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On complex symmetric sequence of symbols of Toeplitz operators

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Abstract

We give and follow (E.Ko,J.E,Lee[15] a characterization of a complex symmetric sequence of symbols of Toeplitz operator $T_{\sum \varphi_m}$ on the Hardy space H^2. Moreover, if $T_{\sum \varphi_m}$ is a complex symmetric sequence of symbols of Toeplitz operator, we provide a necessary and sufficient condition for $T_{\sum \varphi_m}$ to be normal. Finally, we investigate$T_{\sum \varphi_m}$ with finite sequence of symbols.

Introduction

For $\mathcal{L}(\mathcal{H})$ be the algebra of bounded linear operators on a separable complex Hilbert space $\mathcal{H}$. For an operator $T \in \mathcal{L}(\mathcal{H})$, let T^* denote the adjoint of T. An operator $T \in \mathcal{L}(\mathcal{H})$ is $normal$ if $T^*T = TT^*$, $subnormal$ if there exists a Hilbert space K containing $\mathcal{H}$ and a normal operator N on $\mathcal{K}$ such that $N\mathcal{H} \subset \mathcal{H}$ and $T = N_{|\mathcal{H}}$, and $hyponormal$ if $T^*T - TT^* \geq 0$.

A $conjugation$ on $\mathcal{H}$ is an antilinear operator $C : \mathcal{H} \to \mathcal{H}$ with $C^2 = I$ which satisfies $\langle Cx, Cy \rangle = \langle y, x \rangle$ for all $x, y \in \mathcal{H}$. For a conjugation C, there exists an orthonormal basis $\{e_n\}_{n=0}^{\infty}$ for $\mathcal{H}$ such that $Ce_n = e_n$ for all n (see [4] and [6]). We call an operator $T \in \mathcal{L}(\mathcal{H})$ complex symmetric if there exists a conjugation C on $\mathcal{H}$ such that $T = CT^*C$. The class of complex symmetric operators includes all normal operators, Hankel operators, truncated Toeplitz operators, and Volterra integration operators. see [6,7,12,11]. Let L^2 be the Lebesgue (Hilbert) space on the unit circle $\partial\mathbb{D}$, and let L^∞ be the Banach space of all essentially bounded functions on $\partial\mathbb{D}$. Then it is well-known that $\{e_n(z) = z_n : n = 0, \pm1, \pm2, \pm3, \dots\}$ is an orthonormal basis for L^2. If $f_m \in L^2$, then the sequence of functions f_m is expressed as $f_m(z) = \sum_{n=-\infty}^{\infty} \sum \hat{f}_m(n) z^n$ where $\hat{f}_m(n)$ denotes the nth Fourier

coefficient of the sequence of f_m. The Hilbert Hardy space, denoted by H^2, consists of all functions f_m analytic on the open unit disk $\mathbb{D}$ with the power series representation

$$f_m(z) = \sum_{n=0}^{\infty} \sum a_n^m z^n \text{ where } \sum_{n=0}^{\infty} \sum |a_n^m|^2 < \infty,$$

or equivalently, with $\sup_{0<r<\infty} \left(\frac{1}{2\pi}\int_0^{2\pi} \sum \left|f_m(re^{i\theta})\right|^2 d\theta\right) < \infty$. It is clear that $H^2 = \overline{\text{span}\{z^n : n = 0, 1, 2, 3, \dots\}}$. For any $\varphi_m \in L^\infty$, the Toeplitz operator $T_{\sum \varphi_m}: H^2 \to H^2$ is defined by the formula

$$T_{\sum \varphi_m} f_m = P\left(\sum \varphi_m f_m\right)$$

for $f_m \in H^2$ where P denotes the orthogonal projection of L^2 onto H^2. It is known that $T_{\sum \varphi_m}$ is bounded if and only if $\varphi_m \in L^\infty$ and, in which case, $\left\|T_{\sum \varphi_m}\right\| = \sum \|\varphi_m\|_\infty$. A Toeplitz operator $T_{\sum \varphi_m}$ is called analytic if $\varphi_m \in H^\infty$, i.e., φ_m is a sequence of bounded analytic functions on the unit disc $\mathbb{D}$ and coanalytic if $\varphi_m \in H^\infty$ where φ_m denotes the complex conjugate of φ_m. For $\lambda \in \mathbb{D}$, the reproducing kernel K_λ for H^2 is given by $K_\lambda(z) := \frac{1}{1-\bar{\lambda}z}$ and $\langle f_m, K_\lambda \rangle = f_m(\lambda)$ for all $f_m \in H^2$.

The study of complex symmetric operators and Toeplitz operators provides deep and important connections with various problems in the field of quantum mechanics (see [5,13,1]). In 1960s, A. Brown and P. Halmos [2] proved that $T_{\sum \varphi_m}$ is normal if and only if $\varphi_m = \alpha + \beta_\rho$ where ρ is a real-valued function in L^∞ and $\alpha, \beta \in \mathbb{C}$. In 1970, P. Halmos [9] raised for the problem of characterizing subnormal Toeplitz operators. It is apparent that every normal or analytic $T_{\sum \varphi_m}$ is subnormal. In general, $T_{\sum \varphi_m}$ may not be a complex symmetric operator. However, if $T_{\sum \varphi_m}$ is a complex symmetric and hyponormal operator, then $T_{\sum \varphi_m}$ is normal from [14]. Recently, K. Guo and S. Zhu [8] have raised the following interesting question.

Question. Characterize a complex symmetric sequence of Toeplitz operator on the Hardy space H^2 of the unit disk.

Eungil Ko , Gi Eun Lee (see {15}) study properties of complex symmetric sequence of Toeplitz operators $T_{\sum \varphi_m}$ on the Hardy space H^2. We provide a characterization of such an operator $T_{\sum \varphi_m}$ and we give a necessary and sufficient condition for a complex symmetric

sequence of Toeplitz operators $T_{\sum \varphi_m}$ to be normal. We examine these $T_{\sum \varphi_m}$ with finite symbols.

2. Complex symmetric sequence of Toeplitz operators

We study complex symmetric sequence of Toeplitz operators $T_{\sum \varphi_m}$ on the Hardy space H^2. In particular, we give a characterization of such operators. We first start with the following theorem (see {15}).

Theorem 2.1. For $\varphi_m \in L^\infty$, let $T_{\sum \varphi_m}$ be a complex symmetric operator on H^2. If $T_{\sum \varphi_m}$ is analytic or coanalytic, then φ_m is either identically zero on $\mathbb{D}$ or a nonzero constant function on $\mathbb{D}$.

Proof. Let φ_m be not identically zero on $\mathbb{D}$ and let $T_{\sum \varphi_m}$ be analytic. If $\sum \varphi_m(\lambda) = 0$ for some λ in $\mathbb{D}$, then $\sum \varphi_m(z) \neq 0$ for all z in some open set U of $\mathbb{D}$ which does not contain λ. Since $T^*_{\sum \varphi_m} K_\lambda = \sum \overline{\varphi_m(\lambda)} K_\lambda = 0, \left\| T_{\sum \varphi_m} C K_\lambda \right\| = \left\| T^*_{\sum \varphi_m} K_\lambda \right\| = 0$, and so $T_{\sum \varphi_m} C K_\lambda(z) = 0$ for all $z \in \mathbb{D}$. Moreover, since $\varphi_m \in H^\infty, \sum \varphi_m(z) C K_\lambda(z) = 0$ for all $z \in \mathbb{D}$. This means that $C K_\lambda(z) = 0$ for all $z \in U$ and hence $C K_\lambda \equiv 0$ on $\mathbb{D}$ by the identity theorem, which is a contradiction. Hence φ_m does not vanish on $\mathbb{D}$.

Now fix $\alpha \in \mathbb{D}$. Then

$$C T^*_{\sum \varphi_m} K_\alpha - T_{\sum \varphi_m} C K_\alpha = C \sum \overline{\varphi_m(\alpha)}\, K_\alpha - T_{\sum \varphi_m} C K_\alpha$$

$$= \sum \varphi_m(\alpha) C K_\alpha - \sum P(\varphi_m C K_\alpha)$$

$$= \sum [\varphi_m(\alpha) - \varphi_m]\, C K_\alpha = 0. \qquad (1)$$

Since K_α does not vanish on $\mathbb{D}$, it follows that $C K_\alpha$ does not vanish on D either. Hence $\varphi_m \equiv \varphi_m(\alpha)$ for the fixed $\alpha \in \mathbb{D}$ from (1) and therefore, φ_m is a nonzero constant function on $\mathbb{D}$. On the other hand, if $T_{\sum \varphi_m}$ is coanalytic, then φ_m is in H^∞. Hence φ_m must be a constant function or $\varphi_m = 0$ by a similar argument.

We recall that for $u_m \in H^2$ with the power series representation $u_m(z) = \sum_{n=0}^{\infty} \sum a^m_{n\,n} z^n$, we define the function $\tilde{u}_m$ on the boundary of $\mathbb{D}$ by $\tilde{u}_m(e^{i\theta}) := \sum_{n=0}^{\infty} \sum a^m_n e^{in\theta}$. A function $u_m \in H^2$ is called inner if $\sum |\tilde{u}_m(e^{i\theta})| = 1$ for almost all θ. We remark that if $\varphi_m \in H^\infty$, then $T_{\sum \varphi_m}$ is subnormal. Moreover, in this case, if $T_{\sum \varphi_m}$ is

complex symmetric, then it must be normal from [14, Lemma 3.1]. Hence, from [2, Corollary, p. 98], $\varphi_m = \alpha + \beta_\rho$ where ρ is a real-valued function in L^∞ and $\alpha, \beta \in \mathbb{C}$. From Theorem 2.1, we know that $\alpha = \beta = 0$ or $\alpha = 0$ and $\beta = 0$.

