \left| D_t^l D_x^q \varphi(t,x) \right| < \infty , \qquad l, q = 0,1,2, - - -$$

Thus $E(R^n)$ will denote the space of all $\varphi \in E(R^n)$ with support contained in S_a and S_b. Moreover, we say that f is a linear canonical-Mellin transformable if it is a member of E^*, the dual space of E.

2.3 Distributional Generalized Linear Canonical-Mellin Transform

The distributional Linear Canonical-Mellin transform of $f(t,x) \in E^*(R^n)$ is defined by

$$L_A M\{f(t,x)\} = F^A M(u,s) = \langle f(t,x), K(t,x,u,s) \rangle$$

where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $ad - bc = 1$ and

$$K(t,x,u,s) = \sqrt{\frac{1}{2j\pi b}}\, e^{\frac{j}{2}\left[\frac{a}{b}t^2 - \left(\frac{2}{b}\right)tu + \frac{d}{b}u^2\right]} x^{s-1},\ b \neq 0, s > 0.$$

The right-hand side of (1) is meaningful because $K(t,x,u,s) \in E$ and $f(t,x) \in E^*$.

3. Convolution Theorem

In this section, we first give the convolution theorem for the linear canonical-Mellin transform and then its commutative ad associative properties.

Theorem: If $F^A M(u,s)$ and $G^A M(u,s)$ are the Linear Canonical-Mellin transforms of $f(t,x)$ and $g(t,x)$ respectively, then

$\sqrt{2j\pi b}\, e^{\frac{-j}{2}\left(\frac{d}{b}u^2\right)} F^A M(u,s) G^A M(u,s)$ is a Linear Canonical-Mellin transforms of $h(t,x)$, i.e.

$$L_A M\{h(t,x)\} = \sqrt{2j\pi b}\, e^{\frac{-j}{2}\left(\frac{d}{b}u^2\right)} F^A M(u,s) G^A M(u,s)$$

where $h(t,x) = f(t,x) * g(t,x)$

$$= \int_{-\infty}^{\infty} \int_0^{\infty} f(y,z)\, g\left(t-y, \frac{x}{z}\right) e^{-j\frac{a}{b}y(t-y)} dy \frac{dz}{z}$$

Proof: By definition, we have

$$L_A M\{h(t,x)\} = H^A M(u,s)$$

$$= \sqrt{\frac{1}{2j\pi b}} \int_{-\infty}^{\infty} \int_{0}^{\infty} h(t,x)\, e^{\frac{j}{2}\left(\frac{a}{b}t^2 - \frac{2}{b}tu + \frac{d}{b}u^2\right)} x^{s-1} dt dx$$

(3.1)

and $h(t,x) = f(t,x) * g(t,x)$

$$= \int_{-\infty}^{\infty} \int_{0}^{\infty} f(y,z)\, g\left(t-y, \frac{x}{z}\right) e^{-j\frac{a}{b}y(t-y)} dy \frac{dz}{z}$$

(3.2)

Substituting eq^n. (3.2) in eq^n. (3.1), we get

$$L_A M\{h(t,x)\}$$

$$= \sqrt{\frac{1}{2j\pi b}} \int_{-\infty}^{\infty} \int_{0}^{\infty} e^{\frac{j}{2}\left(\frac{a}{b}t^2 - \frac{2}{b}tu + \frac{d}{b}u^2\right)} x^{s-1} dt dx \left[\int_{-\infty}^{\infty} \int_{0}^{\infty} f(y,z)\, g\left(t-y, \frac{x}{z}\right) e^{-j\frac{a}{b}y(t-y)} dy \frac{dz}{z}\right]$$

$$= \sqrt{\frac{1}{2j\pi b}} \int_{-\infty}^{\infty} \int_{0}^{\infty} f(y,z)\, dy \frac{dz}{z} \left[\int_{-\infty}^{\infty} \int_{0}^{\infty} g\left(t-y, \frac{x}{z}\right) e^{-j\frac{a}{b}y(t-y)} e^{\frac{j}{2}\left(\frac{a}{b}t^2 - \frac{2}{b}tu + \frac{d}{b}u^2\right)} x^{s-1} dt dx\right]$$

(3.3)

Substituting $t - y = v$ and $\frac{x}{z} = w \;\Rightarrow t = v + y$ and $x = zw \Rightarrow dt = dv$ and $dx = z dw$

