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Properties of Compact Weighted Composition Operators on Analytic Function Spaces

[1]Manal Elzian Mohamed
College of Science and Art, Department of Mathematics / king Khalid University, Mahayil, Saudi Arabia

[2]Hajir Abedlmulla Mohammed
Gulf College department of computer science

Abstract

Let $H(\Omega)$ denote a functional Hilbert space of analytic functions on a domain Ω. Let $\omega : \Omega \rightarrow C$ and $\emptyset : \Omega \rightarrow \Omega$ be such that $\omega f o \emptyset$ is in H(Ω) for every f in $H(\Omega)$. The operator wCφ given by $f \rightarrow \omega f o \emptyset$ is called a weighted composition operator on H(Ω). In this paper we characterize such operators and those for which $(\omega C_{\emptyset})^*$ is a composition operator. Compact weighted composition operators on some functional Hilbert spaces are also characterized. We give sufficient conditions for the compactness of such operators on weighted Dirichlet spaces.
Keywords : Weighted Composition Operator, analytic functions , Dirichlet spaces, Hilbert space

Introduction

A Hilbert spaceH(Ω) of analytic functions on a domain , is called a functional Hilbert space provided the point evaluation $f \rightarrow f(x)$ is continuous for every x in Ω . The Hardy space H^2 and the Bergman space $L^2_\alpha(D)$ are the well-known examples of functional Hilbert spaces. An application of the Riesz representation theorem shows that for every $x \in \Omega$. there is a vector k_x in H(Ω) such that $f(x) =< f, k_x >$for all f in H(Ω). Let $K = \{ k_x : x \in \Omega\}$. An operator T on H(Ω) is a composition operator if and only if K is invariant underT^* [1].In fact,$T^* k_x = k_{\emptyset(x)}$, where$T = C_{\emptyset}$. It is a multiplication operator if and only if the elements of K are eigenvectors of T^* [2]. In this case $T^* k_x = \overline{\psi(x)}_{k_x}$, where $T = M_\psi$ is the operator of multiplication by ψ . An operator T on H(Ω) is a weighted composition operator if and only if $T^* K \subset \widetilde{K}$, where $\widetilde{K} = \{\lambda k_x | \lambda \in C, x \in \Omega\}$. In this case $T^* k_x = \overline{\omega(x)}\, k_{\emptyset(x)}$. where $T = \omega C_{\emptyset}$.

We note that the Hardy space H^2 can be identified as the space of functions f analytic in the open unit disc D such that

$$\|f\|^2 = \sup_{0<r<1} \int_{-\pi}^{\pi} \left|f(re^{i\theta})\right|^2 d\theta < \infty$$

Actually, if $f \in H^2$ and $f(z) = \sum a_n z^n$ then $\|f\|^2 = \sum |a_n|^2$. Moreover, if $f \in H^2$ then

$$\langle f, g\rangle = \sum a_n \overline{b_n}$$

where $g(x) = \sum b_n z^n$. For, $\lambda \in D$ the function $k_\lambda(z) = (1 - \bar{\lambda} z)^{-1}$ is the reproducing kernel for λ.

Let G be a bounded open subset of the complex plane C. For $1 \leq p \leq \infty$,

the Bergman space of G, $L_a^p(G)$ is the set of all analytic functions $f : G \to C$ such that $\int_G |f|^p dA < \infty$, where $dA(z) = 1|\pi r dr d\theta$. is the usual area measure on G. Note that $L_a^p(G)$ is closed in $L^p(G)$ and it is therefore a Banach space. When $G = D$ the inner product in $L_a^2(G)$ is given by

$$\langle f, g\rangle = \sum \frac{a_n \overline{b_n}}{n+1}$$

Where $f = \sum a_n z^n$ and $g = \sum b_n z^n$. Therefore $k_\lambda(z) = (1 - \bar{\lambda} z)^{-2}$ is the reproducing kernel for *the point* $\lambda \in D$.