Corollary 2.2. (see[15]) If φ_m is a nonconstant inner function on $\mathbb{D}$, then $T_{\sum \varphi_m}$ is not a complex symmetric operator with conjugation $\mathbb{C}$.

Proof. If φ_m is a constant function on $\mathbb{D}$, then $T_{\sum \varphi_m}$ is clearly a complex symmetric operator. Otherwise, the proof immediately follows from Theorem 2.1.

We denote by $(\varphi_m)_+$ and $(\varphi_m)_-$ the positive and negative parts of φ_m, respectively:

$$(\varphi_m)_+(z) = \sum_{n=1}^{\infty} \sum \hat{\varphi}_m(n) z^n, (\varphi_m)_-(z) = \sum_{n=1}^{\infty} \sum \overline{\hat{\varphi}_m(-n)} z^n, \text{and} \sum (\varphi_m)_0(z) = \sum \hat{\varphi}_m(0) e_0.$$

Hence $\varphi_m = (\varphi_m)_+ + (\varphi_m)_0 + \overline{(\varphi_m)_-}$. In the following lemma, we give a new conjugation on the Hardy space H^2. (see[15])

Lemma 2.3. For every ξ and θ, let $C_{\xi,\theta} : H^2 \to H^2$ be defined by

$$C_{\xi,\theta}\left(\sum f_m(z)\right) = e^{i\xi} \sum \overline{f_m(e^{i\theta}\overline{z})}.$$

Then $C_{\xi,\theta}$ is a conjugation on H^2. Moreover, $C_{\xi,\theta}$ and $C_{\overline{\xi},\overline{\theta}}$ are unitarily equivalent where $(\overline{\xi}, \overline{\theta})$ satisfies the equation $\tilde{\xi} - k\tilde{\theta} = -\xi + k\theta - 2n\pi$ for every $k \in \mathbb{N}$ and $n \in \mathbb{Z}$.

Proof. Let $C_{\xi,\theta} : H^2 \to H^2$ be defined by $C_{\xi,\theta}(\sum f_m(z)) = e^{i\xi} \sum \overline{f_m(e^{i\theta}\overline{z})}$ for all ξ and θ. Then $C_{\xi,\theta}$ is clearly antilinear. Since

$$C_{\xi,\theta}^2 \sum f_m(z) = C_{\xi,\theta}\left(\sum e^{i\xi}\overline{f_m(e^{i\theta}\overline{z})}\right) = e^{i\xi}e^{i\xi}\overline{\sum f_m(e^{i\theta}e^{i\theta}\overline{z})} = \sum f_m(z),$$

it is involutive. Moreover, we get that

$$\langle C_{\xi,\theta} \sum f_m(z), C_{\xi,\theta} \sum g_m(z)\rangle = \langle e^{i\xi}\overline{f_m(e^{i\theta}\overline{z})}, e^{i\xi}\overline{g_m(e^{i\theta}\overline{z})}\rangle$$

$$= \int_{\partial\mathbb{D}} \sum g_m(e^{i\theta}\overline{z})\overline{f_m(e^{i\theta}\overline{z})}\, dm(z). \qquad (2)$$

Define a map Φ on $\partial\mathbb{D}$ as $\Phi(z) = e^{i\theta}\overline{z}$.

Then $\Phi(z)$ is continuous on $\partial\mathbb{D}$ and equation (2) becomes

$$\int_{\partial\mathbb{D}} \sum g_m(\Phi(z)) f_m(\Phi(z))\, d\mu = \int_{\partial\mathbb{D}} \sum g_m(z)\overline{f_m(z)}\, d\mu \circ \Phi^{-1} = \sum \langle g_m(z), f_m(z)\rangle$$

due to the fact that $\Phi^{-1}(\partial\mathbb{D}) = \partial\mathbb{D}$. Thus $C_{\xi,\theta}$ is isometric and hence $C_{\xi,\theta}$ is a conjugation on H^2. On the other hand, if we define $V : H^2 \to H^2$ by $V \sum h_m(z) = e^{i\xi} \sum h_m(e^{i\theta}z)$ for every ξ and θ, then $V^* \sum h_m(z) = e^{-i\xi} h_m(e^{-i\theta}z)$ so that V becomes unitary. Since $(\overline{\xi}, \overline{\theta})$ satisfies the equation $\tilde{\xi} - k\tilde{\theta} = -\xi + k\theta - 2n\pi$, it follows that for $h_m(z) = \sum_{j=0}^{\infty} \sum a_j^m z_j$,

$$\langle [e^{i\xi} h_m(e^{-i\theta}\overline{z}) - e^{-i\overline{\xi}}\ h_m(e^{i\overline{\theta}}\ \overline{z})], \overline{z}^k\rangle$$

$$= \sum \left\langle \left[e^{i\xi} \sum_{j=0}^{\infty} a_j^m \left(e^{-i\theta}\overline{z}\right)^j - e^{-i\xi} \sum_{j=0}^{\infty} a_j^m \left(e^{i\overline{\theta}}\ \overline{z}\right)^j \right], \overline{z}^k \right\rangle$$

$$= \left\langle \left[e^{i\xi} \sum_{j=0}^{\infty} \sum a_j^m\, e^{-ij\theta} \overline{z}^j - e^{-i\overline{\xi}} \sum_{j=0}^{\infty} \sum a_j^m\, e^{ij\overline{\theta}}\ \overline{z}^j \right], \overline{z}^k \right\rangle$$

$$= \sum a_k^m \left[e^{i(\xi - k\theta)} - e^{i\left(-\overline{\xi} + k\overline{\theta}\right)} \right] = 0$$

when a_k^m is nonzero. Thus we have $\sum e^{-i\xi}\overline{h_m(e^{-i\theta}z)} = \sum e^{i\overline{\xi}}\ \overline{h_m(e^{i\overline{\theta}}\ \overline{z})}$ and therefore $V^* C_{\xi,\theta} V\, h_m(z) = V^* C_{\xi,\theta} \sum [e^{i\xi} h_m(e^{i\theta}z)] = V^* \sum [e^{i\xi} \overline{e^{i\xi} h_m\left(e^{i\theta}\overline{e^{i\theta}}z\right)}]$

$$= V^* \sum \overline{h_m(\overline{z})} = e^{-i\xi} \sum \overline{h_m(e^{-\iota\theta}\overline{z})} = \sum C_{\tilde{\xi},\tilde{\theta}}\, h_m(z)$$

for all $h_m \in H^2$. This means that $V\, C_{\tilde{\xi},\tilde{\theta}} = C_{\tilde{\xi},\tilde{\theta}} V$ for all ξ and θ. Hence $C_{\xi,\theta}$ and $C_{\tilde{\xi},\tilde{\theta}}$ are unitarily equivalent.

We next investigate conditions for a Toeplitz operator $T_{\sum \varphi_m}$ to be complex symmetric with the conjugation $C_{\xi,\theta}$ on H^2. (see[15])

Note that $C_{\xi,\theta}\left(\sum T_{\sum \varphi_m} f_m\right) = C_{\xi,\theta} P(\sum_m \varphi_m\, (\sum_{n=-\infty}^{\infty} \hat{f}_m(n)\, z^n)$

Theorem 2.4. For $\varphi_m(z) = \sum_{n=-\infty}^{\infty} \sum \hat{\varphi}_m(n) z^n \in L^\infty$, let $T_{\sum \varphi_m}$ be a Toeplitz operator on H^2. Then the following statements are equivalent:

(i) $T_{\sum \varphi_m}$ is complex symmetric with the conjugation $C_{\xi,\theta}$.

(ii) $\sum \varphi_m(-n) = \sum \hat{\varphi}_m(n)\lambda_n$ for all $n \in \mathbb{Z}$ with $|\lambda| = 1$.

(iii) $\sum \varphi_m(z) = \sum(\varphi_m)_0 + \sum_{n=1}^{\infty} \sum \varphi_m(n)(z^n + \lambda_n \overline{z}^n)$ with $|\lambda| = 1$.

(iv) $\sum \varphi_m(z) = \sum(\varphi_m)_+(z) + \sum(\varphi_m)_0 + \sum(\varphi_m)_+(e^{i\theta}\bar{z})$ for $(\varphi_m)_+ \in zH^2$ and some θ.

