Solution of Fractional Summation Difference Equation of finite delay in fuzzy Metric Space

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Abstract:-

Fractional difference equations provide a powerful framework for modeling discrete dynamical systems with memory and hereditary characteristics. On the other hand, fuzzy metric spaces incorporate uncertainty and imprecision that naturally arises in control, engineering, population models, and biological systems. Motivated by the necessity to combine memory with uncertainty in a discrete environment, this paper investigates the existence and uniqueness of solutions for a class of nonlinear fractional summation difference equations of finite delay in fuzzy metric spaces. Using a fixed-point approach supported by the properties of fractional sum operators, sufficient conditions are derived to guarantee unique mild solutions. An illustrative example is provided to demonstrate the applicability of the developed theory.

Keywords: Fractional difference equations, fuzzy metric space, finite delay, existence, uniqueness, fixed point theorem.

Introduction:-

Fractional calculus extends traditional differentiation and integration to non-integer orders, enabling the modeling of processes with memory and hereditary properties. Its discrete counterpart, fractional summation difference equations, plays a crucial role in describing discrete-time systems such as control processes, image analysis, and biological dynamics (16,3,21). When such systems incorporate finite delays, their analysis becomes more intricate due to the interaction between memory effects and time delays, leading to nonlocal and complex dynamic behavior (23,25).

In many real-world applications, uncertainties and vagueness are inherent in system parameters or measurements. To address this, fuzzy metric spaces—introduced by Zadeh (1965) and developed further by Rus (17) and Muresan (14)—provide a mathematical framework that captures imprecision by associating degrees of closeness between elements. Integrating fractional difference equations with fuzzy metric spaces allows the modeling of uncertain, memory-dependent discrete systems with delays, which arise naturally in fields such as engineering, biology, and control theory (1, 20).

This paper aims to establish the existence and uniqueness of mild solutions for nonlinear fractional summation difference equations of finite delay within a fuzzy metric space. Using the Banach fixed-point theorem and properties of fractional summation operators, sufficient conditions are derived that ensure the existence of unique mild solutions. Furthermore, illustrative examples validate the theoretical results and highlight their applicability.

The study contributes to the growing theory of fractional discrete systems under uncertainty by unifying fractional calculus, difference equations, and fuzzy analysis. The results not only extend previous findings in Banach spaces but also lay a foundation for future work on impulsive, stochastic, and hybrid fuzzy fractional systems.

Definitions and Preliminaries :-

We begin by recalling essential definitions in fuzzy metric spaces **and** fractional sum operators**.**

Definition 1 (Fuzzy Metric Space)

A triple $(X, M,*)$ is called a fuzzy metric space where

- X is a non empty set
- * is a continuous t-norm
- $M: X \times X \times (0, \infty) \to [0,1]$ satisfying

1. $M(x, y, t) > 0$
2. $M(x, y, t) = 1 \Leftrightarrow x = y$
3. $M(x, y, t) = M(y, x, t)$
4. $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$
5. $M(x, y,*)$ is continuous

Definition 2 (Gamma Function)

$$\Gamma(\alpha) = \int_0^\infty e^{-s} s^{\alpha-1} ds , \qquad \alpha > 0$$

Definition 3 (Fractional Sum Operator)

For $f: \mathbb{N} \to \mathbb{R}$, fractional sum of $\alpha > 0$ is

$$\triangle^{-\alpha} f(k) = \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{k-1} (k - j - 1)^{(\alpha-1)} f(j), \qquad k \in \mathbb{N}$$

The fractional difference of order $\alpha \in (0,1)$ is defined by

$$\triangle^{-\alpha} x(k) = \triangle \ \ \triangle^{- \ 1-\alpha)} x(k$$

These operators characterize systems whose evolution depends on present and weighted past states

Problem Formulation

Consider a non linear fractional summation difference equation with finite delay

$$\triangle^{\alpha}\, x(k) = f\big(k, x(k), x(k-\tau)\big)\ ,\ k \in \mathbb{N}$$

With initial delay conditions $x(k) = \phi(k)\ ,\ k \in \{-\tau, \ldots\ldots, 0\}$

where

- $0 < \alpha < 1,$
- $\tau \in \mathbb{N}$ indicates the delay,
- $f: \mathbb{N} \times X \times X \to X$ is fuzzy bounded

Problem Solution:-

We assume the following.

(H1) There exists a metric d on X which induces the same topology as the fuzzy metric M

(H2) (Lipschitz) there are constants $L_1, L_2 \geq 0$ such that for every $k \in \mathbb{N}$ and all $u, u', v, v' \in X$,

$$d\big(f(k,u,v), f(k,u',v')\big) \leq L_1 d(u,u') + L_2 d(v,v').$$

Set $L = L_1 + L_2$.