Let $\lambda_\alpha (\alpha > -1)$ be the finite measure defined on D by $d\lambda_\alpha(z) = (1 - |z|^2)^\alpha dA(z)$ .For $\alpha > -1$ and $0 < p < 1$ the weighted Bergman space A_α^p. is the collection of all functions f analytic in D for which $\|f\|_{p,\alpha}^p = \int_D |f|^2 \, d\lambda_\alpha < \infty$. The weighted Dirichlet space D_α $(\alpha > -1)$ is the collection of all analytic functions f in D for which the derivative $\tilde{f}$ belongs to A_α^2. Note that A_α^p,is a Banach space for $p \geq 1$, and a Hilbert space for $p = 2$ [3]. The Dirichlet space D_α. is a Hilbert space in the norm

$$\|f\|_{D_\alpha}^2 = |f(0)|^2 + \int_D \left|\acute{f}\right|^2 d\lambda_\alpha$$

For these spaces the unit ball is a normal family and the point evaluation is bounded. Also, $f(z) = \sum a_n z^n$ analytic in D belongs to A_α^2 if and only if $\sum (n+1)^{-1-\alpha} |a_n|^2 < \infty$, and to D_α,if and only if $\sum (n+1)^{1-\alpha} |a_n|^2 < \infty$. We also note that if . $\alpha > -1$, then $D \subset A_\alpha^2$,and the inclusion map is continuous.

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A function ϕ on D is said to have an angular derivative at $\zeta \in \partial D$ if there exist a complex number c and a point $\omega \in \partial D$ such that $(\phi(z) - \omega)|(z - \zeta)$ tends to c as z tends to ζ over any triangle in D with one vertex at ζ. Define$d(\zeta) = \lim\inf_{z\to\zeta} \frac{1-|\phi(z)|}{1-|z|}$, where z tends unrestrictedly to ζ through D. By [4] the existence of an angular derivative at $\zeta \in \partial D$ is equivalent to $d(\zeta) < 1$.

Proposition (1.1) If ϕ is analytic in D with ϕ (D) ⊂ D, then C_ϕ,is bounded on A_α^p for all $0 < p < 1$ and $\alpha > -1$. Also, if $\omega \in H^\infty$ then ωC_ϕ.is

bounded on A_α^p ,for all $0 < p < 1$ and $\alpha > -1$.

In this paper we characterize such operators and those for which $(\omega C_\emptyset)^*$ is a composition operator. We also study the boundedness and compactness of the weighted composition operators on $A_\propto^p$ or $D_\propto$. The relationship between the compactness of such operators and a special class of measures on the unit disc, Carleson measures, is shown. The main result is to determine, in terms of geometric properties of $\emptyset$ and ω, when$\omega C_\emptyset$is a compact operator on weighted Dirichlet spaces. For Bergman spaces we attack the problem in terms of an angular derivative of $\emptyset$ and an angular limit of ω. We obtain some sufficient conditions for weighted Dirichlet spaces. Finally, we would like to acknowledge the fact that we are borrowing heavily the techniques of the proofs of [5]

1. Adjoint of Weighted Composition Operators

In this section we investigate when the adjoint of a weighted composition operator on some functional Hilbert space is a composition operator.

Theorem (2.1)

Let $T = \omega C_\phi$. be a weighted composition operator on A_α^p. $\alpha > -1$. Then $T^* = C_\psi$ if and only if $\omega = k_\lambda$and $\phi(z) = az(1 - \bar{\lambda}z)^{-1}$, where $\lambda = \psi(0)$ and $a$is a suitable constant. In particular, ψ has the form $\psi(z) = \bar{a}z + \lambda$.

Proof

Assume$(\omega C_\phi)^* = C_\psi$ .Then $(\omega C_\phi)^* k_x = C_\psi k_x \; or \; \overline{\omega(x)} k_{\phi(x)}(y) = k_x \circ \psi(y)$. It follows that

$$\frac{\overline{\omega(x)}}{\left(1 - \overline{\phi(x)}y\right)^{\alpha+2}} = \frac{1}{\left(1 - \bar{x}\psi(y)\right)^{\alpha+2}} \quad x, y \in D$$

In short $\left(1 - \overline{\phi(x)}(y)\right)^{\alpha+2} = \overline{\omega(x)}\left(1 - \bar{x}\psi(y)\right)^{\alpha+2}. If\ we\ put\ y = 0\ and$

