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New Separation Axioms in Soft Topological Ordered Spaces

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Abstract

In this paper, we deal with the ordered soft separation axioms using soft points on soft topological ordered space and discuss the characterizations and properties of them. Also, we prove some significant results which associate both types of the introduced ordered axioms with some notions such as soft topological and soft hereditary properties. Finally, some examples and properties are given to explore differences between separation axioms given in [3]. In fact these separation axioms are stronger than other separation axioms.

Keywords: Soft set, soft point and ordered soft separation axioms.

Mathematics Subject Classification (2010): 54A10, 54B05, 54D10, 54D15, 54D30.

1 Introduction

In 1965, Nachbin [15] defined a topological ordered space by adding a partial order relation to the structure of a topological space. So it can be considered that the topological ordered spaces are one of the generalizations of the topological spaces. McCartan [11] utilized monotone neighborhoods to introduce and study ordered separation axioms.

For the vagueness and uncertainty of real life problems, there are several mathematical tools such as fuzzy sets, intuitionistic fuzzy sets, rough sets, vague sets etc. There is one more mathematical tool named soft sets which was introduced by Molodsov [14] in 1999. After that it was developed and used in decision making problems by Maji et. al in [13] and [12]. Aktas and Cagman [1] introduced the applications of soft set theory in algebraic structures in 2007. Shabir and Naz [18] investigated soft separation axioms defined for crisp points in 2011. Hussain and Ahmad [6] investigate the properties of soft interior, soft closure and soft boundary in 2011. Nazmul and Samanta [16] studied the neighbourhood properties of soft topological spaces in 2013. Four different types of separation axioms were defined and discussed in [8], [7], [18] and [21]. Singh and Noorie [19] derives the relation among these four types of $T_i, i = 1, 2, 3, 4$ spaces in 2017. In 2020 R. Solai [20] introduced the soft separation axioms using soft points on soft topological space and discuss the characterizations and properties of them.

El-Shafei et al. [3], [4] established two new soft relations, namely partial belong and total non belong. Depend on them and initiated a concept of soft topological ordered space. They introduced the notions of ordered soft separation axioms, namely $P-$soft T_i-ordered, $i = 0, 1, 2, 3, 4$.

The present paper is organized as follows: Section 2 is the preliminary part where some definitions and properties of soft sets and soft topologies are given. In Section 3, we introduce the notion $x_{e_1} \not\leq y_{e_2}$, which are more effective to theoretical and application studies on soft topological spaces, and derive some results related to them. Then, we employ this new relation to introduce new soft separation axioms, namely p^*-soft T_i-spaces

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$(i = 0; 1; 2; 3; 4)$. We point out the relationships among them and present several of their properties. One of the most important point in this section is defining a p^*-soft regular spaces concept, which is weaker than $p-$soft regular space [3], and deeply studying its properties.

2 Preliminaries

In this section, we briefly review some concepts and some related results of soft set, soft points, soft topological space and soft topological ordered space which are needed to use in current paper. For more details about these concepts you can see in [[3], [10], [18]].
Henceforth, X denotes the universe set, E denotes the fixed set of parameters and 2^X denotes the power set of X.

Definition 2.1 *[14] A pair (G, E) is said to be a soft set over X, where $G : E \longrightarrow 2^X$.*

Remark 2.1 *1. For short, we use the notation G_E instead of (G, E),*

2. A soft set can be defined as a set of ordered pairs, $G_E = \{(\alpha, G(\alpha)) : \alpha \in E \text{ and } G(\alpha) \in 2^X\}$.

The family of all soft sets over X denoted by $P(X)^E$.

Definition 2.2 *[13] Let $G_E \in P(X)^E$. Then:*

1. *G_E is called a null soft set, denoted by $\widehat{\phi}$, if $G(\alpha) = \emptyset$, $\forall \alpha \in E$.*
2. *G_E is called an absolute soft set, denoted by X_E, if $G(\alpha) = X$, $\forall \alpha \in E$.*

Definition 2.3 *[2] Let $G_E, N_E \in P(X)^E$. Then:*

1. *N_E is called a soft subset of G_E, denoted by $N_E \sqsubseteq G_E$, if $N(\alpha) \subseteq G(\alpha)$, $\forall \alpha \in E$.*
2. *N_E and G_E are called equal, denoted by $N_E = G_E$, if $N_E \sqsubseteq G_E$ and $G_E \sqsubseteq N_E$.*
3. *The union of N_E and G_E is a soft set H_E, denoted by $N_E \sqcup G_E$, defined by $H(\alpha) = N(\alpha) \cup G(\alpha)$, $\forall \alpha \in E$.*
4. *The intersection of N_E and G_E is a soft set H_E, denoted by $N_E \sqcap G_E$, defined by $H(\alpha) = N(\alpha) \cap G(\alpha)$, $\forall \alpha \in E$.*

Definition 2.4 *[18] Let $G_E, N_E \in P(X)^E$. Then:*

1. *The difference of N_E and G_E is a soft set H_E, denoted by $H_E = N_E - G_E$, defined by $H(\alpha) = N(\alpha) - G(\alpha)$, $\forall \alpha \in E$.*
2. *The complement of N_E, denoted by N_E^c, defined by $N^c(\alpha) = (N(\alpha))^c$, $\forall \alpha \in E$.*

Definition 2.5 *[16] The soft set $N_E : E \to 2^X$ given by:*

$$N(e) = \begin{cases} \{a\}, & \text{if } e = \alpha; \\ \phi, & \text{if } e \in E - \{\alpha\}. \end{cases}$$

is called a soft point and denoted by a^α.
The family of all soft sets over X denoted by $Sp(X)_E$.

Definition 2.6 *[17] A soft point a^α is said to be belonging to the soft set N_E, denoted by $a^\alpha \widehat{\in} N_E$, if for the member $\alpha \in E$, $a^\alpha(\alpha) \subseteq N(\alpha)$.*

Definition 2.7 *[3, 14] For a soft set N_E over X and $a \in X$, we say*

1. $a \in N_E$ *if* $a \in N(\alpha)$, *for each* $\alpha \in E$ *and* $a \notin N_E$ *if* $a \notin N(\alpha)$, *for some* $\alpha \in E$.

2. $a \Subset N_E$ *if* $a \in N(\alpha)$, *for some* $\alpha \in E$ *and* $a \notin N_E$ *if* $a \notin N(\alpha)$, *for each* $\alpha \in E$.