Proof. (i) ⇔ (ii): Let $C_{\xi,\theta}(\sum f_m(z)) = e^{i\xi} \sum \overline{f_m(e^{i\theta}\overline{z})}$ for all ξ and θ. Then

$$C_{\xi,\theta}\left(\sum_{k=0}^{\infty} \sum a_k^m z^k\right) = \mu \sum_{k=0}^{\infty} \sum \lambda^k \overline{a_k^m} z^k$$

for all $\sum_{k=0}^{\infty} \sum a_k^m z^k \in H^2$ with $|\lambda| = |\mu| = 1$. Assume that $T_{\sum \varphi_m}$ is complex symmetric with the conjugation $C_{\xi,\theta}$. Since

$$C_{\xi,\theta}\,(T_{\sum \varphi_m} z^k) = C_{\xi,\theta}\, P\left(\sum_{n=-\infty}^{\infty} \sum \hat{\varphi}_m(n) z^{n+k}\right) = C_{\xi,\theta}\left(\sum_{n=-k}^{\infty} \sum \hat{\varphi}_m(n) z^{n+k}\right)$$
$$= \mu \sum_{n=-k}^{\infty} \sum \overline{\hat{\varphi}_m(n)} \lambda^{n+k} z^{n+k} \tag{3}$$

and $T^*_{\sum \varphi_m} C_{\xi,\theta} z^k = T_{\sum \varphi_m}(\mu\lambda^k z^k) = P\left(\sum_{n=-\infty}^{\infty} \sum \overline{\hat{\varphi}_m(n)} \mu\lambda^k z^{-n+k}\right)$

$$= P\left(\sum_{n=-\infty}^{\infty} \sum \overline{\hat{\varphi}_m(-n)} \mu\lambda^k z^{n+k}\right) = \mu \sum_{n=-k}^{\infty} \sum \overline{\hat{\varphi}_m(-n)} \lambda^k z^{n+k} \tag{4}$$

for all nonnegative integers k, it follows that $\hat{\varphi}_m(-n) = \hat{\varphi}_m(n)\lambda^n$ for all $n \in \mathbb{Z}$ with $|\lambda| = 1$. Conversely, suppose that $\hat{\varphi}_m(-n) = \hat{\varphi}_m(n)\lambda^n$ for all $n \in \mathbb{Z}$ with $|\lambda| = 1$. Since $|\mu| = 1$, it follows that μ is nonzero. From (3) and (4), we have

$$\left(C_{\xi,\theta} T_{\sum \varphi_m} - T^*_{\sum \varphi_m} C_{\xi,\theta}\right) z^k$$
$$= \sum_{n=-k}^{\infty} \sum \overline{\hat{\varphi}_m(n)} \mu\lambda^{n+k} z^{n+k} - \sum_{n=-k}^{\infty} \sum \overline{\hat{\varphi}_m(-n)} \mu\lambda^k z^{n+k}$$

$$= \sum_{n=-k}^{\infty} \sum \left[\overline{\hat{\varphi}_m(n)}\lambda^{n+k} - \overline{\hat{\varphi}_m(-n)}\lambda^{k}\right]\mu z^{n+k}$$

$$= \sum_{n=-k}^{\infty} \sum \left[\overline{\hat{\varphi}_m(n)}\lambda^{n+k} - \overline{\hat{\varphi}_m(n)}\lambda^{n+k}\right]\mu z^{n+k} = 0.$$

Hence $T_{\sum \varphi_m}$ is a complex symmetric operator with the conjugation $C_{\xi,\theta}$.

(i) ⇔ (iii): Let $T_{\sum \varphi_m}$ be complex symmetric with the conjugation $C_{\xi,\theta}$. Then, from assertion (ii),

$$\varphi_m(z) = \sum_{n=1}^{\infty} \sum \hat{\varphi}_m(n) z^n + \sum (\varphi_m)_0 + \sum_{n=1}^{\infty} \sum \hat{\varphi}_m(-n)\overline{z}^n$$

$$= \sum_{n=1}^{\infty} \sum \hat{\varphi}_m(n) z^n + \sum (\varphi_m)_0 + \sum_{n=1}^{\infty} \sum \hat{\varphi}_m(n)\lambda^n \overline{z}^n$$

$$= \sum (\varphi_m)_0 + \sum_{n=1}^{\infty} \sum \hat{\varphi}_m(n)\left(z^n + \lambda^n \overline{z}^n\right) \tag{5}$$

with $|\lambda| = 1$, and therefore we have the statement (iii).

Conversely, if $\sum \varphi_m(z) = \sum (\varphi_m)_0 + \sum_{n=1}^{\infty} \sum \hat{\varphi}_m(n)\left(z^n + \lambda^n \overline{z}^n\right)$ with $|\lambda| = 1$, then (5) implies that

$$\hat{\varphi}_m(-n) = \sum \langle \varphi_m, e_{-n} \rangle = \left\langle \left[\sum (\varphi_m)_0 + \sum_{n=1}^{\infty} \sum \hat{\varphi}_m(n)\left(z^n + \lambda^n \overline{z}^n\right)\right], e_{-n} \right\rangle$$

$$= \sum \left\langle \left[\sum_{n=1}^{\infty} \hat{\varphi}_m(n) z^n + \sum (\varphi_m)_0 + \sum_{n=1}^{\infty} \hat{\varphi}_m(n)\lambda^n \overline{z}^n\right], e_{-n} \right\rangle = \lambda^n \sum \hat{\varphi}_m(n)$$

for all $n \in \mathbb{Z}$. By (i) ⇔ (ii), $T_{\sum \varphi_m}$ is complex symmetric with the conjugation $C_{\xi,\theta}$.

(i) ⇔ (iv): Since $T_{\sum \varphi_m}$ is complex symmetric with the conjugation $C_{\xi,\theta}$, it follows from (ii) that

$$(\varphi_m)_-(z) = \sum_{n=1}^{\infty} \sum \overline{\hat{\varphi}_m(n)\lambda^n \bar{z}^n} = \overline{\sum_{n=1}^{\infty} \sum \hat{\varphi}_m(n) e^{i\theta n} \overline{z}^n} = \sum \overline{(\varphi_m)_+(e^{i\theta}\bar{z})}$$

for some θ. Therefore,$\sum \varphi_m(z) = \sum(\varphi_m)_+(z) + \sum(\varphi_m)_0 + \sum(\varphi_m)_+(e^{i\theta}\bar{z})$.

Conversely, assume that $\sum \varphi_m(z) = \sum(\varphi_m)_+(z) + \sum(\varphi_m)_0 + \sum(\varphi_m)_+(e^{i\theta}\overline{z})$ for some θ. Then

$$\hat{\varphi}_m(-n) = \sum \langle [(\varphi_m)_+ + (\varphi_m)_0 + (\varphi_m)_+(e^{i\theta}\bar{z})], e_{-n}\rangle$$

$$= \sum \langle \left[\sum_{n=1}^{\infty} \hat{\varphi}_m(n)z^n + (\varphi_m)_0 + \sum_{n=1}^{\infty} \hat{\varphi}_m(n)\lambda^n\overline{z}^n\right], e_{-n}\rangle = \lambda^n \sum \hat{\varphi}_m(n)$$

for all $n \in \mathbb{Z}$. By (i) ⇔ (ii), we know that $T_{\sum \varphi_m}$ is complex symmetric with the conjugation $C_{\xi,\theta}$.

Notation 2.5. We denote by C_1 and C_2 $C_1 = C_{0,0}$ and $C_2 = C_{0,\pi}$

from Lemma 2.3. (Note that $C_{0,0}$ and $C_{0,\pi}$ are unitarily equivalent from Lemma 2.3.) Since $C_1 e_n(z) = \overline{e_n(\overline{z})} = e_n(z)$ for all nonnegative integers $n, e_n(z) = z^n$ is a C_1-real orthonormal basis for H^2. We also know that $U = C_1C_2$ is a unitary operator with $U^* = C_2C_1$. Furthermore, U is complex symmetric with both conjugations C_1 and C_2. If $\xi = \pi, \theta = 0$ and $\xi = \pi, \theta = \pi$ in Lemma 2.3, we denote $C_{\pi,0}$ and $C_{\pi,\pi}$ by C_3 and C_4, respectively.

We obtain the precise forms of symbol functions with respect to conjugations C_1 and C_2, respectively.

Corollary 2.6. For $\varphi_m(z) = \sum_{n=-\infty}^{\infty} \sum \hat{\varphi}_m(n)z^n \in L^\infty$, let $T_{\sum \varphi_m}$ be a Toeplitz operator on H^2. Then the following statements hold:

(i) $T_{\sum \varphi_m}$ is complex symmetric with the conjugation C_1 if and only if

$$\varphi_m(z) = \sum (\varphi_m)_0 + 2\sum_{n=1}^{\infty} \sum \hat{\varphi}_m(n)\mathrm{Re}\{z^n\}.$$

(ii) $T_{\sum \varphi_m}$ is complex symmetric with the conjugation C_2 if and only if

$$\varphi_m(z) = \sum (\varphi_m)_0 + 2\sum_{k=1}^{\infty} \hat{\varphi}_m(2k)\mathrm{Re}\{z^{2k}\} + 2i\sum_{k=1}^{\infty} \hat{\varphi}_m(2k-1)\mathrm{Im}\{z^{2k-1}\}.$$

Corollary 2.7. (see[15]) under the same hypotheses as in Theorem 2.4, the following statements hold:

(i) If $T_{\sum \varphi_m}$ is a complex symmetric operator with conjugations C_1 and C_2, then $\sum \hat{\varphi}_m(2k-1) = 0$ for all positive integers k.

(ii) If $\varphi_m(z) = \phi_m(z) + \alpha + \phi_m(\bar{z})$ and $\psi_m(z) = \phi_m(z) + \beta + \phi_m(-z)$ for $\phi_m \in zH^2$ and $\alpha, \beta \in \mathbb{C}$, then $T_{\sum \varphi_m}$ and $T_{\sum \psi_m}$ are complex symmetric operators on H^2.