$\psi(0)=\lambda$ we have 1= $\overline{\omega(x)}(1-\lambda\bar{x})^{\alpha+2}$ .Therefore $\omega=k_\lambda$.We also have $(1-\lambda\bar{x})\left(1-\overline{\phi(x)}(y)\right)=1-\bar{x}\psi(y)$ for all $x,y\in D$. Hence $\overline{\phi(x)}y+\lambda\bar{x}+\lambda\overline{\phi(x)}\bar{x}(y)=\bar{x}\psi(y). New \quad xy\neq 0, then\overline{\phi(x)}\ (1-\lambda\bar{x})(\bar{x})^{-1}=\left(\psi(y)-\psi(0)\right)y^{-1}$. Sine the right-hand side is independent of x, it should be a constant, say $\bar{a}$, $a\in C$.Therefore $\psi(z)=\bar{a}z+\lambda$. and $\psi(z)=az(1-\bar{\lambda}z)^{-1}$.

Conversely, suppose $T=\omega C_\phi$, where $\omega=k_\lambda$. and $\phi(x)=ax(1-\bar{\lambda}x)^{-1}$ $a\in C$ Then

$$T^*k_y(x)=\overline{\omega(y)}k_{\phi(y)}(x)=\frac{1}{(1-\lambda\bar{y})^{\alpha+2}}\cdot\frac{1}{\left(1-\overline{\phi(y)}x\right)^{\alpha+2}}$$

$$T^*k_y(x)=\overline{\omega(y)}k_{\phi(y)}(x)=\frac{1}{(1-\lambda\bar{y})^{\alpha+2}}\cdot\frac{1}{\left(1-\overline{\phi(y)}x\right)^{\alpha+2}}$$

$$=\frac{1}{(1-\lambda\bar{y})^{\alpha+2}}\cdot\frac{1}{(1-\overline{ay}(1-\lambda\bar{x})^{-1}x)^{\alpha+2}}=\frac{1}{(1-\lambda\bar{y}-\overline{ay}x)^{\alpha+2}}$$

$$=C_\psi k_y(x)$$

$where \quad \psi(x)=\bar{a}x+\lambda$

Theorem (2.2) [6]

If . is a no constant analytic function defined on the unit disc D with $.\phi(D)\subset D$ such that C_ϕ^* is subnormal on H^2 (and not normal), then there is a number c with $|c|=1$ for which $\lim_{\rho\to 1}\phi(\rho c)=c\ and$ and $\lim_{\rho\to 1^-}\acute{\phi}(\rho c)=s<1$. Moreover, if ϕ is analytic in a neighborhood of c, then C_ϕ^* is subnormal on H^2 if and only if

$$\phi(x)=\frac{(r+s)z+(1-s)c}{r(1-s)\bar{c}z-(1+sr)}$$

for some $\mathrm{r,s}$ with $0\leq \mathrm{r}\leq 1$ and $0<s<1$. Here, as above, $\mathrm{s}=\acute{\phi}(\mathrm{c})$:

Corollary (2.3)

If $\omega=k_\lambda$and $.\phi(z)=szk_\lambda(z)$ with $0<s<1$ and $\lambda=(1-s)c$, where c is the number indicated in (Theorem (1.1.3)), then ωC_ϕ. Is subnormal on H^2.

3. A Weighted Shift Analogy

As we shall see, for suitable w and . the operator$(\mathrm{wC}_\emptyset)^*$. (as well as the operator $\mathrm{wC}_\emptyset$) has an invariant subspace on which it is similar to a weighted shift.

We begin by defining the notions of forward and backward iteration sequences, see also [7].

Definition (3.1)

A non-constant sequence $\{z_k\}_{k=0}^{\infty}$is a B-sequence for . if $\phi(z_k) = z_{k-1}$, $k = 1,2,\ldots$ A nonconstant sequence $\{z_k\}_{k=0}^{\infty}$ or $\{z_k\}_{k=-\infty}^{\infty}$ is an F-sequence for ϕ if $\phi(z_k) = z_{k+1}$for all k.