The notations $\in, \notin, \Subset$ *and* $\notin$ *are respectively read belong, non-belong, partial belong and total non-belong relations.*

Definition 2.8 *[18] A soft topology on* X *is a collection* τ *of soft sets over* X *under* E *satisfying the following axioms:*

1. $\widehat{\phi}$ *and* X_E *belong to* τ,

2. *The union of any member of soft sets in* τ *belongs to* τ,

3. *The intersection of any two soft sets in* τ *belongs to* τ.

The triple (X, τ, E) *is said to be a soft topological space over* X. *Every member of* τ *is called soft open and its relative complement is called soft closed.*

Definition 2.9 *[22] A soft subset* W_E *of a soft topological space* (X, τ, E) *is called soft neighbourhood of* $x_e \in Sp(X)_E$, *if there exists an open soft set* N_E *such that* $x_e \widehat{\in} N_E \sqsubseteq W_E$.

Definition 2.10 *[18] Let* (X, τ, E) *be a soft topological space and* $F_E \in P(X)^E$. *Then, the soft closure of* F_E, *denoted by* $scl(F_E)$, *defined by:* $scl(F_E) = \sqcap\{G_E : F_E \sqsubseteq G_E$ *and* G_E *is a closed soft set in* $X\}$.

Definition 2.11 *[6] Let* (X, τ, E) *be a soft topological space and* $F_E \in P(X)^E$. *Then, the soft interior of* F_E, *denoted by* $sint(F_E)$, *defined by:* $sint(F_E) = \sqcup\{G_E : G_E \sqsubseteq F_E$ *and* $G_E \in \tau\}$.

Definition 2.12 *[10] A binary relation* $\leq$ *on* X *is said to be a partial order relation if* $\leq$ *is reflexive, anti-symmetric and transitive.*
$\{(a, a) :$ *for every* $a \in X\}$ *is the equality relation on* X, *and it is denoting by* $\blacktriangle$.

Definition 2.13 *[15] A triple* $(X, \tau, \leq)$ *is said to be a topological ordered space if* (X, τ) *is a topological space and* $(X, \leq)$ *is a partially ordered set.*

Definition 2.14 *[3] A triple* $(X, E, \leq)$ *is said to be a partially ordered soft space, where* $\leq$ *is a partial order relation on* X.

Definition 2.15 *[3] Let* $(X, E, \leq)$ *be a partially ordered soft space. The definition of an increasing soft operator* $i : (P(X)^E, \leq) \to (P(X)^E, \leq)$ *and a decreasing soft operator* $d : (P(X)^E, \leq) \to (P(X)^E, \leq)$ *as follows, for each soft set* N_E *in* $P(X)^E$:

1. $i(N_E) = (iN)_E$, *where* iN *is a mapping of* E *into* X *given by*
   $iN(\alpha) = i(N(\alpha)) = \{a \in X : b \leq a$, *for some* $b \in N(\alpha)\}$.

2. $d(N_E) = (dN)_E$, *where* dN *is a mapping of* E *into* X *given by*
   $dN(\alpha) = d(N(\alpha)) = \{a \in X : a \leq b$, *for some* $b \in N(\alpha)\}$.

Definition 2.16 *[3] A soft subset* N_E *of a partially ordered soft space* $(X, E, \leq)$ *is increasing (resp. decreasing) if* $N_E = i(N_E)$ (*resp.* $N_E = d(N_E)$).

Definition 2.17 *[3] A quadrable system* $(X, \tau, E, \leq)$ *is said to be a soft topological ordered space (briefly,* $STOS$*) if* (X, τ, E) *is a soft topological space and* $(X, E, \leq)$ *is a partially ordered soft space.*

Definition 2.18 *[3] A soft subset W_E of an $STOS(X, \tau, E, \leq)$ is called increasing (resp. decreasing) soft neighbourhood of $x \in X$ if W_E is soft neighbourhood of x and increasing (resp. decreasing).*

Definition 2.19 *[3] An STOS $(X, \tau, E, \leq)$ is said to be:*

1. *Lower (resp. upper)$P-$soft T_1-ordered if for every distinct points $x, y \in X$ there exists an increasing (resp. a decreasing) soft neighbourhood W_E of x such that $y \notin W_E$.*
2. *$P-$soft T_0-ordered if it is Lower $P-$soft T_1-ordered or Upper $P-$soft T_1-ordered.*
3. *$P-$soft T_1-ordered if it is Lower $P-$soft T_1-ordered and Upper $P-$soft T_1-ordered.*
4. *$P-$soft T_2-ordered if for every distinct points $x, y \in X$ there exist disjoint soft neighbourhoods W_E and V_E of x and y respectively, such that W_E is increasing and V_E is decreasing.*
5. *Lower (resp. upper) $P-$soft regularly ordered if for each decreasing (resp. increasing) soft closed set H_E and $x \in X$ such that $x \notin H_E$, there exist disjoint soft neighbourhoods W_E of H_E and V_E of x such that W_E is decreasing (resp. increasing) and V_E is increasing (resp. decreasing).*
6. *$P-$soft regularly ordered if it is both lower $P-$soft regularly ordered and upper $P-$soft regularly ordered.*
7. *Lower (resp. upper) $P-$soft T_3-ordered if it is both lower (resp. upper) $P-$soft T_1-ordered and lower (resp. upper) $P-$soft regularly ordered.*
8. *$P-$soft T_3-ordered if it is both lower $P-$soft T_3-ordered and upper $P-$soft T_3-ordered.*
9. *Soft normally ordered if for each disjoint soft closed sets F_E and H_E such that F_E is increasing and H_E is decreasing, there exist disjoint soft neighbourhoods W_E of F_E and V_E of H_E such that W_E is increasing and V_E is decreasing.*
10. *$P-$soft T_4-ordered if it is soft normally ordered and $P-$soft T_1-ordered.*

Definition 2.20 *[3] Let $Y \subseteq X$ and $(X, \tau, E, \leq)$ be an STOS. Then $(Y, \tau_Y, E, \leq_Y)$ is called soft ordered subspace of $(X, \tau, E, \leq)$ provided that (Y, τ_Y, E) is soft subspace of (X, τ, E) and $\leq_Y = \leq \cap Y \times Y$.*

Lemma 2.1 *[3] If W_E is an increasing (resp. a decreasing) soft subset of an $STOS(X, \tau, E, \leq)$, then $W_E \sqcap Y$ is an increasing (resp. a decreasing) soft subset of a soft ordered subspace $(Y, \tau_Y, E, \leq_Y)$.*

