

The Forcing Circular Metric Dimension of A Graph

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Abstract

Let $G = (V, E)$ be a simple graph and u, v be any two vertices of G . Then the circular distance between u and v denoted by $D^c(u, v)$ and is defined by

$$D^c(u, v) = \begin{cases} D(u, v) + d(u, v) & \text{if } u \neq v \\ 0 & \text{if } u = v \end{cases}$$

where $D(u, v)$ and $d(u, v)$ are detour distance and distance between u and v respectively. Let $W = \{w_1, w_2, \dots, w_k\} \subset V(G)$ and $v \in V(G)$. The representation $cr(v/W)$ of v with respect to W is the k -tuple $(D^c(v, w_1), D^c(v, w_2), \dots, D^c(v, w_k))$. Then W is called a circular resolving set if different vertices of G have different representations with respect to W . A circular resolving set of minimum cardinality is called a $cdim$ -set for G and the cardinality of the $cdim$ -set is known as the *circular metric dimension* of G , represented by $cdim(G)$. Let W be a minimum $cdim$ -set of G . A subset of $T \subseteq W$ is called a forcing subset of W if W is the unique minimum $cdim$ -set containing T . A forcing circular resolving subset for W of minimum cardinality is a minimum forcing circular resolving subset of W . The forcing circular metric dimension of W , $f_{cdim}(W)$ in G is the cardinality of a minimum forcing circular resolving subset of W . The forcing circular metric dimension of G , $f_{cdim}(G) = \min \{f_{cdim}(W)\}$, where the minimum is taken over all minimum circular resolving subset of W . In this article forcing circular metric dimension of some standard graphs are determined.

Keywords: circular distance, circular resolving set, circular metric dimension, forcing circular resolving set, forcing circular metric dimension.

AMS Subject Classification: 05C12

1. Introduction and Preliminaries

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. The order of a graph G is $|V(G)|$, its number of vertices denoted by n . The size of a graph G is $|E(G)|$, its number of edges

For the graph G given in Figure 2.1, let $W=\{v_1, v_2, v_3, v_4, v_5, v_6\}$. Then

$$cr(v_1/W) = (0,5,5,5,5,5)$$

$$cr(v_2/W) = (5,0,5,7,5,4)$$

$$cr(v_3/W) = (5,5,0,2,5,5)$$

$$cr(v_4/W) = (7,7,2,0,7,7)$$

$$cr(v_5/W) = (5,5,5,7,0,5)$$

$$cr(v_6/W) = (5,4,5,7,5,0)$$

Clearly $W_1=\{v_1, v_2, v_3\}$, $W_2=\{v_1, v_2, v_4\}$, $W_3=\{v_1, v_2, v_5\}$, $W_4=\{v_2, v_3, v_4\}$, $W_5=\{v_2, v_3, v_5\}$, $W_6=\{v_1, v_5, v_6\}$, $W_7=\{v_4, v_5, v_6\}$, $W_8=\{v_1, v_3, v_6\}$, $W_9=\{v_1, v_4, v_6\}$, $W_{10}=\{v_3, v_4, v_6\}$, $W_{11}=\{v_3, v_5, v_6\}$, $W_{12}=\{v_2, v_4, v_5\}$ are the only $cdim$ -set of G such that $f_{cdim}(W_i)=3$, so that $f_{cdim}(G) = 3$

Observation: 2.3. For a connected graph G , $0 \leq f_{cdim}(G) \leq cdim(G)$.

Theorem 2.4. For the path $G = P_n (n \geq 2)$, $f_{cdim}(G) = 1$.

Proof. Let $V(P_n) = \{v_1, v_2, \dots, v_n\}$. Then $W_1 = \{v_1\}$ and $W_2 = \{v_n\}$ are the only two $cdim$ -sets of G such that $f_{cdim}(W_1) = f_{cdim}(W_2) = 1$. Hence $f_{cdim}(G) = 1$. ■

Theorem 2.5. For the cycle $G = C_n (n \geq 3)$, $f_{cdim}(G) = n - 1$.

Proof. Let $(C_n) = \{v_1, v_2, \dots, v_n\}$. It is easily seen that $W = V(G) - \{x\}$, where $x \in V(G)$ is a $cdim$ -set of G so that $f_{cdim}(W) = n - 1$. Since x is arbitrary, $f_{cdim}(G) = n - 1$. ■

Theorem 2.6. For the complete graph $G = K_n, n \geq 2$, $f_{cdim}(G) = n - 1$.