Theorem(3.2)

If $\{z_j\}_{j=0}^{\infty}$ is a B-sequence for, and

$$\frac{1-|z_j|}{1-|z_{j-1}|} \le r \ < \ 1$$

for all j, then $\{z_j\}_{j=0}^{\infty}$ gives rise to an invariant subspace of $(\omega C_\phi)^*$. On which it is similar to a backward weighted shift

Proof

. Let $\{z_j\}$ be a B-sequence as in the statement of the (theorem. By [8,p. 203]), $\{z_j\}$is an interpolating sequence. Let $u_j = (1-|z_j|^2)^{\frac{1}{2}} k_j$,where k_j denotes the reproducing kernel at z_j .We keep this notation throughout the rest of this section. Let $\mathcal{M}$be the closed linear span of $\{u_j\}$. By [7], $\{u_j\}$ is a basic sequence in $\mathcal{M}$ equivalent to an orthonormal basis. Since.

$$(\omega C_\phi)^* u_j = \left(1-|z_j|^2\right)^{\frac{1}{2}} \overline{\omega(z_j)} k_{j-1} = \overline{\omega(z_j)} \left(\frac{1-|z_j|^2}{1-|z_{j-1}|^2}\right)^{\frac{1}{2}} u_{j-1}$$

$(\omega C_\phi)^*|\mathcal{M}$ is similar to a backward weighted shift with weights

$$\left\{ \left(\frac{1-|z_{j+1}|}{1-|z_j|}\right)^{\frac{1}{2}} \overline{\omega(z_{j+1})} \right\}$$

Recall that if φ is analytic in D with ∅(D) ⊂ D and ∅ is not an analytic elliptic automorphism of D, then there is a unique fixed point a of ∅ (with |a| ≤ 1) such that |∅⁻ (a)| ≤ 1. We will call the distinguished fixed point a the Denjoy–Wolff point [8] of ∅. We note that if |a| = 1 then 0 < ∅⁻ (a) ≤ 1, and if |a| < 1 then 0 ≤ |∅⁻ (a)| < 1

Corollary (3.3)

If ϕ has a Denjoy–Wolff point a in ∂D with $.\acute{\phi}(a) \ < \ 1$ then every F-sequence for ϕ gives rise to an invariant subspace of $(\omega C_\phi)^*$ on which it is similar to a forward weighted shift with weights

$$\left\{\left(\frac{1-|z_{j-1}|^2}{1-|z_j|^2}\right)^{\frac{1}{2}}\frac{}{\omega(z_{j-1})}\right\}$$

Corollary (3.4)

For $0 < s < 1$ let $\omega = k_{1-s}$ and $.\phi(z) = szk_{1-s}(z)$.

Then ωC_ϕ. has an invariant subspace M such that $\omega C_\phi|\mathcal{M}$ is similar to a weighted shift.

Proof.

Let $\psi(z) = sz + (1-s)$. Then 1 is a Denjoy–Wolff point for. Also, $\acute{\psi}(1) = s < 1$. So by Corollary (3.3) every F-sequence for gives rise to an invariant subspace of C_ψ^*. on which it is similar to a weighted shift. Now by Theorem (2.1) $C_\psi^* = \omega C_\phi$. where $\omega = k_{1-s}$ and . $\phi(z) = szk_{1-s}(z)$. The proof is now complete.

We note that if ϕ has a Denjoy–Wolff point a in ∂D with $\acute{\phi}(1) < 1$, then for real θ, C_ϕ, is similar to $e^{i\theta}$ C_ϕ. [7]. In fact, much more is true. For the proof of the next corollary see [7].

Corollary (3.5).

If ϕ is an analytic map of the disc to itself, $\phi(1) = 1$ and $\acute{\phi}(1) < 1$, then for any function w for which ωC_ϕ. is bounded we have ωC_ϕ. similar to $\lambda\omega C_\phi$. For $|\lambda| = 1$

4. Compactness on Weighted Bergman Spaces

In this section we will focus our attention on the relationship between compact weighted composition operators and a special class of measures on the unit disc. First, we will recall a few definitions. For $0 < \delta \leq 2$ and $\zeta \in \partial D$ let

$S(\zeta, \delta) = \{z \in D : |z - \zeta| < \delta\}$.