Proof. (i) By Theorem 2.4, we have $\sum \hat{\varphi}_m(n)[1-(-1)^n] = 0$ so that $\sum \hat{\varphi}_m(2k-1) = 0$ for all positive integers k.

(ii) The proof directly follows from Theorem 2.4.

We know from Theorem 2.4 that the matrices (a_{ij}^m) and $(\widetilde{a_{ij}^m})$ for complex symmetric operators $T_{\sum \varphi_m}$ with conjugation C_1 and C_2 with respect to the basis $\{z^n : n = 0, 1, 2, \dots\}$ are given by

$$(a_{ij}^m) = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & \cdots & \cdots \\ a_1 & a_0 & a_1 & a_2 & \ddots & \cdots \\ a_2 & a_1 & a_0 & a_1 & \ddots & \ddots \\ a_3 & a_2 & a_1 & a_0 & \ddots & \ddots \\ \cdots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \cdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}^m, \quad (\widetilde{a_{ij}^m}) = \begin{pmatrix} a_0 & -a_1 & a_2 & -a_3 & \cdots & \cdots \\ a_1 & a_0 & -a_1 & a_2 & \ddots & \cdots \\ a_2 & a_1 & a_0 & -a_1 & \ddots & \ddots \\ a_3 & a_2 & a_1 & a_0 & \ddots & \ddots \\ \cdots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \cdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}^m,$$

where $a_k^m = \hat{\varphi}_m(k)$ for positive integer k. In this case, $(a_{ij}^m)^t = (a_{ij}^m)$ and $(\widetilde{a_{ij}^m})^t \neq (\widetilde{a_{ij}^m})$ where t denotes the transpose. A simple observation of these matrices immediately gives us the following corollary (see[15]).

Corollary 2.8. If (a_{ij}^m) and $(\widetilde{a_{ij}^m})$ are matrices in the previous notes, then the following properties hold:

(i) (a_{ij}^m) is self-adjoint if and only if $a_k^m = \overline{a_k^m}$ for all $k = 0, 1, 2, \dots$.

(ii) $(\widetilde{a_{ij}^m})$ is self-adjoint if and only if $a_{2k}^m = \overline{a_{2k}^m}$ and $a_{2k+1}^m = -\overline{a_{2k+1}^m}$ for all $k = 0, 1, 2, \dots$.

Remark 2.9. For any $\varphi_m \in L^\infty$, let $T_{\sum \varphi_m}$ be a Toeplitz operator on H^2. Then we get the same results as in Corollary 2.6:

(i) $T_{\sum \varphi_m}$ is complex symmetric with the conjugation C_3 if and only if

$$\varphi_m(z) = \sum(\varphi_m)_0 + 2\sum_{n=1}^{\infty}\sum\hat{\varphi}_m(n)\mathrm{Re}\{z^n\}.$$

(ii) $T_{\sum\varphi_m}$ is complex symmetric with the conjugation C_4 if and only if

$$\varphi_m(z) = \sum(\varphi_m)_0 + 2\sum_{k=1}^{\infty}\sum\hat{\varphi}_m(2k)\mathrm{Re}\{z^{2k}\} + 2i\sum_{k=1}^{\infty}\sum\hat{\varphi}_m(2k-1)\mathrm{Im}\{z^{2k-1}\}.$$

We observe from Theorem 2.4 that if $\overline{\hat{\varphi}_m(n)} = \hat{\varphi}_m(n)$ for all $n \in \mathbb{Z}$, then the bilinear form$\sum[T_{\varphi_m}f_m, g_m]$ defined by $\sum[T_{\varphi_m}f_m, g_m] = \sum\langle T_{\varphi_m}f_m, C_{\xi,\theta}g_m\rangle$ is bounded and symmetric. Indeed,

$$\sum[T_{\varphi_m}f_m, g_m] = \sum\langle T_{\varphi_m}f_m, C_{\xi,\theta}g_m\rangle = \sum\langle T^*_{\varphi_m}f_m, C_{\xi,\theta}g_m\rangle = \sum\langle f_m, C_{\xi,\theta}T_{\varphi_m}g_m\rangle$$
$$= \sum[f_m, T_{\varphi_m}g_m]$$

from Theorem 2.4. Moreover, we have

$$\sum|[T_{\varphi_m}f_m, g_m]| = \left|\sum\langle T_{\varphi_m}f_m, C_{\xi,\theta}g_m\rangle\right| \le \sum\|T_{\varphi_m}f_m\|\,\|C_{\xi,\theta}g_m\|$$
$$\le \sum\|T_{\varphi_m}\|\,\|f_m\|\|g_m\| = \sum\|\varphi_m\|_\infty\|f_m\|\|g_m\|.$$

If $(\varphi_m)_- = (\varphi_m)_+$ in Theorem 2.10, then $T_{\sum\varphi_m}$ is clearly normal from [2] or [10, p. 157]. In the following theorem, for a complex symmetric operator $T_{\sum\varphi_m}$ with the conjugation $C_{\xi,\theta}$, we provide a necessary and sufficient condition for $T_{\sum\varphi_m}$to be normal. Furthermore, we examine symbol functions of normal and complex symmetric sequence of Toeplitz operators. (see[15])

Theorem 2.10. Let φ_m be in L^∞ such that $\sum\varphi_m = \sum(\varphi_m)_+ + \sum(\varphi_m)_0 + \sum(\varphi_m)_-$ where $(\varphi_m)_+$ and $(\varphi_m)_-$ are defined previously. If $T_{\sum\varphi_m}$ is complex symmetric with the conjugation $C_{\xi,\theta}$, then $T_{\sum\varphi_m}$is normal if and only if for some ζ, $\overline{\hat{\varphi}_m(n)} = e^{i(\zeta+n\theta)}\hat{\varphi}_m(n)$ for all positive integers

Proof. If $T_{\sum\varphi_m}$ is complex symmetric with the conjugation $C_{\xi,\theta}$, then it follows from Theorem 2.4 that $\hat{\varphi}_m(-n) = \hat{\varphi}_m(n)e^{in\theta}$ for some θ. It is known from [2] or [10, p. 157] that if $\varphi_m \in L^\infty$, then

$T_{\sum\varphi_m}$ is normal if and only if $(\varphi_m)_- = \lambda(\varphi_m)_+$ for some $\lambda \in C$ with

$$|\lambda| = 1. \qquad (6)$$

From this, we conclude that $T_{\sum \varphi_m}$ is normal if and only if for some $\zeta, \overline{\hat{\varphi}_m(n)} = e^{i(\zeta+n\theta)}\,\hat{\varphi}_m(n)$ for all positive integers n.

In the corollary stated below, the symbols φ_m of (i) and (ii) are exactly the Brown and Halmos type functions of [2] (see[15]).

Corollary 2.11. Let φ_m be in L^∞ such that $\sum \varphi_m = \sum(\varphi_m)_+ + \sum(\varphi_m)_0 + \sum(\varphi_m)_-$. Then the following statements hold:

(i) If $T_{\sum \varphi_m}$ is complex symmetric with the conjugation C_1, then $T_{\sum \varphi_m}$ is normal if and only if $\sum \varphi_m(z) = \sum(\varphi_m)_0 + 2e^{-i\frac{\theta}{2}}\,\mathrm{Re}\left\{\sum_{n=1}^{\infty} \sum e^{i\frac{\theta}{2}}\hat{\varphi}_m(n)\mathrm{Re}(z^n)\right\}$ for some θ.

(ii) If $T_{\sum \varphi_m}$ is complex symmetric with the conjugation C_2, then $T_{\sum \varphi_m}$ is normal if and only if a symbol function φ_m has the form:

$$\varphi_m(z) = \sum(\varphi_m)_0 + 2e^{-i\frac{\theta}{2}}\,\mathrm{Re}\left\{\sum_{k=1}^{\infty} \sum e^{i\frac{\theta}{2}}\,\hat{\varphi}_m(2k)\mathrm{Re}(z^{2k})\right\} + 2ie^{-i\frac{\theta}{2}}\,Im\left\{\sum_{k=1}^{\infty} \sum e^{i\frac{\theta}{2}}\,\hat{\varphi}_m(2k-1)\mathrm{Im}(z^{2k-1})\right\}.$$

Proof. (i) Assume that $T_{\sum \varphi_m}$ is complex symmetric with the conjugation C_1. It follows from Theorem 2.10 that $T_{\sum \varphi_m}$ is normal if and only if $\hat{\varphi}_m(n) = \lambda\hat{\varphi}_m(n)$ for all positive integers n and for some λ with $|\lambda| = 1$. By Corollary 2.6(i), we get that $\varphi_m(z) = \sum(\varphi_m)_0 + 2\sum_{n=1}^{\infty} \sum \hat{\varphi}_m(n)\mathrm{Re}\{z^n\}$. Moreover, since $T_{\sum \varphi_m}$ is normal, it follows that

$$\varphi_m(z) = \sum(\varphi_m)_0 + 2\sum_{n=1}^{\infty} \sum \hat{\varphi}_m(n)\mathrm{Re}\{z^n\}$$

$$= \sum(\varphi_m)_0 + 2\sum_{n=1}^{\infty} \sum \hat{\varphi}_m(n)\mathrm{Re}\{z^n\} + \sum_{n=1}^{\infty} \sum \overline{e^{i\theta}\hat{\varphi}_m(n)}\mathrm{Re}\{z^n\}$$

$$= \sum(\varphi_m)_0 + 2e^{-i\frac{\theta}{2}}\,\mathrm{Re}\left\{\sum_{n=1}^{\infty} \sum e^{i\frac{\theta}{2}}\,\hat{\varphi}_m(n)\mathrm{Re}(z^n)\right\}$$

for some θ. The converse statement follows from [2].