3 Ordered Soft Separation Axioms

Definition 3.1 *Let x_{e_1} and y_{e_2} be two soft points in a partially ordered space $(X, E, \leq)$. We say that $x_{e_1} \not\leq y_{e_2}$ if and only if $(x \not\leq y$ or $e_1 \neq e_2)$.*

Definition 3.2 *A soft subset W_E of an $STOS(X, \tau, E, \leq)$ is called increasing (resp. decreasing) soft neighbourhood of $x^e \in Sp(X)_E$ if W_E is soft neighbourhood of x^e and increasing (resp. decreasing).*

Definition 3.3 *An STOS $(X, \tau, E, \leq)$ is said to be:*

1. *Lower P^*-soft T_1-ordered if for every pair of soft points x_{e_1}, y_{e_2} such that $x_{e_1} \not\leq y_{e_2}$ there exists an increasing soft neighbourhood W_E of x_{e_1} such that $y_{e_2} \widehat{\notin} W_E$.*
2. *Upper P^*-soft T_1-ordered if for every pair of soft points x_{e_1}, y_{e_2} such that $x_{e_1} \not\leq y_{e_2}$ there exists a decreasing soft neighbourhood W_E of y_{e_2} such that $x_{e_1} \widehat{\notin} W_E$.*
3. *P^*-soft T_0-ordered if it is Lower P^*-soft T_1-ordered or upper P^*-soft T_1-ordered.*

4. *P^*-soft T_1-ordered if it is Lower P^*-soft T_1-ordered and upper P^*-soft T_1-ordered.*

5. *P^*-soft T_2-ordered if for every pair of soft points x_{e_1}, y_{e_2} such that $x_{e_1} \not\leq y_{e_2}$ there exist disjoint soft neighbourhoods W_E and V_E of x_{e_1} and y_{e_2} respectively, such that W_E is increasing and V_E is decreasing.*

Proposition 3.1 *Every P^*-soft T_i-ordered space $(X, \tau, E, \leq)$ is P^*-soft $T_{i-1}-$ordered, for $i = 1; 2$.*
Proof. *The proof comes immediately from definition 3.2*

We present two examples to illustrate that the converse of the above proposition fails.

Example 3.1 *Let $E = \{e_1, e_2\}, \leq = \blacktriangle \cup \{(x, y)\}$ be a partial order relation on $X = \{x, y\}$ and $\tau = \{X_E, \widehat{\phi}, F_E^1, F_E^2, ..., F_E^7\}$ where,*
$F_E^1 = \{(e_1, \emptyset), (e_2, \{y\})\}$.
$F_E^2 = \{(e_1, \{y\}), (e_2, \emptyset)\}$.
$F_E^3 = \{(e_1, X), (e_2, \emptyset)\}$.
$F_E^4 = \{(e_1, \emptyset), (e_2, X)\}$.
$F_E^5 = \{(e_1, \{y\}), (e_2, X)\}$.
$F_E^6 = \{(e_1, X), (e_2, \{y\})\}$.
$F_E^7 = \{(e_1, \{y\}), (e_2, \{y\})\}$.

For the soft points $x_{e_1}, y_{e_2}, x_{e_1} \not\leq y_{e_2}$, there exists an increasing soft neighbourhood F_E^3 of x_{e_1} such that $y_{e_2} \widehat{\notin} F_E^3$. For the soft points $x_{e_1}, x_{e_2}, x_{e_1} \not\leq x_{e_2}$, there exists an increasing soft neighbourhood F_E^3 of x_{e_1} such that $x_{e_2} \widehat{\notin} F_E^3$. For the soft points $x_{e_2}, x_{e_1}, x_{e_2} \not\leq x_{e_1}$, there exists an increasing soft neighbourhood F_E^4 of x_{e_2} such that $x_{e_1} \widehat{\notin} F_E^4$. For the soft points $x_{e_2}, y_{e_1}, x_{e_2} \not\leq y_{e_1}$, there exists an increasing soft neighbourhood F_E^4 of x_{e_2} such that $y_{e_1} \widehat{\notin} F_E^4$. For the soft points $y_{e_1}, x_{e_2}, y_{e_1} \not\leq x_{e_2}$, there exists an increasing soft neighbourhood F_E^7 of y_{e_1} such that $x_{e_2} \widehat{\notin} F_E^7$. For the soft points $y_{e_1}, y_{e_2}, y_{e_1} \not\leq y_{e_2}$, there exists an increasing soft neighbourhood F_E^3 of y_{e_1} such that $y_{e_2} \widehat{\notin} F_E^3$. For the soft points $y_{e_2}, x_{e_1}, y_{e_2} \not\leq x_{e_1}$, there exists an increasing soft neighbourhood F_E^5 of y_{e_2} such that $x_{e_1} \widehat{\notin} F_E^5$. For the soft points $y_{e_2}, y_{e_1}, y_{e_2} \not\leq y_{e_1}$, there exists an increasing soft neighbourhood F_E^1 of y_{e_2} such that $y_{e_1} \widehat{\notin} F_E^1$. For the soft points $y_{e_1}, x_{e_1}, y_{e_1} \not\leq x_{e_1}$, there exists an increasing soft neighbourhood F_E^5 of y_{e_1} such that $x_{e_1} \widehat{\notin} F_E^5$. For the soft points $y_{e_2}, x_{e_2}, y_{e_2} \not\leq x_{e_2}$, there exists an increasing soft neighbourhood F_E^1 of y_{e_2} such that $x_{e_2} \widehat{\notin} F_E^1$.

Then $(X, \tau, E, \leq)$ is lower P^*-soft T_1-ordered. So it is P^*-soft T_0-ordered. On the other hand, every decreasing soft neighbourhood of x_{e_1} containing y_{e_1}. Thus $(X, \tau, E, \leq)$ is not P^*-soft T_1-ordered.