One can show that the λα-measure of the semi disc S(ζ, δ) is comparable with $\delta^{\alpha+2}$ ($\alpha > -1$). We can now give

Definition(4.1)

compact α - Carleson measure) Let $\alpha > -1$ and suppose . is a finite positive Borel measure on D. We call μ an α-Carleson measure if

$$\|\mu\|_\alpha = \sup \mu(S(\xi,\delta))/\delta^{\alpha+2} < \infty \quad , \qquad S(\xi,\delta) = \{z \in D: |z-\xi| < \delta\}$$

where the supremum is taken over all $\xi \in \partial D$ and $0 < \delta \leq 2$. If, in addition

$$\lim_{\delta\to 0} \sup_{\xi\in\partial D} \mu(S(\xi,\delta))/\delta^{\alpha+2} = 0$$

then we call μ a compact α - Carleson measure.

The next theorem is stated and proved in [5]. Since we refer to it several times, we state it without proof.

Theorem (4.2)

Fix $0 < p < \infty$ and $\alpha > -1$ and let μ be a finite positive Borel measure on D. Then μ is an α-Carleson measure if and only if $A_\alpha^p \subset L^P(\mu)$. In this case the inclusion map $I_\alpha.: A_\alpha^p \to L^P(\mu)$ is a bounded operator with a norm comparable with $\|\mu\|_\alpha$ .If μ is an α- Carleson measure, then I_α is compact if and only if μ is compact.

In the next theorem we extend the result of [5, Corollary 4.4] by characterizing the compact weighted composition operators on the spaces $A\,_\alpha^p$in terms of Carleson measures

Theorem(4.3)

Fix $0 < p < \infty$ and $\alpha > -1$.Then $\omega C_\phi$is a bounded (compact) operator on A_α^p ,if and only if the measure $\mu_{\alpha,P} \circ \phi^{-1}$is an α- Carleson (compact α-Carleson) measure. Here $d\mu_{\alpha,P} = |\omega|^P d\lambda_\alpha$.

Proof.

We know that for every $f \in A_\alpha^p$ By(Theorem (4,2) ωC_ϕ. is bounded on A_α^p if and only if $\mu_{\alpha,P} \circ \phi^{-1}$ is an α-Carleson measure.

Now equip the space A_α^p. with the metric of $L^p(\mu_{\alpha,P} \circ \phi^{-1})$ and call this (usually incomplete) space X. The above equation shows that ωC_ϕ, induces an isometry S of X into A_α^p. Thus $\omega C_\phi = SI_\alpha$. is compact if and only if I_α is. An application of (Theorem (4.2)) completes the proof.

A modification of the proof of Theorem (5.3) of [5] will give

Theorem (4.4)

Suppose $\alpha > -1$, $p > 0$.

(i) If $\omega C_\phi$is a compact operator on A_α^p , then . does not have an angular derivative at those points of ∂D at which ω has a nonzero angular limit.

(ii) Suppose w has a zero angular limit at any point of ∂D at which .has an angular derivative; then ωC_ϕ is compact.

5. Boundedness on Weighted Dirichlet Spaces

In this section we study the relationship between the boundedness of weighted composition operators on weighted Dirichlet spaces and a special class of measures on the unit disc. We recall that $D_1 = H^2$ and if α > 1 then $D_\alpha = A_{\alpha-2}^2$ and the characterization of bounded (compact) weighted composition operators on D_α for α > 1 is given in Theorem (4.3).. However, for −1 < α < 1, an obvious necessary condition for ωC_α to be bounded on Dα is that $\omega = \omega C_\alpha \in$I D_α. In the following, we characterize the boundedness of such operators.

Theorem (5.1)

Suppose $\omega \in D_\alpha$ Then $\omega C_\phi$is bounded on D_α if the measures $\mu_\alpha \circ \phi^{-1}$ and $v_\alpha \circ \phi^{-1}$are α-Carleson measures, where $d\mu_\alpha = |\acute{\omega}|^2 d\lambda_\alpha$ and $dv_\alpha = |\omega|^2 |\acute{\phi}|^2 d\lambda_\alpha$

Proof.