Therefore, eq^n. (3) becomes

$$L_A M\{h(t,x)\}$$

$$= \sqrt{\frac{1}{2j\pi b}} \int_{-\infty}^{\infty} \int_{0}^{\infty} f(y,z)\, dy \frac{dz}{z} \left\{\int_{-\infty}^{\infty} \int_{0}^{\infty} g(v,w)\, e^{-j\frac{a}{b}vy} e^{\frac{j}{2}\left[\frac{a}{b}(v+y)^2 - \frac{2}{b}(v+y)u + \frac{d}{b}u^2\right]} (zw)^{s-1} z\, dv dw\right\}$$

$$= \sqrt{\frac{1}{2j\pi b}} \int_{-\infty}^{\infty} \int_{0}^{\infty} z^{s-1} f(y,z)\, dy dz \left\{\int_{-\infty}^{\infty} \int_{0}^{\infty} g(v,w)\, e^{-j\frac{a}{b}vy} e^{\frac{j}{2}\left[\frac{a}{b}(v+y)^2 - \frac{2}{b}(v+y)u + \frac{d}{b}u^2\right]} w^{s-1} dv dw\right\}$$

(3.4)

Now, $e^{\frac{j}{2}\left[\frac{a}{b}(v+y)^2 - \frac{2}{b}(v+y)u + \frac{d}{b}u^2\right]}$

$$= e^{\frac{j}{2}\left[\frac{a}{b}(v^2 + 2vy + y^2) - \frac{2}{b}vu - \frac{2}{b}yu + \frac{d}{b}u^2\right]}$$

$$= e^{\frac{j}{2}\left(\frac{a}{b}v^2-\frac{2}{b}vu+\frac{d}{b}u^2\right)}e^{\frac{j}{2}\left(\frac{a}{b}y^2-\frac{2}{b}yu\right)}e^{j\frac{a}{b}vy}$$

$$= e^{j\frac{a}{b}vy}e^{\frac{j}{2}\left(\frac{a}{b}v^2-\frac{2}{b}vu+\frac{d}{b}u^2\right)}e^{\frac{j}{2}\left(\frac{a}{b}y^2-\frac{2}{b}yu+\frac{d}{b}u^2\right)}e^{\frac{-j}{2}\left(\frac{d}{b}u^2\right)} \quad (3.5)$$

Substituting (3.5) in (3.4), we get

$$L_AM\{h(t,x)\} = \sqrt{\frac{1}{2j\pi b}}\int_{-\infty}^{\infty}\int_{0}^{\infty} z^{s-1}f(y,z)\,dydz \times \left\{\int_{-\infty}^{\infty}\int_{0}^{\infty} g(v,w)\,e^{\frac{j}{2}\left(\frac{a}{b}v^2-\frac{2}{b}vu+\frac{d}{b}u^2\right)}e^{\frac{j}{2}\left(\frac{a}{b}y^2-\frac{2}{b}yu+\frac{d}{b}u^2\right)}e^{\frac{-j}{2}\left(\frac{d}{b}u^2\right)}w^{s-1}dvdw\right\}$$

$$= \sqrt{2j\pi b}.\,e^{\frac{-j}{2}\left(\frac{d}{b}u^2\right)}\left\{\sqrt{\frac{1}{2j\pi b}}\int_{-\infty}^{\infty}\int_{0}^{\infty} f(y,z)\,e^{\frac{j}{2}\left(\frac{a}{b}y^2-\frac{2}{b}yu+\frac{d}{b}u^2\right)}z^{s-1}dydz\right\} \times \left\{\sqrt{\frac{1}{2j\pi b}}\int_{-\infty}^{\infty}\int_{0}^{\infty} g(v,w)\,e^{\frac{j}{2}\left(\frac{a}{b}v^2-\frac{2}{b}vu+\frac{d}{b}u^2\right)}w^{s-1}dvdw\right\}$$

$$= \sqrt{2j\pi b}.\,e^{\frac{-j}{2}\left(\frac{d}{b}u^2\right)}F^AM(u,s)G^AM(u,s)$$

3.1 Commutative Property:

If $L_AM\{f(t,x)\} = F^AM(u,s)$ and $L_AM\{g(t,x)\} = G^AM(u,s)$ then as per the commutative property:

$$L_AM\{(f*g)(t,x)\} = \sqrt{2j\pi b}e^{\frac{-j}{2}\left(\frac{d}{b}u^2\right)}F^AM(u,s)G^AM(u,s) \quad \text{and} \quad (3.1.1)$$

$$L_AM\{(g*f)(t,x)\} = \sqrt{2j\pi b}e^{\frac{-j}{2}\left(\frac{d}{b}u^2\right)}G^AM(u,s)F^AM(u,s) \quad (3.1.2)$$

i.e., $(f*g)(t,x) = (g*f)(t,x)$

Proof: Consider,

$$s(t,x) = (g*f)(t,x) = \int_{-\infty}^{\infty}\int_{0}^{\infty} g(y,z)\,f\left(t-y,\frac{x}{z}\right)e^{-j\frac{a}{b}y(t-y)}dy\,\frac{dz}{z}$$