Example 3.2 *Example 4.7 which is given in [3], is P^*-soft T_1-ordered, but it is not P^*-soft T_2-ordered.*

Theorem 3.1 *Let $(X, \tau, E, \leq)$ be an STOS. Then the following three statements are equivalent:*

1. *$(X, \tau, E, \leq)$ is upper (resp. lower) P^*-soft T_1-ordered,*

2. *For any two soft points $x_{e_1}, y_{e_2} \in Sp(X)_E$ such that $x_{e_1} \not\leq y_{e_2}$, there is a soft open set G_E containing y_{e_2}(resp. x_{e_1}) in which $x_{e_1} \not\leq z_{e_3}$(resp. $z_{e_3} \not\leq y_{e_2}$) for every $z_{e_3} \widehat{\in} G_E$,*

3. *For any soft point $x_e, i(x_e)$(resp. $d(x_e)$) is soft closed.*

Proof. *$(1 \to 2)$ Consider $(X, \tau, E, \leq)$ is an upper P^*- soft T_1-ordered space and let x_{e_1}, y_{e_2} be two soft points such that $x_{e_1} \not\leq y_{e_2}$. Then there exists a decreasing soft neighbourhood W_E of y_{e_2} such that $x_{e_1} \widehat{\notin} W_E$. Putting $G_E = sint(W_E)$. Suppose that $G_E \not\sqsubseteq i(x_{e_1})^c$. Then there exists $z_{e_3} \widehat{\in} G_E$ and $z_{e_3} \widehat{\notin} i(x_{e_1})^c$. Therefore $z_{e_3} \widehat{\in} i(x_{e_1})$ and this implies that $x_{e_1} \leq z_{e_3}$. Now, $z_{e_3} \widehat{\in} G_E \sqsubseteq W_E$ implies that $x_{e_1} \widehat{\in} W_E$. But this contradicts that $x_{e_1} \widehat{\notin} W_E$. Thus $G_E \sqsubseteq i(x_{e_1})^c$. Hence $x_{e_1} \not\leq z_{e_3}$, for every $z_{e_3} \widehat{\in} G_E$.*

$(2 \to 3)$ Consider x_{e_1} is a soft point and let $y_{e_2} \widehat{\in} (i(x_{e_1}))^c$. Then $x_{e_1} \not\leq y_{e_2}$. Therefore there exists a soft open set G_E containing y_{e_2} such that $G_E \sqsubseteq (i(x_{e_1}))^c$. Since x_{e_1} and y_{e_2} are chosen arbitrary, then a soft set $(i(x_{e_1}))^c$ is soft open, for any point x_{e_1}. Hence $i(x_{e_1})$ is soft closed, for any $x_{e_1} \in Sp(X)_E$.

$(3 \to 1)$ *Let x_{e_1}, y_{e_2} be a soft points such that $x_{e_1} \not\leq y_{e_2}$. Obviously, $i(x_{e_1})$ is increasing and by hypothesis, $i(x_{e_1})$ is soft closed. Then $i(x_{e_1})^c$ is a decreasing soft open set satisfies that $y_{e_2} \widehat{\in} i(x_{e_1})^c$ and $x_{e_1} \not\widehat{\in} i(x_{e_1})^c$. Hence the proof is completed.*

A similar proof can be given for the case between parentheses.

Proposition 3.2 *Every P^*-soft T_i-ordered space $(X, \tau, E, \leq)$ is $P-$soft T_i-ordered.*
Proof. *We prove the proposition when $i = 0$ and the other two cases are proven similarly. Suppose $(X, \tau, E, \leq)$ is not $P-$soft T_0-ordered, then it is not Lower $P-$soft T_1-ordered nor upper $P-$soft T_1-ordered. say it is not Lower $P-$soft T_1-ordered, implies there exists two points x, y in X such that $x \not\leq y$, and for every increasing soft neighbourhood W_E of x satisfies $y \Subset W_E$. Then $y \in W_E$ for some $e \in E$. Therefore $x_{e_1} \not\leq y_e \forall e \in E$ and $x_{e_1}, y_e \widehat{\in} W_E$. Hence $(X, \tau, E, \leq)$ is not lower P^*-soft T_1-ordered. Thus $(X, \tau, E, \leq)$ is not P^*-soft T_0-ordered.*

It can be given some examples to illustrate that the converse of the above proposition fails. However, for the sake of economy, we consider $e_1 = e_2$ in definition 3.1 and suffice with Example 4.6 in [3].

Definition 3.4 *Let $(X, E, \leq)$ be a partially ordered space. A soft point a_e is called:*

1. *A smallest soft element of X_E provided that $a_e \leq x_e, \forall x_e \widehat{\in} X_E$.*
2. *A largest soft element of X_E provided that $x_e \leq a_e, \forall x_e \widehat{\in} X_E$.*

Proposition 3.3 *If a_e is one of the smallest soft element of a finite lower P^*-soft T_1-ordered space $(X, \tau, E, \leq$ $)$, then a_e is decreasing soft closed.*
Proof. *Since a_e is the smallest soft element for $e \in E$. Then $a_e \leq x_e, \forall x_e \widehat{\in} X_E$. Hence a_e is decreasing soft closed.*

Proposition 3.4 *If a_e is one of the largest soft element of a finite upper P^*-soft T_1-ordered space $(X, \tau, E, \leq)$, then a_e is increasing soft closed.*
Proof. *Since a_e is the largest soft element for $e \in E$. Then $x_e \leq a_e, \forall x_e \widehat{\in} X_E$. Hence a_e is increasing soft closed.*

Proposition 3.5 *If a_e is a smallest (resp. a largest) soft element of a finite P^*-soft T_1-ordered space $(X, \tau, E, \leq)$, then a_e is decreasing (resp. increasing) soft open.*
Proof. *Since a_e is a smallest soft element for $e \in E$. Then $a_e \leq x_e, \forall x_e \widehat{\in} X_E$. By the anti-symmetric of $\leq$, we have $x_e \not\leq a_e, \forall x_e \widehat{\in} X_E$. By hypothesis, there is a decreasing soft neighbourhood W_E of a_e such that $x_e \not\widehat{\in} W_E$. It follows that $a_e = \sqcap W_E$. Since X is finite, then a_e is decreasing soft open.*

A similar proof can be given for the case between parentheses.

Proposition 3.6 *A finite $STOS(X, \tau, E, \leq)$ is P^*-soft T_1-ordered if and only if it is P^*-soft T_2-ordered.*

Proof. *Necessity: For each $y_{e'} \widehat{\in} (i(x_e))^c$, we have $d(y_{e'})$ is soft closed. Since X is finite, then $\sqcup_{y_{e'} \widehat{\in} (i(x_e))^c} d(y_{e'})$ is soft closed. Therefore $(\sqcup_{y_{e'} \widehat{\in} (i(x_e))^c} d(y_{e'}))^c$ is a soft open set. Thus $(X, \tau, E, \leq)$ is a P^*-soft T_2-ordered space.*
Sufficiency: It follows immediately from Proposition 3.1.