Assume $\mu_\alpha \circ \phi^{-1}$ and $v_\alpha \circ \phi^{-1}$ are α-Carleson measures. Then, for every f in D_α we have $\acute{f} \in A_\alpha^p \in L^2(v_\alpha \circ \phi^{-1})$ by Theorem (4.2) For every f in D_α we have $(\omega C_\phi f)^/ = \omega f \circ \phi + \omega(f \circ \phi)^/$. We now have

$$\left\|\omega(f \circ \phi)^/\right\|_{2,\alpha}^p = \int |\omega|^2 |\acute{\phi}|^2 |\acute{f} \circ \phi|^2 d\lambda_\alpha = \int |\acute{f} \circ \phi|^2 dv_\alpha = \int |\acute{f}|^2 dv_\alpha \circ \phi^{-1} < \infty$$

therefore, $\omega(f \circ \phi)^/ \in A_{\alpha-2}^2$ Note also that

$$\int |\acute{\omega}|^2 |f \circ \phi|^2 d\lambda_\alpha = \int |f \circ \phi|^2 d\mu_\alpha = \int |f|^2 d\mu_\alpha \circ \phi^{-1}$$

Since $f \in D_\alpha \subset A_\alpha^2 \subset L^2(\mu_\alpha \circ \phi^{-1})$, we have $\int |\acute{\omega}|^2 |f \circ \phi|^2 d\lambda_\alpha < \infty$.

Combining these two observations we conclude that $(\omega C_\phi f)^/ \in A_\alpha^2$ for every f in D_α Therefore $\omega C_\phi f \in D_\alpha$. and ωC_ϕ is bounded on D_α

6. Compactness on Dirichlet Spaces

The main result of this section concerns sufficient conditions for the compactness of weighted composition operators on Dirichlet spaces D_α. We would like to investigate whether an analogue of Theorem 4.3, the Carleson measure characterization of compact weighted composition operators, holds for Dirichlet spaces

Theorem (6.1)

Compactness on Dirichlet Spaces) If $\mu_\alpha \circ \phi^{-1}$ and $v_\alpha \circ \phi^{-1}$ are compact,α-Carleson measures, where $d\mu_\alpha = |\acute{\omega}|^2 d\lambda_\alpha$ and $dv_\alpha = |\omega|^2 |\acute{\phi}|^2 d\lambda_\alpha$, then ωC_ϕ is compact on D_α for $\alpha > -1$.

Proof

Let X denote the space D_α taken in the metric induced by$\|.\|_1$ defined by

$$\|f\|_1^2 = (\|f\|_2 + \|f\|_3)^2 + |\omega(0) f \circ \phi(0)|^2$$

where $\|f\|_2^2 = \int_D |f|^2 d\mu \circ \phi^{-1}$ and $\|f\|_3^2 = \int_D |\acute{f}|^2 dv_\alpha \circ \phi^{-1}$ $(f \in D_\alpha)$

Let $I : D_\alpha \to X$ be the identity map and define $S : X \to D_\alpha$. By $Sf = wf \circ \phi$. So $\omega C_\phi = SL$. To show that S is a bounded operator , let $f \in X$. Then

$$\|Sf\|_{D_\alpha}^2 = \int_D \left|\omega(f\circ\phi)^{/}\right|^2 d\lambda_\alpha + |\omega(0)f\circ\phi(0)|^2$$

$$\leq \left(\|\acute{\omega}f\circ\phi\|_{2,\alpha} + \left\|\omega\acute{\phi}(\acute{f}\circ\phi)\right\|_{2,\alpha}\right)^2 + |\omega(0)f\circ\phi(0)|^2$$

We use the change of variable formula to get

$$\int_D |\acute{\omega}|^2|f\circ\phi|^2 d\lambda_\alpha = \int_D |f|^2 d\mu_\alpha\circ\phi^{-1} = \|f\|_2^2$$

And

$$\int_D |\omega|^2\left|\acute{\phi}\right|^2\left|\acute{f}\circ\phi\right|^2 d\lambda_\alpha = \int_D \left|\acute{f}\right|^2 dv_\alpha\circ\phi^{-1} = \|f\|_3^2$$

Thus we have

$$\|Sf\|_{D_\alpha}^2 \leq (\|f\|_2 + \|f\|_3)^2 + |\omega(0)f\circ\phi(0)|^2 = \|f\|_1^2$$

Hence $|S| \leq 1$ and S is bounded. If we show that I is compact, then ωC_ϕ is compact and the proof is complete.