By definition, we have

$$L_AM\{s(t,x)\}$$

$$= \sqrt{\frac{1}{2j\pi b}}\int_{-\infty}^{\infty}\int_{0}^{\infty}\left[\int_{-\infty}^{\infty}\int_{0}^{\infty} g(y,z)\,f\left(t-y,\frac{x}{z}\right)e^{-j\frac{a}{b}y(t-y)}dy\,\frac{dz}{z}\right]e^{\frac{j}{2}\left(\frac{a}{b}t^2-\frac{2}{b}tu+\frac{d}{b}u^2\right)}x^{s-1}dtdx \quad (3.1.3)$$

Substituting $t-y=v\,,\frac{x}{z}=w\Rightarrow t=v+y,x=zw\Rightarrow dt=dv,dx=zdw$

Therefore, equation (3.1.3) becomes

$$L_AM\{s(t,x)\}=$$
$$\sqrt{\frac{1}{2j\pi b}}\int_{-\infty}^{\infty}\int_{0}^{\infty}z^{s-1}\,g(y,z)dydz\left[\int_{-\infty}^{\infty}\int_{0}^{\infty}f(v,w)e^{-j\frac{a}{b}vy}e^{\frac{j}{2}\left[\frac{a}{b}(v+y)^2-\frac{2}{b}(v+y)u+\frac{d}{b}u^2\right]}w^{s-1}dvdw\right]$$

By rearranging, we obtain

$$L_AM\{s(t,x)\}=\sqrt{2j\pi b}e^{\frac{-j}{2}\left(\frac{d}{b}u^2\right)}\left(\sqrt{\frac{1}{2j\pi b}}\int_{-\infty}^{\infty}\int_{0}^{\infty}g(y,z)e^{\frac{j}{2}\left(\frac{a}{b}y^2-\frac{2}{b}yu+\frac{d}{b}u^2\right)}z^{s-1}dydz\right)\times$$
$$\left(\sqrt{\frac{1}{2j\pi b}}\int_{-\infty}^{\infty}\int_{0}^{\infty}f(v,w)e^{\frac{j}{2}\left(\frac{a}{b}v^2-\frac{2}{b}vu+\frac{d}{b}u^2\right)}w^{s-1}dvdw\right)$$
$$=\sqrt{2j\pi b}e^{\frac{-j}{2}\left(\frac{d}{b}u^2\right)}G^AM(u,s)F^AM(u,s)$$

Therefore, from (3.1.1) and (3.1.2), we have

$$(f*g)(t,x)=(g*f)(t,x)$$

This proves that the convolution theorem of Linear Canonical-Mellin Transform satisfies commutative law.

3.2 Associative Property:

If $L_AM\{f(t,x)\}=F^AM(u,s)$, $L_AM\{g(t,x)\}=G^AM(u,s)$ and $L_AM\{h(t,x)\}=H^AM(u,s)$, then as per the associative property:

$$L_AM\{\big((f*g)*h\big)(t,x)\}=2j\pi be^{-j\frac{d}{b}u^2}F^AM(u,s)G^AM(u,s)H^AM(u,s)$$
(3.2.1)

And

$$L_AM\{\big(f*(g*h)\big)(t,x)\}=2j\pi be^{-j\frac{d}{b}u^2}F^AM(u,s)G^AM(u,s)H^AM(u,s)$$
(3.2.2)

i.e., $\big((f*g)*h\big)(t,x)=\big(f*(g*h)\big)(t,x)$

Proof: Consider,

$$p(t,x)=\big((f*g)*h\big)(t,x)=(m*h)(t,x)$$

Where $m(t,x) = (f * g)(t,x) = \int_{-\infty}^{\infty}\int_{0}^{\infty} f(y,z)\, g\left(t-y, \frac{x}{z}\right) e^{-j\frac{a}{b}y(t-y)} dy \frac{dz}{z}$

(3.2.3)

And

$$p(t,x) = \big((f * g) * h\big)(t,x) = (m * h)(t,x)$$

$$= \int_{-\infty}^{\infty}\int_{0}^{\infty} m(v,w)\, h\left(t-v, \frac{x}{w}\right) e^{-j\frac{a}{b}v(t-v)} dv \frac{dw}{w}$$

(3.2.4)