(ii) Let $T_{\sum \varphi_m}$ be complex symmetric with the conjugation C_2. Then, by Theorem 2.10, $T_{\sum \varphi_m}$ is normal if and only if $(-1)^n \hat{\varphi}_m(n) = \lambda \hat{\varphi}_m(n)$ for all positive integers n and for some λ with $|\lambda| = 1$. From Corollary 2.6(ii), we obtain that

$$\varphi_m(z) = \sum (\varphi_m)_0 + 2\sum_{k=1}^{\infty} \sum \hat{\varphi}_m(2k)\mathrm{Re}\{z^{2k}\} + 2i \sum_{k=1}^{\infty} \sum \hat{\varphi}_m(2k-1)\mathrm{Im}\{z^{2k-1}\}.$$

Moreover, since $T_{\sum \varphi_m}$ is normal,

$$\varphi_m(z) = \sum (\varphi_m)_0 + 2\sum_{k=1}^{\infty} \sum \hat{\varphi}_m(2k)\mathrm{Re}\{z^{2k}\} + 2i \sum_{k=1}^{\infty} \sum \hat{\varphi}_m(2k-1)\mathrm{Im}\{z^{2k-1}\}$$

$$= \sum (\varphi_m)_0 + \sum_{k=1}^{\infty} \sum \hat{\varphi}_m(2k)\mathrm{Re}\{z^{2k}\} + \sum_{k=1}^{\infty} \sum \overline{e^{i\theta}\hat{\varphi}_m(2k)}\mathrm{Re}\{z^{2k}\}$$

$$+ i \sum_{k=1}^{\infty} \sum \hat{\varphi}_m(2k-1)\mathrm{Im}\{z^{2k-1}\} - i \overline{\sum_{k=1}^{\infty} \sum e^{i\theta}\hat{\varphi}_m(2k-1)\,\mathrm{Im}\{z^{2k-1}\}}$$

$$= (\varphi_m)_0 + 2e^{-i\frac{\theta}{2}}\,\mathrm{Re}\left\{\sum_{k=1}^{\infty} \sum e^{i\frac{\theta}{2}}\hat{\varphi}_m(2k)\mathrm{Re}(z^{2k})\right\}$$

$$+ 2ie^{-i\frac{\theta}{2}}\,Im\left\{\sum_{k=1}^{\infty} \sum e^{i\frac{\theta}{2}}\hat{\varphi}_m(2k-1)\mathrm{Im}(z^{2k-1})\right\}$$

for some θ. The converse statement follows from [2].

We consider some more applications of Theorem 2.10. (see[15]).

Corollary 2.12. For $\varphi_m(z) = \sum_{n=-\infty}^{\infty} \sum \hat{\varphi}_m(n)z^n \in L^{\infty}$, let $T_{\sum \varphi_m}$ be a Toeplitz operator on H^2. If $T_{\sum \varphi_m}$ is complex symmetric with the conjugation $C_{\xi,\theta}$, then $T_{\sum \varphi_m}$ is unitary if and only if for some ζ, $\hat{\varphi}_m(n) = e^{i(\zeta+n\theta)}\hat{\varphi}_m(n)$ for all positive integers n and

$$\sum_{n=-k}^{\infty} \sum \hat{\varphi}_m(n)\overline{\hat{\varphi}_m(n+k-l)} = \sum_{n=-k}^{\infty} \sum \hat{\varphi}_m(-n)\overline{\hat{\varphi}_m(-(n+k-l))}$$

$$= \begin{cases} 1 \text{ if } k = l, \\ 0 \text{ if } k \neq l \end{cases}$$

for all positive integers l, k.

Proof. Let $T_{\sum \varphi_m}$ be complex symmetric with the conjugation $C_{\xi,\theta}$. If $T_{\sum \varphi_m}$is unitary, then $\hat{\varphi}_m(n) = e^{i(\zeta+n\theta)}\,\hat{\varphi}_m(n)$ for some ζ and for all positive integers n from Theorem 2.10. Since $T_{\sum \varphi_m}{}^{*}\, T_{\sum \varphi_m} = I$, it follows that

$$1 = \langle T^*_{\sum \varphi_m} T_{\sum \varphi_m} z^k, z^k\rangle = \langle P\left(\sum_{n=-\infty}^{\infty} \sum \hat{\varphi}_m(n) z^{n+k}\right), P\left(\sum_{m=-\infty}^{\infty} \sum \hat{\varphi}_m(m) z^{m+k}\right)\rangle$$

$$= \langle \sum_{n=-k}^{\infty} \sum \hat{\varphi}_m(n) z^{n+k}, \sum_{m=-k}^{\infty} \sum \hat{\varphi}_m(m) z^{m+k}\rangle = \sum_{n=-k}^{\infty} \sum |\hat{\varphi}_m(n)|^2 \qquad (7)$$

and if $k \neq l$, then

$$0 = \langle T^*_{\sum \varphi_m} T_{\sum \varphi_m} z^k, z^l\rangle = \langle P\left(\sum_{n=-\infty}^{\infty} \sum \hat{\varphi}_m(n) z^{n+k}\right), P\left(\sum_{m=-\infty}^{\infty} \sum \hat{\varphi}_m(m) z^{m+l}\right)\rangle$$

$$= \langle \sum_{n=-k}^{\infty} \sum \hat{\varphi}_m(n) z^{n+k}, \sum_{m=-l}^{\infty} \sum \varphi_m(m) z^{m+l}\rangle$$

$$= \sum_{n=-k}^{\infty} \sum \hat{\varphi}_m(n)\overline{\hat{\varphi}_m(n + k - l)} \qquad (8)$$

for all positive integers k, l. Similarly, since $T_{\sum \varphi_m} T^*_{\sum \varphi_m} = I$, we get that $\sum_{n=-k}^{\infty} \sum |\hat{\varphi}_m(-n)|^2 = 1$ and $\sum_{n=-k}^{\infty} \sum \hat{\varphi}_m(-n)\overline{\hat{\varphi}_m\big(-(n + k - l)\big)} = 0$ if $k \neq l$.

Conversely, let

$\sum_{n=-k}^{\infty} \sum \hat{\varphi}_m(-n)\overline{\hat{\varphi}_m\big(-(n + k - l)\big)} = \begin{cases} 1 \text{ if } k = l, \\ 0 \text{ if } k \neq l \end{cases}$ for all positive integers l, k, and $\sum \overline{\hat{\varphi}_m(n)} = e^{i(\zeta+n\theta)} \sum \hat{\varphi}_m(n)$ for all positive integers n. By Theorem 2.10, we conclude that $T^*_{\sum \varphi_m} T_{\sum \varphi_m} = T_{\sum \varphi_m} T^*_{\sum \varphi_m}$. Moreover, we have

$$\sum \langle T^*_{\sum \varphi_m} T_{\sum \varphi_m} z^k, z^l \rangle = \sum_{n=-k}^{\infty} \sum \hat{\varphi}_m(n) z^{n+k}, \sum_{m=-l}^{\infty} \sum \hat{\varphi}_m(m) z^{m+l}$$

$$= \begin{cases} \sum_{n=-k}^{\infty} \sum |\hat{\varphi}_m(n)|^2 = 1 \text{ if } k = l, \\ \sum_{n=-k}^{\infty} \sum \hat{\varphi}_m(n) \overline{\hat{\varphi}_m(n+k-l)} = 0 \text{ if } k \neq l \end{cases}$$

for all positive integers k, l. Hence $T_{\sum \varphi_m} T^*_{\sum \varphi_m} = I$ and therefore $T_{\sum \varphi_m}$ is isometric and unitary. If $T_{\sum \varphi_m} = T^*_{\sum \varphi_m}$ there is self-adjoint, if $\|T_{\sum \varphi_m}\| \leq 1$ then it is a contraction and if $T^*_{\sum \varphi_m} T_{\sum \varphi_m} - T_{\sum \varphi_m} T^*_{\sum \varphi_m} \geq 0$ then it is hyponormal.

Corollary 2.13. (see[15]) for $\varphi_m(z) = \sum_{n=-\infty}^{\infty} \sum \hat{\varphi}_m(n) z^n \in L^\infty$, let $T_{\sum \varphi_m}$ be a Toeplitz operator on H^2. Then the following arguments hold:

(i) If $\hat{\varphi}_m(-n) = \hat{\varphi}_m(n)$ for all $n \geq 1$ and $\hat{\varphi}_m(k) = e^{i\theta} \hat{\varphi}_m(k)$for some positive integer k and for all θ, then $T_{\sum \varphi_m}$ is nonnormal and complex symmetric.

(ii) If $\hat{\varphi}_m(-n) = \hat{\varphi}_m(n)(-1)^n$ for all $n \geq 1$ and $(-1)^n \hat{\varphi}_m(k) = e^{i\theta} \hat{\varphi}_m(k)$for some positive integer k and for all θ, then $T_{\sum \varphi_m}$ is nonnormal and complex symmetric.