Now, we use the idea of [6] to prove that I is compact. It is enough to show that each sequence (f_n) in D_α. That converges uniformly to zero on compact subsets of D must be norm convergent to zero in X.

Fix $0 < \delta < 1$ and let μ_δ and $v_\alpha$be the restriction of the measures $\mu_\alpha\circ\phi^{-1}$ and $v_\alpha\circ\phi^{-1}$ to the annulus $1-\delta \leq |Z| < 1$. Observe that the α-Carleson norm of μ_δ and v_δ satisfy

$$\|\mu_\delta\|_\alpha \leq c_1\, sup\mu_\alpha\circ\phi^{-1}(S(\xi,r))/r^{\alpha+2}$$

and

$$\|v_\delta\|_\alpha \leq c_2\, sup v_\alpha\circ\phi^{-1}(S(\xi,r))/r^{\alpha+2}$$

where the supremum is taken over all $0 < r <$. and $\xi \in \partial D$, and c_1 , c_2 are positive constants which depend only on α Since $\mu_\alpha\circ\phi^{-1}$ and $v_\alpha\circ\phi^{-1}$ are compact α-Carleson measures, the right-hand sides of the above two inequalities, which we denote by $\epsilon_1(\delta)$ and $\epsilon_2(\delta)$, respectively, tend to zero as $\delta \to 0$. So we have

$$\|f_n\|_2^2 = \int_{|Z|<1-\delta} |f_n|^2 dv_\alpha\circ\phi^{-1} + \int_D |f_n|^2 d\mu_\delta \leq O(1) + k_1\epsilon_1(\delta)\|f_n\|_{2,\alpha}^2$$

and in the same manner

$$\|f_n\|_3^2 \leq o(1) + k_2\epsilon_2(\delta)\left\|\acute{f}_n\right\|_{2,\alpha}^2$$

where k_1 and $k_2$are constants depending only on α. We recall that the estimate of the first terms comes from the uniform convergence of (f_n) to zero on $|Z| \leq 1-\delta$,and the estimate of the second terms comes from the first part of [6]. Theorem(3.4) Since $\epsilon_i(\delta) \to 0$ as $\delta \to 0$,i = 1,2, and $\omega(0) f_n \circ \phi(0) \to 0$, we have $\|f_n\|_1 \to 0$, which completes the proof.

Theorem (6.2)

If $\acute{\omega}$ has a zero angular limit at any point of ∂D at which ϕ has an angular derivative, then $\mu_\alpha \circ \phi^{-1}$is a compact α −Carleson measure .Here $d\mu_\alpha = |\acute{\omega}|^2 d\lambda_\alpha$.

Proof

Suppose $\acute{\omega}$ has a zero angular limit at those points of ∂D at which ϕ has an angular derivative. Choose $0 < \gamma < \alpha$. with $r = 2 - (\alpha - \gamma) > 0$.

For $0 < \delta < 2$ define

$$\epsilon(\delta) = \sup\left\{\frac{1 - |z|^2 \acute{\omega}(z)}{1 - |\phi(z)|^2} : 1 - |z| \le \delta\right\}$$

Since $\acute{\omega}$ has a zero angular limit at those points of ∂D at which , has an angular derivative we have $lim_{\delta \to 0} \epsilon(\delta) = 0$. With no loss of generality assume that $\phi(0) = 0$. Fix $S = S(\xi . \delta)$. By the Schwartz Lemma and definition of $\epsilon(\delta)$ we have

$$|\acute{\omega}(Z)|(1 - |z|^2) \le (1 - |\phi(z)|^2)\epsilon(\delta) \le 2\delta\epsilon(\delta)$$

whenever $\phi(z) \in S$. So we have

$$\mu_\alpha \circ \phi^{-1}(S) = \int_{\phi^{-1}(S)} |\acute{\omega}(Z)|^2 (1 - |z|^2)^\alpha d\lambda(Z)$$

$$\le (2\delta\epsilon(\delta))^{\alpha-\gamma} \int_{\phi^{-1}(S)} |\acute{\omega}(Z)|^r (1 - |\phi(z)|^2)^\gamma d\lambda(Z)\ (2\epsilon(\delta))^{\alpha-\gamma} \delta^{\alpha-\gamma} \mu_{r,\gamma} \circ \phi^{-1}(S)$$