Theorem 3.2 *The property of being a P^*-soft T_i-ordered space is hereditary, for $i = 0, 1, 2$.*

Proof. *Let $(Y, \tau_Y, E, \leq_Y)$ be a soft ordered subspace of a P^*-soft T_2-ordered space $(X, \tau, E, \leq)$. If $x_e, y_{e'} \widehat{\in} Y_E$ such that $x_e \leq_Y y_{e'}$, then $x_e \leq y_{e'}$. So by hypothesis, there exist disjoint soft neighbourhoods W_E*

and V_E of x_e and $y_{e\prime}$, respectively ,such that W_E is increasing and V_E is decreasing. Setting $U_E = Y_E \sqcap W_E$ and $G_E = Y_E \sqcap V_E$, then from lemma 2.1, we obtain that U_E is an increasing soft neighbourhood of x_e and G_E is a decreasing soft neighbourhood of $y_{e\prime}$. Since the soft neighbourhoods U_E and G_E are disjoint, it follows that $(Y, \tau_Y, E, \leq_Y)$ is P^*-soft T_2-ordered.
The theorem can be proven similarly in case of $i = 0, 1$.

Definition 3.5 *An $STOS(X, \tau, E, \leq)$ is said to be:*

1. *Lower (resp. upper) P^*-soft regularly ordered if for each decreasing (resp. increasing) soft closed set H_E and $x_e \in Sp(X)_E$ such that $x_e \widehat{\notin} H_E$, there exist disjoint soft neighbourhoods W_E of H_E and V_E of x_e such that W_E is decreasing (resp. increasing) and V_E is increasing (resp. decreasing).*
2. *P^*-soft regularly ordered if it is both lower P^*-soft regularly ordered and upper P^*-soft regularly ordered.*
3. *Lower (resp. upper) P^*-soft T_3-ordered if it is both lower (resp. upper) P^*-soft T_1-ordered and lower (resp. upper) P^*-soft regularly ordered.*
4. *P^*-soft T_3-ordered if it is both lower P^*-soft T_3-ordered and upper P^*-soft T_3-ordered.*

Theorem 3.3 *An $STOS(X, \tau, E, \leq)$ is lower (resp. upper) P^*-soft regularly ordered if and only if for all $x_e \in Sp(X)_E$ and every increasing (resp. decreasing) soft open set U_E containing x_e there is an increasing (resp. decreasing) soft neighbourhood V_E of x_e satisfies that $scl(V_E) \sqsubseteq U_E$.*
Proof. *Necessity: Let $x_e \in Sp(X)_E$ and U_E be an increasing soft open set containing x_e. Then, U_E^c is decreasing soft closed such that $x_e \widehat{\notin} U_E^c$. By hypothesis, there exist disjoint soft neighbourhoods V_E of x_e and W_E of U_E^c such that V_E is increasing and W_E is decreasing. So there is a soft open set G_E such that $U_E^c \sqsubseteq G_E \sqsubseteq W_E$. Since $V_E \sqsubseteq W_E^c$, then $V_E \sqsubseteq W_E^c \sqsubseteq G_E^c \sqsubseteq U_E$ and since G_E^c is soft closed, then $scl(V_E) \sqsubseteq G_E^c \sqsubseteq U_E$.*

Sufficiency: Let $x_e \in Sp(X)_E$ and H_E be a decreasing soft closed set such that $x_e \widehat{\notin} H_E$. Then H_E^c be an increasing soft open set containing x_e. So that, by hypothesis, there is an increasing soft neighbourhood V_E of x_e such that $scl(V_E) \sqsubseteq H_E^c$. Consequently, $(scl(V_E))^c$ is a soft open set containing H_E. Suppose that $V_E \sqcap d((scl(V_E))^c) \neq \widehat{\phi}$. Then there exists $y_{e'} \in Sp(X)_E$ such that $y_{e'} \widehat{\in} V_E$ and $y_{e'} \widehat{\in} d((scl(V_E))^c)$. So there exists $z_{e''} \widehat{\in} (cl^s(V_E))^c$ satisfies that $y_{e'} \leq z_{e''}$. This means that $z_{e''} \widehat{\in} V_E$. But this contradicts the disjointedness between V_E and $d((scl(V_E))^c)$. Thus $V_E \sqcap d((scl(V_E))^c) = \widehat{\phi}$. This completes the proof.

A similar proof can be given for the case between parentheses.

Proposition 3.7 *The following three properties are equivalent if $(X, \tau, E, \leq)$ is P^*-soft regularly ordered:*

1. *$(X, \tau, E, \leq)$ is P^*-soft T_2-ordered,*
2. *$(X, \tau, E, \leq)$ is P^*-soft T_1-ordered,*
3. *$(X, \tau, E, \leq)$ is P^*-soft T_0-ordered.*

Proof. *The direction $(1) \to (2) \to (3)$ is obvious.*
To prove that $(3) \to (1)$, let $x_e, y_{e'} \in Sp(X)_E$ such that $x_e \not\leq y_{e'}$. Since $(X, \tau, E, \leq)$ is P^*-soft T_0-ordered, then it is lower P^*-soft T_1-ordered or upper P^*-soft T_1-ordered. Say it is upper P^*-soft T_1-ordered. From theorem 3.1, we have that $i(x_e)$ is soft closed. Obviously, $i(x_e)$ is increasing and $y_{e'} \widehat{\in} i(x_e)$. Since $(X, \tau, E, \leq)$ is P^*-soft regularly ordered, then there exist disjoint soft neighbourhoods W_E and V_E of $y_{e'}$ and $i(x_e)$, respectively, such that W_E is decreasing and V_E is increasing. Thus $(X, \tau, E, \leq)$ is P^*-soft T_2-ordered.

Corollary 3.1 *The following three properties are equivalent if $(X, \tau, E, \leq)$ is lower (resp. upper) P^*-soft regularly ordered:*

1. $(X, \tau, E, \leq)$ *is* P^*-*soft* T_2-*ordered,*

2. $(X, \tau, E, \leq)$ *is* P^*-*soft* T_1-*ordered,*

3. $(X, \tau, E, \leq)$ *is lower* (*resp. upper*) P^*-*soft* T_0-*ordered.*

Definition 3.6 *An* $STOS(X, \tau, E, \leq)$ *is said to be:*

1. *Soft normally ordered if for each disjoint soft closed sets* F_E *and* H_E *such that* F_E *is increasing and* H_E *is decreasing, there exist disjoint soft neighbourhoods* W_E *of* F_E *and* V_E *of* H_E *such that* W_E *is increasing and* V_E *is decreasing.*

2. P^*-*soft* T_4-*ordered if it is soft normally ordered and* P^*-*soft* T_1-*ordered.*

Theorem 3.4 *An* $STOS(X, \tau, E, \leq)$ *is soft normally ordered if and only if for every decreasing* (*resp. increasing*) *soft closed set* F_E *and every decreasing* (*resp. increasing*) *soft open neighbourhood* U_E *of* F_E*, there is a decreasing* (*resp. an increasing*) *soft neighbourhood* V_E *of* F_E *satisfies that* $scl(V_E) \sqsubseteq U_E$.