Substituting the value of $m(t,x)$ from (3.2.3) in (3.2.4), we obtain

$$p(t,x) = \int_{-\infty}^{\infty}\int_{0}^{\infty}\left[\int_{-\infty}^{\infty}\int_{0}^{\infty} f(y,z)\, g\left(v-y, \frac{w}{z}\right) e^{-j\frac{a}{b}y(v-y)} dy \frac{dz}{z}\right] h\left(t-v, \frac{x}{w}\right) e^{-j\frac{a}{b}v(t-v)} dv \frac{dw}{w}$$

Applying Linear Canonical-Mellin Transform, we get

$$L_A M\{p(t,x)\} = \sqrt{\frac{1}{2j\pi b}} \int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty} f(y,z)\, g\left(v-y, \frac{w}{z}\right) \times$$

$$h\left(t-v, \frac{x}{w}\right) e^{-j\frac{a}{b}y(v-y)} e^{-j\frac{a}{b}v(t-v)} e^{\frac{j}{2}\left(\frac{a}{b}t^2 - \frac{2}{b}tu + \frac{d}{b}u^2\right)} x^{s-1} dy \frac{dz}{z} dv \frac{dw}{w} dt dx$$

(3.2.5)

Substituting $t - v = l, \frac{x}{w} = l' \Rightarrow t = l + v, x = wl' \Rightarrow dt = dl, dx = w dl'$

Therefore, (3.2.5) becomes

$$L_A M\{p(t,x)\} = \sqrt{\frac{1}{2j\pi b}} \int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty} f(y,z)\, g\left(v-y, \frac{w}{z}\right) e^{-j\frac{a}{b}y(v-y)} dy \frac{dz}{z} dv \frac{dw}{w} \times$$

$$\left[\int_{-\infty}^{\infty}\int_{0}^{\infty} h(l,l') e^{-j\frac{a}{b}vl} e^{\frac{j}{2}\left[\frac{a}{b}(l+v)^2 - \frac{2}{b}(l+v)u + \frac{d}{b}u^2\right]} (wl')^{s-1} w\, dl\, dl'\right]$$

$$= e^{\frac{-j}{2}\left(\frac{d}{b}u^2\right)} H^A M(u,s) \int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty} f(y,z)\, g\left(v-y, \frac{w}{z}\right) e^{-j\frac{a}{b}y(v-y)} \times$$

$$e^{\frac{j}{2}\left(\frac{a}{b}v^2 - \frac{2}{b}vu + \frac{d}{b}u^2\right)} w^{s-1} dy \frac{dz}{z} dv dw$$

(3.2.6)

Substituting $v - y = n, \frac{w}{z} = n'$, (3.2.6) becomes

$$L_A M\{p(t,x)\} = e^{\frac{-j}{2}\left(\frac{d}{b}u^2\right)} H^A M(u,s) \int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty} f(y,z)\, g(n,n') e^{-j\frac{a}{b}yn} \times$$

$$e^{\frac{j}{2}\left[\frac{a}{b}(y+n)^2-\frac{2}{b}(y+n)u+\frac{d}{b}u^2\right]}(zn')^{s-1}dydzdndn'$$

$$= 2j\pi be^{-j\frac{d}{b}u^2}H^AM(u,s)\left(\sqrt{\frac{1}{2j\pi b}}\int_{-\infty}^{\infty}\int_0^{\infty}f(y,z)e^{\frac{j}{2}\left(\frac{a}{b}y^2-\frac{2}{b}yu+\frac{d}{b}u^2\right)}z^{s-1}dydz\right)\times$$

$$\left(\sqrt{\frac{1}{2j\pi b}}\int_{-\infty}^{\infty}\int_0^{\infty}g(n,n')e^{\frac{j}{2}\left(\frac{a}{b}n^2-\frac{2}{b}nu+\frac{d}{b}u^2\right)}n'^{s-1}dndn'\right)$$

$$\Rightarrow L_AM\{((f*g)*h)(t,x)\} = 2j\pi be^{-j\frac{d}{b}u^2}F^AM(u,s)G^AM(u,s)H^AM(u,s)$$

(3.2.7)

Similarly, it can be proved that

$$L_AM\{(f*(g*h))(t,x)\} = 2j\pi be^{-j\frac{d}{b}u^2}F^AM(u,s)G^AM(u,s)H^AM(u,s)$$

(3.2.8)

Therefore, from (3.2.7) and (3.2.8), we have

$$((f*g)*h)(t,x) = (f*(g*h))(t,x)$$

This proves that the convolution theorem of Linear Canonical-Mellin Transform satisfies associative law.

4. Conclusion

In this paper, the authors first proved convolution theorem for the linear canonical-Mellin transform and later commutative and associative properties associated with the proposed convolution theorem for linear canonical-Mellin transform.

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