Here $d\mu_{r,\gamma}(z) = |\acute{\omega}(Z)|^2 d\lambda_\gamma(z)$ Now by (Proposition (1.1) and Theorem (1.1.11) $\mu_{r,\gamma} \circ \phi^{-1}$ is a γ-Carleson measure. Thus there exists a constant k independent of $\xi . \delta$ such that $\mu_{r,\gamma} \circ \phi^{-1}(S) \le k\delta^{\gamma+2}$.

Hence $\mu_{\alpha,\gamma} \circ \phi^{-1}(S) \le k(2\epsilon(\delta))^{\alpha-\gamma}\delta^{\alpha+2}$.Since$\epsilon(\delta) \to 0$ as$\delta \to 0$, is therefore a compact α-Carleson measure and the proof is complete.

Theorem (6.3)

Suppose w has a zero angular limit at any point of ∂D at which ϕ has an angular derivative. If in addition, for some $-1 < \gamma < \alpha$, the measure $\eta_\gamma \circ \phi^{-1}$is a α-Carleson measure, where $d\eta_\gamma(z) = |\omega|^{2-\alpha+\gamma} |\acute{\phi}|^2 d\lambda_\gamma$,then $v_\alpha \circ \phi^{-1}$,is a compact α −Carleson measure ($dv_\alpha(z) = |\omega|^2 |\acute{\phi}|^2 d\lambda_\alpha$)

Proof

For $0 < \delta < 2$ define

$$\rho(\delta) = \sup\left\{\frac{1-|z|^2|\omega(z)|}{1-|\phi(z)|^2} : 1-|z| \le \delta\right\}$$

By the argument of the proof of (Theorem (6,2)) $lim_{\delta\to 0}\rho(\delta) = 0$. Also, we Have $|\omega(Z)|(1-|z|^2) \le (1-|\phi(z)|^2)\rho(\delta) \le 2\delta\rho(\delta)$,whenever $\phi(z) \in S(\xi,\delta)$.

Thus

$$v_\alpha \circ \phi^{-1}(S) = \int_{\phi^{-1}(S)} |\omega|^2 \left|\acute{\phi}(z)\right|^2 (1-|z|^2)^\alpha d\lambda(Z) M d\lambda(Z) = (2\rho(\delta))^{\alpha-\gamma}\eta_\gamma \circ \phi^{-1}(S)$$

Now we use the hypothesis that $\eta_\gamma \circ \phi^{-1}$ is a γ-Carleson measure; so there exists a constant k independent of ξ and δ such that $\eta_\gamma \circ \phi^{-1}(S) \le k\delta^{\alpha-\gamma}$.

Thus $v_\alpha \circ \phi^{-1}(S) \le k(2\rho(\delta)^{\alpha-\gamma})\delta^{\grave{\alpha}+2}$. Since $\rho(\delta) \to 0$ as $\delta \to 0$, $v_\alpha \circ \phi^{-1}$ therefore a compact $\alpha -$ Carleson measure, and the proof is complete.

Theorem (6.4)

Let $\acute{\omega} \in H^\infty$ and $\phi \in D_\alpha$ Assume ω and $\acute{\omega}$ have a zero angular limit at any point of ∂D at which . has an angular derivative. If, in addition, for some $-1 < \gamma < \alpha$, the measure $\eta_\gamma \circ \phi^{-1}$ is a γ-Carleson measure, then ωC_ϕ,is compact on D_α Here $d\eta_\gamma = |\omega|^{2-\alpha+\gamma}\left|\acute{\phi}\right|^2 d\lambda_\gamma$.

Proof. By**(**Theorems (6.1) and (6.3), the measures $\mu_\alpha \circ \phi^{-1}$and $v_\alpha \circ \phi^{-1}$ are compact α-Carleson measures. Thus Theorem (6.1) shows that ωC_ϕ is compact on D_α.

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