Proof. The proof is similar to the Theorem 2.5. ■

Theorem 2.7. Let G be the graph obtained from $K_{1,n-1} (n \geq 3)$ by subdividing the end edges exactly once. Then $f_{cdim}(G) = n - 2$.

Proof. It is easily seen that $W = V(G) - \{x, u\}$, where $u \in \{v_1, v_2, \dots, v_{n-1}\}$ is a $cdim$ -set of G and so $f_{cdim}(W) = n - 2$. Since u is arbitrary, $f_{cdim}(G) = n - 2$. ■

Theorem 2.8. For the graph $G = K_{n_1} + P_{n_2}$, where $n_1, n_2 \geq 2$. $f_{cdim}(G) = 1$.

Proof. Let $V(K_{n_1}) = \{u_1, u_2, \dots, u_{n_1}\}$ and $V(P_{n_2}) = v_1, v_2, \dots, v_{n_2}$. Let $S = V(K_{n_1}) \cup \{x\}$, where $x \in V(P_{n_2})$. Without loss of generality, let $x = u_1$. $cr(u_2/S) = cr(u_{n_1}/S) =$ Then S is a $cdim$ -set of G so that $cdim(G) = n_1 + 1$. By Theorem $f_{cdim}(G) \leq \dim(G) - |V(K_{n_1})| = n + 1 - n = 1$. Since $cdim$ -set of G is not unique, $f_{cdim}(G) = 1$. ■

137

$$cr(y_{s-1}/W) = (r+s-1, r+s-1, \overset{\cdot}{r+s-1}, \dots, r+s-1, r+s-1, \dots, 0)$$

↓
(r+s-1)th place

$$cr(y_s/W) = (r+s-1, r+s-1, r+s-1, \dots, r+s-1, r+s-1, \dots, r+s-1).$$

Since the representation are distinct, W is a circular resolving set of G so that $cdim(G) \leq r+s-1$. We demonstrate that $cdim(G) = r+s-1$. Consider however, that $cdim(G) \leq r+s-2$. If so, a circular resolving set S' exists such that $|S'| \leq r+s-2$. As a result, there are at least two vertices, $u, v \in V \setminus S'$ such that $cr(u/S') = cr(v/S') = (r+s-1, r+s-1, r+s-1, \dots, r+s-1)$, which is incoherent. As a result, $cdim(G) = r+s-1$. It is easily seen that $W = V(G) - \{y_s\}$, where $y_s \in V(G)$ is a $cdim$ -set of G so that $f_{cdim}(W) = r+s-1$. Since y_s is arbitrary, $f_{cdim}(G) = r+s-1$. ■

Theorem 2.10. For the graph $G = C_n \circ K_1$, ($n \geq 4$), $f_{cdim}(G) = n$.

Proof. Let $V(C_n) = \{u_1, u_2, \dots, u_n\}$ and $\{v_1, v_2, \dots, v_n\}$ be the set of end vertices of G . Let $W = \{v_1, v_2, \dots, v_n\}$. Then the circular metric representations n tuples are as follows

$$\begin{aligned} cr(v_1/W) &= (0, n+4, n+4, \dots, n+4) \\ cr(v_2/W) &= (n+4, 0, n+4, \dots, n+4) \\ &\vdots \\ cr(v_n/W) &= (n+4, n+4, \dots, n+4, 0) \\ cr(u_1/W) &= (2, n+2, n+2, n+2, \dots, n+2) \\ cr(u_2/W) &= (n+2, 2, n+2, \dots, n+2) \\ cr(u_3/W) &= (n+2, n+2, 2, \dots, n+2) \\ &\vdots \\ cr(u_n/W) &= (n+2, n+2, \dots, n+2, 2). \end{aligned}$$

W is a circular resolving set of G since the representations are distinct and as a result, $cdim(G) \leq n$. We establish that $cdim(G) = n$. Consider, however, that $cdim(G) \leq n-1$. If so, a circular resolving set S' exists such that $|S'| \leq n-1$. Therefore, either two end vertices or at least two cut vertices of G belongs to $V \setminus S'$. Allow $u, v \in V \setminus S'$. If G 's end vertices are u and v , then $cr(u/S') = cr(v/S') = (n+2, n+4, n+4, \dots, n+4)$. If G 's cut vertices are u and v , then $cr(u/S') = cr(v/S') = (n, n+2, n+2, \dots, n+2)$. Which is incongruous. Consequently, $cdim(G) = n$. It is easily seen that $W = \{v_1, v_2, \dots, v_n\}$, where $v_i \in V(G)$ is a $cdim$ -set of G so that $f_{cdim}(W) = n$. Since v_i is arbitrary, $f_{cdim}(G) = n$. ■

Theorem 2.11. For the graph $G = K_n \circ K_1$, ($n \geq 4$), $f_{cdim}(G) = n$.