Proof. *Necessity: Let* F_E *be a decreasing soft closed set and* U_E *be a decreasing soft open neighbourhood of* F_E*. Then,* U_E^c *is an increasing soft closed set and* $F_E \sqcap U_E^c = \widehat{\phi}$*. Since* $(X, \tau, E, \leq)$ *is soft normally ordered, then there exist disjoint a decreasing soft neighbourhood* V_E *of* F_E *and an increasing soft neighbourhood* W_E *of* U_E^c*. Since* W_E *is soft neighbourhood of* U_E^c*, then there exists a soft open set* H_E *such that* $U_E^c \sqsubseteq H_E \sqsubseteq W_E$*. Consequently,* $W_E^c \sqsubseteq H_E^c \sqsubseteq U_E$ *and* $V_E \sqsubseteq W_E^c$*. So it follows that* $scl(V_E) \sqsubseteq scl(W_E^c) \sqsubseteq H_E^c \sqsubseteq U_E$*. Thus* $F_E \sqsubseteq scl(V_E) \sqsubseteq scl(W_E^c) \sqsubseteq H_E^c \sqsubseteq U_E$*. Hence the necessary part holds.*

Sufficiency: Let F_E^1 *and* F_E^2 *be two disjoint soft closed sets such that* F_E^1 *is decreasing and* F_E^2 *is increasing. Then* F_E^{2c} *is a decreasing soft open set containing* F_E^1*. By hypothesis, there exists a decreasing soft neighbourhood* V_E *of* F_E^1 *such that* $scl(V_E) \sqsubseteq F_E^c$*.*

Setting $H_E = X_E - scl(V_E)$*. This means that* H_E *is a soft open set containing* F_E^2*. Obviously,* $F_E^2 \sqsubseteq H_E, F_E^1 \sqsubseteq V_E$ *and* $H_E \sqcap V_E = \widehat{\phi}$*. Now,* $i(H_E)$ *is an increasing soft neighbourhood of* F_E^2*. Suppose that* $i(H_E) \sqcap V_E \neq \widehat{\phi}$*. Then there exists* $x_e \in Sp(X)_E$ *such that* $x_e \widehat{\in} i(H_E)$ *and* $x_e \widehat{\in} V_E = d(V_E)$*. This implies that there exist* $a_{e'} \widehat{\in} H_E$ *and* $b_{e''} \widehat{\in} V_E$ *such that* $a_{e'} \leq x_e$ *and* $x_e \leq b_{e''}$*. As* $\leq$ *is transitive, then* $a_{e'} \leq b_{e''}$*. Therefore* $b_{e''} \widehat{\in} H_E \sqcap V_E$*. This contradicts the disjointedness between* H_E *and* V_E*. Thus* $i(H_E) \sqcap V_E = \widehat{\phi}$*. Hence the proof is completed.*

A similar proof can be given for the case between parentheses.

Proposition 3.8 *Every* P^*-*soft* T_i-*ordered space* $(X, \tau, E, \leq)$ *is* P^*-*soft* $T_{i-1}-$*ordered, for* $i = 3; 4$.

Proof. *From proposition 3.7, we obtain that every* P^*-*soft* T_3-*ordered space is* P^*-*soft* T_2-*ordered. To prove the proposition in case of* $i = 4$*, let* $a_e \in Sp(X)_E$ *and* F_E *be a decreasing soft closed set such that* $a_e \widehat{\notin} F_E$*. Since* $(X, \tau, E, \leq)$ *is* P^*-*soft* T_1-*ordered, then* $i(a_e)$ *is an increasing soft closed set and since* $(X, \tau, E, \leq)$ *is soft normally ordered, then there exist disjoint soft neighbourhoods* W_E *and* V_E *of* $i(a_e)$ *and* F_E*, respectively, such that* W_E *is increasing and* V_E *is decreasing. Therefore,* $(X, \tau, E, \leq)$ *is lower* P^*-*soft regularly ordered. If* F_E *is an increasing soft set, then we prove similarly that* $(X, \tau, E, \leq)$ *is upper* P^*-*soft regularly ordered. Thus* $(X, \tau, E, \leq)$ *is* P^*-*soft regularly ordered. Hence is* P^*-*soft* T_3-*ordered.*

The converse of the above proposition is not always true as illustrated in the following two examples.

Example 3.3 *Let* $E = \{e_\alpha, e_\beta\}$ *be a set of parameters,* $\leq = \blacktriangle \cup \{(1, 2)\}$ *be a partial order relation on the set of natural numbers* N*. Define* $\tau = \{G_E \sqsubseteq N_E$ *such that* $1 \in G_E^c$ *or* $[1_{e_i} \in G_E, i \in \{\alpha, \beta\}]$ *and* G_E^c *is infinite* $\} \cup N_E$.

1. *Clearly* $\widehat{\phi} \in \tau$ *and* $N_E \in \tau$.

2. *To illustrate the second condition "union".*
   If $G_E^{\delta}, F_E^{\lambda} \in \tau$ *for some* $\delta, \lambda \in \Delta$, *where* Δ *is an index set, we have* 3 *cases:*
   (a) *Either* $1 \in G_E^{\delta c}$ *and* $G_E^{\delta c}$ *is infinite. Now* $1 \in \sqcap G_E^{\delta c} = (\sqcup G_E^{\delta})^c$. *So that* $\sqcup G_E^{\delta} \in \tau$.
   (b) *Or* $1 \in G_E^{\delta c}$ *and* $1_{e_i} \widehat{\in} F_E^{\lambda}, i \in \{\alpha, \beta\}$. *So* $1 \notin G_E^{\delta} \sqcap F_E^{\lambda}, \forall e \in E$. *Therefore* $1 \in (G_E^{\delta} \sqcup F_E^{\lambda})^c$. *Hence* $G_E^{\delta} \sqcup F_E^{\lambda} \in \tau$.
   (c) *Or* $1_{e_i} \widehat{\in} F_E^{\lambda}, i \in \{\alpha, \beta\}$. *So* $1_{e_i} \widehat{\in} \sqcup F_E^{\lambda}, i \in \{\alpha, \beta\}$. *Therefore* $\sqcup F_E^{\lambda} \in \tau$.