Proof. The proof follows from Corollaries 2.6 and 2.11.

We now investigate complex symmetric Toeplitz operators with an arbitrary conjugation C on H^2. Moreover, we compare an arbitrary conjugation C of a complex symmetric operator $T_{\sum \varphi_m}$ with $C_{\xi,\theta}$.

Theorem 2.14. [15] let φ_m be in L^∞ such that $\sum \varphi_m = \sum(\varphi_m)_+ + \sum(\varphi_m)_0 + \sum(\varphi_m)_-$. If C is a conjugation on H^2 and $T_{\sum \varphi_m}$ is a complex symmetric operator with conjugation C, then

$$\sum_{i=0}^{k-1} \sum \left[\hat{\varphi}_m(k-i) a_i^m - \hat{\varphi}_m(-(k-i)) \overline{\widetilde{a_k^m}} \gamma_k \right]$$

$$= \sum_{n=1}^{\infty} \sum \left[\hat{\varphi}_m(n) \overline{\widetilde{a_{n+k}^m}} \gamma_k - a_{n+k}^m \hat{\varphi}_m(-n) \right] \qquad (9)$$

for all k where $a_k^m = \langle f_m, z^k \rangle$, $\widetilde{a_k^m} = \langle Cf_m, z^k \rangle$, and $\gamma_k = \langle Cz^j, z^k \rangle$ for all $f_m \in H^2$.

Proof. Let C be a conjugation on H^2 and $T_{\sum \varphi_m}$ be a complex symmetric operator with conjugation C. Direct calculation shows that

$$(\varphi_m)_+ f_m + P(\overline{\varphi_{m\_}} f_m) = CP(\overline{(\varphi_m)_+} Cf_m) + C((\varphi_m)_- Cf_m) \qquad (10)$$

where P denotes the orthogonal projection of L^2 onto H^2. Now, put $f_m(z) = \sum_{j=0}^{\infty} \sum a_j^m z^j \in H^2$ and $Cf_m(z) = \sum_{j=0}^{\infty} \sum \widetilde{a_j^m} z^j \in H^2$ where $a_j^m = \sum \langle f_m, z^j \rangle$ and $\widetilde{a_j^m} = \sum \langle Cf_m, z^j \rangle$. Since $(\varphi_m)_+(z) = \sum_{n=1}^{\infty} \sum \hat{\varphi}_m(n) z^n$ and $(\varphi_m)_-(z) = \sum_{n=1}^{\infty} \sum \overline{\hat{\varphi}_m(-n)} z^n$, it follows that

$$\begin{cases} (\varphi_m)_+ f_m = \sum_{k=1}^{\infty} \left[\sum_{i=0}^{k-1} \sum \hat{\varphi}_m(k-i) a_i^m \right] z^k, \\ P(\varphi_{m\_} f_m) = \sum_{k=1}^{\infty} \left[\sum_{n=1}^{\infty} \sum a_{n+k}^m \hat{\varphi}_m(-n) \right] z^k, \\ P((\varphi_m)_+ Cf_m) = \sum_{k=1}^{\infty} \left[\sum_{n=1}^{\infty} \sum \widetilde{a_{n+k}^m} \hat{\varphi}_m(n) \right] z^k, \\ ((\varphi_m)_- Cf_m) = \sum_{k=1}^{\infty} \left[\sum_{i=0}^{k-1} \sum \overline{\hat{\varphi}_m(-(k-i))} \widetilde{a_i^m} \right] z^k. \end{cases} \qquad (11)$$

Equation (10) implies that

$$\sum \langle \varphi_{m+} f_m(z), z^j \rangle + \sum \langle P\left(\overline{\varphi_{m\_}} f_m(z)\right), z^j \rangle = \sum \langle CP\left(\overline{\varphi_{m+}} Cf_m(z)\right), z^j \rangle + \sum \langle C\left((\varphi_m)_- Cf_m(z)\right), z^j \rangle \qquad (12)$$

for all j. Next, from (11), equation (12) becomes

$$\sum_{i=0}^{j-1} \sum \hat{\varphi}_m(j-i) a_i^m + \sum_{n=1}^{\infty} a_{n+j}^m \hat{\varphi}(-n)$$

$$= \sum \langle CP(\overline{\varphi_m} + Cf_m(z)), z^j \rangle + \sum \langle C\left(\varphi_{m\_} Cf_m(z)\right), z^j \rangle$$

$$= \sum \langle Cz^j, P(\overline{\varphi_m} + Cf_m(z)) \rangle + \sum \langle Cz^j, \varphi_{m\_} Cf_m(z) \rangle \qquad (13)$$

for all j. Since $\{1, z, z^2, \dots\}$ is an orthonormal basis for $H^2, \{C_1, Cz, Cz^2, \dots\}$ is also an orthonormal basis for H^2. Moreover, since $Cz^j \in H^2$, set $Cz^j = \sum_{s=0}^{\infty} \gamma_s z^s$ where $\sum_{s=0}^{\infty} |\gamma_s|^2 = 1$. Then equation (13) becomes

$$\sum_{i=0}^{k-1} \sum \hat{\varphi}_m(k-i) a_i^m + \sum_{n=1}^{\infty} \sum a_{n+k}^m \hat{\varphi}_m(-n)$$

$$= \sum \langle \sum_{s=0}^{\infty} \gamma_s z^s, \sum_{k=1}^{\infty} \left[\sum_{n=1}^{\infty} \sum \widetilde{a_{n+k}^m} \hat{\varphi}_m(n) \right] z^k \rangle$$

$$+ \langle \sum_{s=0}^{\infty} \gamma_s z^s, \sum_{k=1}^{\infty} \left[\sum_{i=0}^{k-1} \sum \overline{\hat{\varphi}_m(-(k-i))}\, \widetilde{a_k^m} \right] z^k \rangle$$

$$= \sum_{n=1}^{\infty} \sum \hat{\varphi}_m(n) \overline{\widetilde{a_{n+k}^m}}\, \gamma_k + \sum_{i=0}^{k-1} \sum \hat{\varphi}_m(-(k-i)) \widetilde{a_k^m} \gamma_k \qquad (14)$$

for all k. Hence we conclude that

$$\sum_{i=0}^{k-1} \sum \left[\hat{\varphi}_m(k-i) a_i^m - \hat{\varphi}_m(-(k-i)) \overline{\widetilde{a_k^m}}\, \gamma_k \right]$$

$$= \sum_{n=1}^{\infty} \sum \left[\hat{\varphi}_m(n) \overline{\widetilde{a_{n+k}^m}} \gamma_k - a_{n+k}^m \hat{\varphi}_m(-n) \right]$$

for all k where $a_k^m = \sum \langle f_m, z^k \rangle$, $\widetilde{a_k^m} = \sum \langle C f_m, z^k \rangle$, and $\gamma_k = \langle Cz^j, z^k \rangle$ for all $f_m \in H^2$.

Corollary 2.15. (see [15]) under the same hypotheses as in Theorem 2.14, if $T_{\sum \varphi_m}$ is a complex symmetric operator with the conjugation $C_{\xi,\theta}$, then

$$\sum_{i=0}^{k-1} \sum \left[\lambda^{k-i} \hat{\varphi}_m(-(k-i)) - \hat{\varphi}_m(k-i) \right] a_i^m$$

$$= \sum_{n=1}^{\infty} \sum \left[\hat{\varphi}_m(-n) - \overline{\lambda^n} \hat{\varphi}_m(n) \right] a_{n+k}^m, \quad (15)$$

where $|\lambda| = 1$.

Proof. Let $h_m(z) = \sum_{k=1}^{\infty} \sum \alpha_m z^k$ and $g_m(z) = \sum \sum_{k=1}^{\infty} \beta_m z^k$, where

$$\alpha_m = \sum_{n=1}^{\infty} \sum \widetilde{a_{n+k}^m} = \sum \hat{\varphi}_m(n) \text{ and } \beta_m = \sum_{i=0}^{k-1} \sum \hat{\varphi}_m\big(-(k-i)\big)\widetilde{a_i^m}.$$

Then $Ch_m(z) = \sum_{k=1}^{\infty} \sum \tilde{\alpha}_m z^k$ and $Cg_m(z) = \sum_{k=1}^{\infty} \sum \tilde{\beta}_m z^k$. If $C = C_{\xi,\theta}$ in Theorem 2.14, then $\widetilde{a_j^m} = \mu\lambda_j a_j^m$ with $|\mu| = |\lambda| = 1$ and so

$$\alpha_m = \sum_{n=1}^{\infty} \sum \mu\lambda^{n+k}\overline{a_{n+k}^m}\hat{\varphi}_m(n) = \text{and } a_k^m \sum_{i=0}^{k-1} \sum \mu\lambda^i \overline{\hat{\varphi}_m\big(-(k-i)\big)}a_i^m.$$

Then $\widetilde{\alpha_m} = \langle\Big(C_{\xi,\theta} \sum_{k=1}^{\infty} \Big[\sum_{n=1}^{\infty} \sum \mu\lambda^{n+k}\overline{a_{n+k}^m\hat{\varphi}_m(n)}\Big] z^k\Big), z^k\rangle$

$$= \langle C_{\xi,\theta} \sum_{k=1}^{\infty} \left[\sum_{n=1}^{\infty} \sum \mu\lambda^{n+k}\overline{a_{n+k}^m\hat{\varphi}_m(n)}\right] z^k\rangle$$

$$= \langle \mu\lambda^k z^k, \sum_{k=1}^{\infty} \left[\sum_{n=1}^{\infty} \sum \mu\lambda^{n+k}\overline{a_{n+k}^m\hat{\varphi}_m(n)}\right] z^k\rangle = \sum_{n=1}^{\infty} \sum \overline{\lambda}^n a_{n+k}^m\hat{\varphi}_m(n)$$

and, similarly, $\tilde{\beta}_m = \sum_{i=0}^{k-1} \sum \lambda^{k-i} \hat{\varphi}_m\big(-(k-i)\big)a_i^m$.