Proof. The proof is similar to the Theorem 2.10. ■

Theorem 2.12. Let G be the middle graph of the path $P_n (n \geq 3)$. Then $f_{cdim}(G) = 0$.

Proof. Let $V(P_n) = \{v_1, v_2, \dots, v_n\}$ and $u_i = v_i v_{i+1}, 1 \leq i \leq n - 1$. Then $(G) = \{v_i, u_j, \{1 \leq i \leq n, 1 \leq j \leq n - 1\}$. In G , v_i is adjacent to u_1, v_n is adjacent to u_{n-1} . v_i is adjacent to $u_{i-1}, u_i, 2 \leq i \leq n - 1$, u_1 is adjacent to v_1, v_2 and u_2, u_i is adjacent to u_{i-1}, u_{i+1}, v_i and v_{i+1} for $2 \leq i \leq n - 2$ and u_{n-1} is adjacent to v_{n-2}, v_{n-1} and v_n .

Evidently, $|V(G)| = 2n - 1$. As a result of $G \neq P_n$, by Theorem 2.8, $cdim(G) \geq 2$. Let $W = \{v_1, v_n\}$, $Cr(v_1/W) = (0, 3n - 2), Cr(v_2/W) = (5, 3n - 4), \dots, Cr(v_{n-1}/W) = (3n - 5, 5), Cr(v_n/W) = (3n - 2, 0), Cr(u_1/W) = (2, 3n - 4), Cr(u_2/W) = (5, 3n - 7), Cr(u_3/W) = (8, 3n - 10), \dots, Cr(u_{n-1}/W) = (3n - 4, 2)$. W is a circular resolving set of G , and since the circular metric representation is distinct, $cdim(G) = 2$. It is easily seen that $W = \{v_1, v_n\}$ is an unique $cdim$ -set of G so that $f_{cdim}(W) = 0$ and hence $f_{cdim}(G) = 0$

■

Theorem 2.13. Let G be the total graph of the path $P_n (n \geq 4)$. Then $f_{cdim}(G) = 0$.

Proof. Let $V(P_n) = \{v_1, v_2, \dots, v_n\}$ and $u_i = v_i v_{i+1}, 1 \leq i \leq n - 1$. Then $(G) = \{v_i, u_j, 1 \leq i \leq n, 1 \leq j \leq n - 1\}$. In G , v_1 is adjacent to v_2 and u_1 ; v_n is adjacent to v_{n-1} and u_{n-1} ; v_i is adjacent to $v_{i-1}, v_{i+1}, u_{i-1}$ and u_i ; $2 \leq i \leq n - 1$, u_1 is adjacent to v_1, v_2 and u_2 , u_i is adjacent to u_{i-1}, u_{i+1}, v_i and v_{i+1} for $2 \leq i \leq n - 2$ and u_{n-1} is adjacent to v_{n-2}, v_n and u_{n-2} . Evidently, $|V(G)| = 2n - 1$. As a result of $G \neq P_n$, $cdim(G) \geq 2$. Let $W = \{v_1, v_n\}$, $Cr(v_1/W) = (0, 3n - 3), Cr(v_2/W) = (2n - 1, 3n - 4), Cr(v_3/W) = (2n, 3n - 5), \dots, Cr(v_{n-1}/W) = (3n - 4, 2n - 1), Cr(v_n/W) = (3n - 3, 0), Cr(u_1/W) = (2n - 1, 3n - 3), Cr(u_2/W) = (2n, 3n - 4), Cr(u_3/W) = (2n + 1, 3n - 5), \dots,$

$Cr(u_{n-1}/W) = (3n - 4, 2n), Cr(u_n/W) = (3n - 3, 2n - 1)$. W is a circular resolving set of G , and since the circular metric representation is distinct, $cdim(G) = 2$. It is easily seen that $W = \{v_1, v_n\}$ is an unique $cdim$ -set of G so that $f_{cdim}(W) = 0$ and hence $f_{cdim}(G) = 0$. ■

Conclusion

This article established a novel distance metric called the forcing circular metric dimension of a graph. We will develop this concept to incorporate more distance considerations in a subsequent investigation.

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