3. *To illustrate the third condition "intersection".*
   If $G_E, F_E \in \tau$, *we have* 3 *cases:*
   (a) *Either* $1 \in G_E^c$ *and* $1 \in F_E^c$. *Now* $1 \in G_E^c \sqcup F_E^c = (G_E \sqcap F_E)^c$. *So that* $G_E \sqcap F_E \in \tau$.
   (b) *Or* $1 \in G_E^c$ *and* $1_{e_i} \widehat{\in} F_E, i \in \{\alpha, \beta\}$. *Therefore* $1 \in G_E^c$ *and* $1 \notin F_E^c$. *So* $1 \in G_E^c \sqcup F_E^c = (G_E \sqcap F_E)^c$. *Thus* $G_E \sqcap F_E \in \tau$.
   (c) *Or* $1_{e_i} \widehat{\in} G_E$ *and* $1_{e_j} \widehat{\in} F_E, i, j \in \{\alpha, \beta\}$, *we have* 4 *cases:*
      i. *Either* $G_E \sqcap F_E = \widehat{\phi}$. *Therefore* $G_E \sqcap F_E \in \tau$.
      ii. *Or* $1_{e_i} \widehat{\in} G_E \sqcap F_E$. *Therefore* $G_E \sqcap F_E \in \tau$.
      iii. *Or* $1_{e_j} \widehat{\in} G_E \sqcap F_E$. *Therefore* $G_E \sqcap F_E \in \tau$.
      iv. *Or* $1_{e_i}, 1_{e_j} \widehat{\in} G_E \sqcap F_E$. *Therefore* $G_E \sqcap F_E \in \tau$.

From the above discussion, we conclude that (N, τ, E) *is a soft topological space. In the following, we illustrate that* $(N, \tau, E, \leq)$ *is* P^*-*soft* T_2-*ordered. We have the following 10 cases:*
For $i, j \in \{\alpha, \beta\}$ *and* $x, y \in N - \{1, 2\}$

1. *Either* $1_{e_i} \not\leq 1_{e_j}, i \neq j$. *Then we define two soft sets* W_E, V_E *as follows* $W_E = \{1, 2\}_{e_i}$ *and* $V_E = 1_{e_j}, i \neq j$. *So* W_E *is an increasing soft open set containing* $1_{e_i}, V_E$ *is a decreasing soft open set containing* 1_{e_j} *and* $W_E \sqcap V_E = \widehat{\phi}$.

2. *Or* $1_{e_i} \not\leq 2_{e_j}, i \neq j$. *Then we define two soft sets* W_E, V_E *as follows* $W_E = \{1, 2\}_{e_i}$ *and* $V_E = \{1, 2\}_{e_j}, i \neq j$. *So* W_E *is an increasing soft open set containing* $1_{e_i}, V_E$ *is a decreasing soft open set containing* 2_{e_j} *and* $W_E \sqcap V_E = \widehat{\phi}$.

3. *Or* $1_{e_i} \not\leq x_{e_j}, \forall i, j$. *Then we define two soft sets* W_E, V_E *as follows* $W_E = \{1, 2\}_{e_i}$ *and* $V(e) = \{x\}, \forall e \in E$. *So* W_E *is an increasing soft open set containing* $1_{e_i}, V_E$ *is a decreasing soft open set containing* x_{e_j} *and* $W_E \sqcap V_E = \widehat{\phi}$.

4. *Or* $2_{e_i} \not\leq 1_{e_j}, \forall i, j$. *Then we define two soft sets* W_E, V_E *as follows* $W_E = \{2\}, \forall e \in E$ *and* $V(e) = 1_{e_j}$. *So* W_E *is an increasing soft open set containing* $2_{e_i}, V_E$ *is a decreasing soft open set containing* 1_{e_j} *and* $W_E \sqcap V_E = \widehat{\phi}$.

5. *Or* $2_{e_i} \not\leq 2_{e_j}, i \neq j$. *Then we define two soft sets* W_E, V_E *as follows* $W_E = \{1, 2\}_{e_i}$ *and* $V_E = \{1, 2\}_{e_j}, i \neq j$. *So* W_E *is an increasing soft open set containing* $2_{e_i}, V_E$ *is a decreasing soft open set containing* 2_{e_j} *and* $W_E \sqcap V_E = \widehat{\phi}$.

6. *Or* $2_{e_i} \not\leq x_{e_j}, \forall i, j$. *Then we define two soft sets* W_E, V_E *as follows* $W_E = \{1, 2\}_{e_i}$ *and* $V(e) = \{x\}, \forall e \in E$. *So* W_E *is an increasing soft open set containing* $2_{e_i}, V_E$ *is a decreasing soft open set containing* x_{e_j} *and* $W_E \sqcap V_E = \widehat{\phi}$.

7. *Or* $x_{e_i} \not\leq 1_{e_j}, \forall i, j$. *Then we define two soft sets* W_E, V_E *as follows* $W_E = \{x\}, \forall e \in E$ *and* $V(e) = 1_{e_j}$. *So* W_E *is an increasing soft open set containing* x_{e_i}, V_E *is a decreasing soft open set containing* 1_{e_j} *and* $W_E \sqcap V_E = \widehat{\phi}$.

8. *Or* $x_{e_i} \not\leq 2_{e_j}, \forall i,j$. *Then we define two soft sets* W_E, V_E *as follows* $W_E = \{x\}, \forall e \in E$ *and* $V(e) = \{1,2\}_{e_j}$. *So* W_E *is an increasing soft open set containing* x_{e_i}, V_E *is a decreasing soft open set containing* 2_{e_j} *and* $W_E \sqcap V_E = \widehat{\phi}$.

9. *Or* $x_{e_i} \not\leq x_{e_j}, i \neq j$. *Then we define two soft sets* W_E, V_E *as follows* $W_E = \{1,x\}_{e_i}$ *and* $V_E = \{1,x\}_{e_j}, i \neq j$. *So* W_E *is an increasing soft open set containing* x_{e_i}, V_E *is a decreasing soft open set containing* x_{e_j} *and* $W_E \sqcap V_E = \widehat{\phi}$.

10. *Or* $x_{e_i} \not\leq y_{e_j}, \forall i,j$. *Then we define two soft sets* W_E, V_E *as follows* $W_E = \{x\}, \forall e \in E$ *and* $V(e) = \{y\}, \forall e \in E$. *So* W_E *is an increasing soft open set containing* x_{e_i}, V_E *is a decreasing soft open set containing* y_{e_j} *and* $W_E \sqcap V_E = \widehat{\phi}$.