Now, equation (9) implies that

$$\sum_{i=0}^{k-1} \sum \left[\lambda^{k-i} \hat{\varphi}_m\big(-(k-i)\big) - \hat{\varphi}_m(k-i)\right] a_i^m$$

$$= \sum_{n=1}^{\infty} \sum \left[\hat{\varphi}_m(-n) - \overline{\lambda}^n \hat{\varphi}_m(n)\right] a_{n+k}^m$$

where $|\lambda| = 1$. Since $T_{\sum \varphi_m}$ is complex symmetric with the conjugation $C_{\xi,\theta}$, we know from Theorem 2.4 that equation (15) always holds.

Remark 2.16. If we put $C_{\xi,\theta} = C_1$ in Corollary 2.15, then $\widetilde{a_j^m} = a_i^m, \widetilde{a_k^m} = \sum_{n=1}^{\infty} \sum a_{n+k}^m \hat{\varphi}_m(n)$, and $\tilde{\beta}_m = \sum_{i=0}^{k-1} \sum \hat{\varphi}_m\big(-(k-i)\big)a_i^m$. Then equation (9) implies that

$$\sum_{i=0}^{k-1} \sum \left[\hat{\varphi}_m\big(-(k-i)\big) - \hat{\varphi}_m(k-i)\right] a_i^m$$

$$= \sum_{n=1}^{\infty} \sum \left[\hat{\varphi}_m(-n) - \hat{\varphi}_m(n)\right] a_{n+k}^m \qquad (16)$$

Since $T_{\sum \varphi_m}$ is complex symmetric with the conjugation C_1, we know from Corollary 2.6 that equation (16) always holds.

Example 2.17. [15] If T_z is a unilateral shift on H^2, then T_z is not a complex symmetric sequence of Toeplitz operator with conjugation C_1. Indeed, since $\varphi_m(z) = z$, it follows that $\sum \hat{\varphi}_m(1) = 1$ and $\sum \hat{\varphi}_m(n) = 0$ for all $n = 1$. Then we obtain from (16) that

$$\sum_{i=0}^{k-1} \sum \left[\hat{\varphi}_m\left(-(k-i)\right) - \hat{\varphi}_m(k-i)\right] a_i^m - \sum_{n=1}^{\infty} \sum [\hat{\varphi}_m(-n) - \hat{\varphi}_m(n)] a_{n+k}^m$$

$$= \sum \left[\hat{\varphi}_m\left(-(k)\right) - \hat{\varphi}_m(k)\right] a_0^m + \cdots + \sum [\hat{\varphi}_m(-1) - \hat{\varphi}_m(1)] a_{k-1}^m$$

$$- \sum [\hat{\varphi}_m(-1) - \hat{\varphi}_m(1)] a_{1+k}^m - \sum [\hat{\varphi}_m(-2) - \hat{\varphi}_m(2)] a_{2+k}^m - \cdots$$

$$= \sum \hat{\varphi}_m(1)(a_{k+1}^m - a_{k-1}^m) \neq 0$$

for some k. Thus (16) does not hold.

3. Complex symmetric sequence of Toeplitz operators with a finite symbol

We examine complex symmetric sequence of Toeplitz operators with finite symbols (see[15]).

Theorem 3.1. Let $\varphi_m(z) = \sum_{n=-m_0}^{N} \sum a_k^m z^n$ where $N \geq m_0 > 0$ and $a_n^m \in \mathbb{C}$ with nonzero m_{-m_0}, m_N . Then $T_{\sum \varphi_{m_0}}$ is complex symmetric with the conjugation $C_{\xi,\theta}$ if and only if $m_0 = N$ and for some $\theta, a_{-n}^m = a_n^m e^{in\theta}$ for all $n = 1, 2, \ldots, N$. In particular, $T_{\sum \varphi_m}$ is normal if and only if $a_{-m_0}^m = a_{m_0}^m e^{im_0\theta}$ and $\overline{a_{-m_0}^m} a_k^m = e^{i(m_0-k)\theta} a_{m_0}^m \overline{a_k^m}$ for all $k = 1, 2, \ldots, m_0 - 1$.

Proof. Let $C_{\xi,\theta}(\sum f_m(z)) = e^{i\xi} \sum f_m(e^{i\theta} z)$. Then $C_{\xi,\theta}(\sum_{k=0}^{\infty} \sum a_k^m z^k) = \mu \sum_{k=0}^{\infty} \sum \lambda^k a_k^m z^k$ for all $\sum_{k=0}^{\infty} \sum a_k^m z^k \in H^2$ with $|\lambda| = |\mu| = 1$. Suppose that $T_{\sum \varphi_m}$ is complex symmetric with the conjugation $C_{\xi,\theta}$. Since

$$C_{\xi,\theta}\left(\sum T_{\sum\varphi_m} z^k\right) = C_{\xi,\theta} P\left(\sum_{n=-m_0}^{N} \sum a_n^m z^{n+k}\right) = C_{\xi,\theta}\left(\sum_{n=-k}^{N} \sum a_n^m z^{n+k}\right)$$

$$= \mu \sum_{n=-k}^{N} \sum a_n^m \lambda^{n+k} z^{n+k} \qquad (17)$$

and

$$T^*_{\sum\varphi_m} C_{\xi,\theta} z^k = \mu T_{\sum\varphi_m}(\lambda^k z^k) = \mu P\left(\sum_{n=-m_0}^{N} \sum a_n^m \lambda^k z^{-n+k}\right)$$

$$= \mu P\left(\sum_{n=-N}^{m_0} \sum a_{-n}^m \lambda^k z^{n+k}\right) = \sum_{n=-k}^{m_0} \sum \overline{a_{-n}^m} \lambda^k z^{n+k}, \qquad (18)$$

it follows that $N = m_0$ and $a_{-n}^m = a_n^m \lambda^n$ for all $n = 1, 2, \ldots, N$ with $|\lambda| = 1$.

Conversely, we assume that $N = m_0$ and $a_{-n}^m = a_n^m \lambda^n$ for all $n = 1, 2, \ldots, N$ with $|\lambda| = 1$. Let $\varphi_m(z) = \sum_{n=-m_0}^{N} \sum a_n^m z^n$. Then

$$(\varphi_m)_+(z) = \sum_{n=1}^{N} \sum a_n^m z^n \text{ and } (\varphi_m)_-(z) = -\sum_{n=-m_0}^{1} \overline{a_n^m} z^n = \sum_{n=1}^{m_0} \sum a_{-n}^m z^n.$$

Since $N = m_0$ and $a_{-n}^m = a_n^m \lambda^n$ for all $n = 1, 2, \ldots, N$ with $|\lambda| = 1$, it follows from (17) and (18) that

$$\sum \left[C_{\xi,\theta}\, T_{\sum\varphi_m} - T^*_{\sum\varphi_m}\, C_{\xi,\theta}\right] z^k = \sum_{n=-k}^{N} \sum \overline{a_n^m} \lambda^{n+k} z^{n+k} - \sum_{n=-k}^{m_0} a_{-n}^m \lambda^k z^{n+k}$$

$$= \sum_{n=-k}^{N} \sum \left[a_n^m \lambda^{n+k} - a_{-n}^m \lambda^k\right] z^{n+k}$$

$$= \sum_{n=-k}^{N} \sum \left[a_n^m \lambda^{n+k} - a_n^m \lambda^{n+k}\right] z^{n+k} = 0.$$

Hence $T_{\sum\varphi_m}$ is complex symmetric with the conjugation $C_{\xi,\theta}$. For a trigonometric polynomial φ_m, it is well known from [3, Corollary 1.5] that $T_{\sum\varphi_m}$ is normal if and only if $|a_{-m_0}^m| = \left|a_{m_0}^m\right|$ with $a_{m_0}^m \neq 0$ and

$$a_{m_0}^m \begin{pmatrix} a_{-1} \\ a_{-2} \\ \vdots \\ a_{-m} \end{pmatrix}^m = a_{-m_0}^m \begin{pmatrix} \overline{a_1} \\ \overline{a_2} \\ \vdots \\ \overline{a_m} \end{pmatrix}^m \tag{19}$$

Since $T_{\sum \varphi_m}$ is complex symmetric with the conjugation $C_{\xi,\theta}$, it follows from the first statement that $a_{-n}^m = a_n^m e^{in\theta}$ for all $n = 1, 2, \ldots, m_0$. Hence from (19), we conclude that $T_{\sum \varphi_m}$ is normal if and only if $a_{-m_0}^m = e^{im_0\theta} \sum a_{m_0}^m$ and $\overline{a_{m_0}^m} a_k^m = e^{i(m_0-k)\theta} \sum a_{m_0}^m \overline{a_k^m}$ for all $k = 1, 2, \ldots, m_0 - 1$.

Corollary 3.2. (see[15]) let $\varphi_m(z) = \sum_{n=-m_0}^{m_0} \sum a_n^m z^n$ for $a_n^m \in \mathbb{C}$ with nonzero $a_{-m_0}^m, a_m$. If $T_{\sum \varphi_m}$ is complex symmetric with the conjugation $C_{\xi,\theta}$, then $T_{\sum \varphi_m}$ is unitary if and only if $a_{-m_0}^m = \sum e^{im_0\theta} a_{m_0}^m$, $\sum \overline{a_{m_0}^m} a_k^m = \sum e^{i(m_0-k)\theta} a_{m_0}^m \overline{a_k^m}$, and

$$\sum_{n=-k}^{m_0} \sum a_n^m \overline{a_{n+k-l}^m} = \sum_{n=-k}^{m_0} \sum a_{-n}^m \overline{a_{-(n+k-l)}^m} = \begin{cases} 1 \text{ if } k = l, \\ 0 \text{ if } k \neq l \end{cases}$$

for all $k, l = 1, 2, \ldots, m_0$.

Proof. Assume that $T_{\sum \varphi_m}$ is complex symmetric with the conjugation $C_{\xi,\theta}$. If $T_{\sum \varphi_m}$ is unitary, then $a_{-m_0}^m = \sum a_{m_0}^m e^{im_0\theta}$ and $\overline{a_{m_0}^m} a_k^m = \sum e^{i(m_0-k)\theta} a_{m_0}^m \overline{a_k^m}$ for all $k = 1, 2, \ldots, m_0 - 1$ from Theorem 3.1. Since $T_{\sum \varphi_m}^* T_{\sum \varphi_m} = I$, it follows that

$$1 = \sum \langle T_{\sum \varphi_m}^* T_{\sum \varphi_m} z^k, z^k \rangle = \sum \langle P\left(\sum_{n=-m_0}^{m} \sum a_n^m z^{n+k} \right), P\left(\sum_{j=-m_0}^{m} a_j^m z^{j+k} \right) \rangle$$

$$= \sum \langle \sum_{n=-k}^{m_0} a_n^m z^{n+k}, \sum_{j=-k}^{m_0} a_j^m z^{j+k} \rangle = \sum_{n=-k}^{m_0} \sum |a_n^m|^2 \tag{20}$$

and, if $k \neq l$, then

$$0 = \sum \langle T_{\sum \varphi_m}^* T_{\sum \varphi_m} z^k, z^l \rangle = \sum \langle P\left(\sum_{n=-m_0}^{m} a_n^m z^{n+k} \right), P\left(\sum_{j=-m_0}^{m} a_j^m z^{j+l} \right) \rangle$$

$$= \sum \langle \sum_{n=-k}^{m_0} a_n^m z^{n+k}, \sum_{j=-l}^{m_0} a_j^m z^{j+l} \rangle = \sum_{n=-k}^{m} \sum a_n^m \overline{a_{(n+k-l)}^m} \tag{21}$$

for all $k, l = 1, 2, \ldots, m_0$. Conversely, if $a^m_{-m_0} = \sum e^{im_0\theta} a^m_{m_0}$, $\overline{a^m_{m_0}} a^m_k = \sum e^{i(m_0-k)\theta} a^m_{m_0} \overline{a^m_k}$, and

$$\sum_{n=-k}^{m_0} \sum a^m_n \overline{a^m_{(n+k-l)}} = \sum_{n=-k}^{m_0} \sum a^m_{-n} \overline{a^m_{-(n+k-l)}} = \begin{cases} 1 \text{ if } k = l, \\ 0 \text{ if } k \neq l \end{cases}$$

for all $k, l = 1, 2, \ldots, m_0$, then $T^*_{\sum \varphi_m} T_{\sum \varphi_m} = T_{\sum \varphi_m} T^*_{\sum \varphi_m}$ from Theorem 3.1. Moreover, we have

$$\sum \langle T_{\sum \varphi_m} T^*_{\sum \varphi_m} z^k, z^l \rangle = \sum \langle \sum_{n=-k}^{m_0} a^m_n z^{n+k}, \sum_{j=-l}^{m_0} a^m_j z^{j+l} \rangle$$

$$= \sum_{n=-k}^{m_0} \sum a^m_n \langle z^{n+k}, \sum_{j=-l}^{m_0} a^m_j z^{j+l} \rangle = \begin{cases} \sum_{n=-k}^{m_0} |a^m_n|^2 = 1 \text{ if } k = l, \\ \sum_{n=-k}^{m_0} a^m_n \overline{a^m_{(n+k-l)}} = 0 \text{ if } k \neq l \end{cases}$$

for all $k, l = 1, 2, \ldots, m_0$. Hence $T_{\sum \varphi_m} T^*_{\sum \varphi_m} = I$. Similarly, we obtain that

$$\sum \langle T_{\sum \varphi_m} T^*_{\sum \varphi_m} z^k, z^l \rangle = \begin{cases} \sum_{n=-k}^{m_0} \sum |a^m_{-n}|^2 = 1 \text{ if } k = l, \\ \sum_{n=-k}^{m_0} \sum a^m_{-n} \overline{a^m_{-(n+k-l)}} = 0 \text{ if } k \neq l \end{cases}$$

for all $k, l = 1, 2, \ldots, m_0$. Hence $T_{\sum \varphi_m} T^*_{\sum \varphi_m} = I$ and therefore $T_{\sum \varphi_m}$ is unitary.

Corollary 3.3. (see[15]) let $\varphi_m(z) = \sum_{n=-m_0}^{m} \sum a^m_n z^n$ for $a^m_n \in \mathbb{C}$ with nonzero $a^m_{-m_0}, a^m_{m_0}$. If $T_{\sum \varphi_m}$ is complex symmetric with the conjugations C_1 and C_2 and it is normal, then $\overline{a^m_{m_0}} a^m_k = a^m_{m_0} \overline{a^m_k}$ when $k + m_0$ is even, $a^m_k = 0$ for $k = 2, 4, 6, \ldots, m_0 - 1$ when m_0 is odd, or $a^m_k = 0$ for $k = 1, 3, 5, \ldots, m_0 - 1$ when m is even.

Proof. Since $a^m_{m_0} \overline{a^m_k} = (-1)^{m_0+k} a^m_{m_0} \overline{a^m_k}$ by Theorem 3.1, it follows that $m_0 + k$ is even or $a^m_k = 0$ when $m_0 + k$ is odd. If $k + m_0$ is even, we get that $\overline{a^m_{m_0}} a^m_k = a^m_{m_0} \overline{a^m_k}$. Now assume that $k + m_0$ is odd. If m_0 is odd, then k is even and so $\sum (1 + (-1)^k) \overline{a^m_{m_0}} a^m_k = 0$. Since $a^m_{m_0}$ is nonzero, this implies that $a^m_k = 0$ for $k = 2, 4, 6, \ldots, m_0 - 1$. If m_0 is even,

then k is odd and hence $\sum(1-(-1)^k)\overline{a_{m_0}^m}a_k^m = 0$. Since $a_{m_0}^m \neq 0$, it follows that $a_k^m = 0$ for $k = 1,3,5,\ldots,m_0 - 1$.

Using Theorem 3.1, we give examples of nonnormal and complex symmetric sequence of Toeplitz operators with a finite symbol (see[15]).

Example 3.4. Suppose that $\varphi_m(z) = iz^3 + z + \overline{z} + i\overline{z}^3$ and $\psi_m(z) = 2z^2 + z + i - \overline{z} + 2\overline{z}^2$. Then the matrices for Toeplitz operators $T_{\sum\varphi_m}$ and $T_{\sum\psi_m}$ with respect to the basis $B = \{z^n : n = 0,1,2,\ldots\}$ are given by

$$[T_{\sum\varphi_m}]_B = \begin{pmatrix} 0 & 1 & 0 & i & 0 & \ddots \\ 1 & 0 & 1 & 0 & i & \ddots \\ 0 & 1 & 0 & 1 & 0 & \ddots \\ i & 0 & 1 & 0 & 1 & \ddots \\ 0 & i & 0 & 1 & 0 & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix},$$

$$[T_{\sum\psi_m}]_B = \begin{pmatrix} i & -1 & 2 & 0 & 0 & \ddots \\ 1 & i & -1 & 2 & 0 & \ddots \\ 2 & 1 & i & -1 & 2 & \ddots \\ 0 & 2 & 1 & i & -1 & \ddots \\ 0 & 0 & 2 & 1 & i & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

Hence, by Theorem 3.1, $T_{\sum\varphi_m}$ and $T_{\sum\psi_m}$ are complex symmetric with respect to the conjugations C_1 and C_2, respectively. But, since $\overline{a_3^m}a_{-1}^m - a_{-3}^m\overline{a_1^m}$ and $\overline{a_2^m}a_{-1}^m - a_{-2}^m a_1^m$ are nonzero, both $T_{\sum\varphi_m}$ and $T_{\sum\psi_m}$ are not normal from Theorem 3.1 or [3].

Example 3.5. (see[15]) let $\varphi_m(z) = e^{i\theta}\left(3z^{n+1} + z^n + \overline{z^n} + 3\overline{z}^{n+1}\right)$ for some θ. Then $T_{\sum\varphi_m}$ is complex symmetric and normal from Theorem 3.1.

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