To illustrate that $(N, \tau, E, \leq)$ *is not lower* P^*-*soft regularly ordered, we define a decreasing soft closed set* H_E *as follows:*
$H_E = \{(e_\alpha, \{1,2,4,5,...\}), (e_\beta, \{1,2,4,5,...\})\}$.
Since $3_{e_\alpha} \widehat{\notin} H_E$ *and there do not exist disjoint soft neighbourhoods* W_E *and* V_E *containing* H_E *and* 3_{e_α}, *respectively, then* $(N, \tau, E, \leq)$ *is not lower* P^*-*soft regularly ordered. Hence* $(N, \tau, E, \leq)$ *is not* P^*-*soft* T_3-*ordered.*

Example 3.4 *Let* $E = \{e_1, e_2, e_3\}$ *be a set of parameters,* $\leq = \blacktriangle \cup \{(1,2)\}$ *be a partial order relation on the set of natural numbers* N *and* $\tau = \{G_E \sqsubseteq N_E$ *such that* $1 \in G_E^c$ *or* $[1 \in G(e_2)$ *and* G_E^c *is finite* $]\}$. *Obviously,* $(N, \tau, E, \leq)$ *is* P^*-*soft* T_2-*ordered. In the following, we illustrate that* $(N, \tau, E, \leq)$ *is* P^*-*soft regularly ordered. A soft subset* H_E *of* $(N, \tau, E, \leq)$ *is soft closed if* $1 \in H_E$ *or* $[1 \notin H(e_2)$ *and* H_E *is finite* $]$.

On the one hand, consider $\widehat{\phi} \neq H_E \neq N_E$ *is a decreasing soft closed set. Then we have the following two cases:*

1. *Either* $1 \in H_E$. *Then for each* $x_e \in Sp(X)_E$ *such that* $x_e \widehat{\notin} H_E$, *we define a soft set* G_E *as follows* $G_E = x_e$. *So* G_E *is an increasing soft open set containing* x_e *and its relative complement is a decreasing soft open set containing* H_E.

2. *Or* $[1 \notin H(e_2)$ *and* H_E *is finite* $]$. *Suppose that* $x_e \widehat{\notin} H_E$. *Then we have the following two cases:*

   (a) *Either* $x_e = 1_{e_2}$. *Then* $2_{e_2} \widehat{\notin} H_E$. *So we define a soft set* G_E *as follows* $G(e) = N - H(e)$, *for each* $e \in E$. *Thus* G_E *is an increasing soft open set containing* 1_{e_2} *and its relative complement is a decreasing soft open set containing* H_E.

   (b) *Or* $x_e \neq 1_{e_2}$. *Then we define a soft set* G_E *as follows* $G(e) = N - H(e)$, *for each* $e \in E$. *Thus* G_E *is an increasing soft open set containing* x_e *and its relative complement is a decreasing soft open set containing* H_E.

   Thus $(N, \tau, E, \leq)$ *is lower* P^*-*soft regularly ordered.*

On the other hand, consider $\widehat{\phi} \neq H_E \neq N_E$ *is an increasing soft closed set. Then we have the following two cases:*

1. *Either* $1 \in H_E$. *Then* $2 \in H_E$. *So for each* $x_e \in Sp(X)_E$ *such that* $x_e \widehat{\notin} H_E$, *we define a soft set* G_E *as follows* $G_E = x_e$. *Thus* G_E *is a decreasing soft open set containing* x_e *and its relative complement is an increasing soft open set containing* H_E.

2. *Or* $[1 \notin H(e_2)$ *and* H_E *is finite* $]$. *Suppose that* $x_e \widehat{\notin} H_E$. *Then we have the following two cases:*

   (a) *Either* $x_e = 1_{e_2}$. *Then we define a soft set* G_E *as follows* $G(e) = N - H(e)$, *for each* $e \in E$. *Thus* G_E *is a decreasing soft open set containing* 1_{e_2} *and its relative complement is an increasing soft open set containing* H_E.

(b) *Or $x_e \neq 1_{e_2}$. If $x_e = 2_{e_2}$, then $1 \notin H_E$. So, by the definition of soft open sets in this soft topology, we obtain that H_E is an increasing soft open set. Obviously, its relative complement is a decreasing soft open set containing x_e. If $x_e \neq 1_{e_2} \neq 2_{e_2}$, then we define a soft set G_E as follows $G(e) = N - H(e)$, for each $e \in E$. Thus G_E is a decreasing soft open set containing x_e and its relative complement is an increasing soft open set containing H_E.*

Thus $(N, \tau, E, \leq)$ is upper P^*-soft regularly ordered.

From the above discussion, we conclude that $(N, \tau, E, \leq)$ is P^*- soft regularly ordered. Hence $(N, \tau, E, \leq)$ is P^*-soft T_3-ordered. To illustrate that $(N, \tau, E, \leq)$ is not soft normally ordered, we define an increasing soft closed set H_E and a decreasing soft closed set F_E as follows:
$H_E = \{(e_1, \{1, 2\}), (e_2, \{3\}), (e_3, \{4\})\}$.
$F_E = \{(e_1, \{3\}), (e_2, \{4\}), (e_3, \{1, 5\})\}$.
Since the two soft closed set are disjoint and there do not exist disjoint soft neighbourhoods W_E and V_E containing H_E and F_E, respectively, then $(N, \tau, E, \leq)$ is not soft normally ordered. Hence $(N, \tau, E, \leq)$ is not P^*-soft T_4-ordered.

4 Conclusion

In this study, we have obtained a new class of ordered soft separation axioms called P^*-soft T_1-ordered spaces, $(i = 0, 1, 2, 3, 4)$. They are formulated with respect to belong and non-belong relations between soft points sets. We have investigated the interrelations between these concepts. Some illustrative examples are given to clarify the obtained relationships and results.
As future works, we shall study these concepts with respect to another generalizations of soft open sets such as pairwise soft open sets. Also, we shall redefine these concepts using partial belong and partial nonbelong relations and investigate their characterizations. Moreover, we will study them on the contents of supra soft topological spaces and minimal soft topological spaces.
In conclusion, we hope that the initiated notions will be beneficial for researchers and scholars to promote and progress the study in soft topology.

5 Declarations

5.1 Availability of data and materials

All data employed was obtained from the herein cited public sources. Please contact author for data requests.

5.2 Contributions

The main idea of this paper was proposed by **S. A. E** , **S. A. K**, and **S. H** analyzed the data. All the authors participated in writing and reviewing of the paper. All authors read and approved the final manuscript.

5.3 Competing interests

The authors declare that they have no competing interest.

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