(H3) The initial history φ is bounded $Sup_{-\tau \leq k \leq 0} d(\varphi(k), x) < \infty$.

Using the shift Fractional sum convention, a sequence x is a mild solution on $[-\tau, N]$ iff
for $k \geq 1$

$$x(k) = \varphi(0) + \triangle^{-\alpha}\,(F_x)(k) = \varphi(0) + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{k-1} (k-j)^{\alpha-1} f(j, x(j), x(j-\tau))$$

Thus solving the evolution is equivalent to finding a fixed point of the operator T defined below

Local existence & uniqueness on finite interval $[-\tau, N]$

Fix an arbitrary integer $N \geq 1$ work in the Banach Space

$C_N = \{x: \{-\tau, -\tau+1, \ldots N\} \to X\}$ equipped with the sup metric induced by d:

$||x - y||_{\infty} = \max d\big(x(k), y(k)\big)$. Define the operator $T: C_N \to C_N$ by

$$(Tx)(k) = \begin{cases} \varphi(k) & , \quad -\tau \le k \le 0, \\ \varphi(0) + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{k-1} (k-j)^{\alpha-1} f(j, x(j), x(j-\tau)), & 1 \le k \le N \end{cases}$$

Claim A – T maps C_N into itself.

This is immediate from the definition T assigns the given history for $k \le 0$ and defines values for $k \in \{1,2,\dots\dots N\}$ via a finite sum of elements in X. Boundedness follows from continuity of f on bounded set and the finiteness of the sum

Claim B – T is a contraction on C_N provided a simple quantitative bound holds.

For any $x, y \in C_N$ and $1 \le k \le N$ we estimate (using the lipschitz property H2)

$$d\big((Tx)(k).(Ty)(k)\big)$$

$$= \frac{1}{\Gamma(\alpha)} d\left(\sum_{j=0}^{k-1} (k-j)^{\alpha-1} f(j, x(j), x(j-\tau)), \sum_{j=0}^{k-1} (k-j)^{\alpha-1} f(j, y(j), y(j-\tau))\right)$$

$$\le \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{k-1} (k-j)^{\alpha-1} f(j, x(j), x(j-\tau)), (k-j)^{\alpha-1} f(j, y(j), y(j-\tau)$$

$$\le \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{k-1} (k-j)^{\alpha-1} (L_1 d(x(j), y(j)) + L_2 d(x(j-\tau), y(j-\tau))$$

$$\le \frac{L}{\Gamma(\alpha)} \sum_{m=1}^{k} m^{\alpha-1} ||x-y||_\infty$$

Taking maximum over $-\tau \le k \le N$ gives

$$\left|T_x - T_y\right|_\infty \le qN||x-y||_\infty \ , \quad qN = \frac{L}{\Gamma(\alpha)} \sum_{m=1}^{k} m^{\alpha-1}$$

If qN < 1 then T is a strict contraction on C_N. By the Banach fixed-point theorem there exists a unique $x^{(N)} \in C_N$ such that $Tx^{(N)} = x^{(N)}$ By (MS) this fixed point is exactly the unique mild solution of the problem on the interval $[-\tau, N]$

Thus **for every N satisfying** qN < 1 we obtain existence and uniqueness on the interval $[-\tau, N]$

Main Result:-

Let $(X, M, *)$ be a complete fuzzy metric space and let d be a metric compatible with M. Suppose $f: \mathbb{N} \times X \times X \to X$ satisfies the Lipschitz condition with constants L_1, L_2 and let $L = L_1 + L_2$. Fix $0 < \alpha < 1$ then there exist an integer $N_0 \ge 1$ such that if

$$\frac{L}{\Gamma(\alpha)} \sum_{m=1}^{k} m^{\alpha-1} < 1,$$

Then the problem $\triangle^{\alpha} x(k) = f\big(k, x(k), x(k-\tau)\big)$, $k \in \mathbb{N}$ $x(k) = \varphi(k)$, $k = -\tau, \dots, 0$

has a unique mild solution $x: \mathbb{Z}_{-\tau}^{\infty} \to X$ In particular, by the stepwise extension described above one obtains a unique global solution on $\mathbb{N}$

Theorem 1 (Existence)

Assume:

$M(f(k,u,v), f(k,u',v'), \lambda t) \geq M(u.u', t) * M(v, v', \mu t)$, for some $\lambda, \mu \in (0,1)$ then the system admits at least one mild fuzzy solution.

Proof:

Using the fractional sum transform: $x(k) = \phi(0) + \triangle^{-} f(k-1, s(k-1), x(k-1-\tau))$,

Define of operator $T: X \to X$, $(Tx)(k) = \phi(0) + \triangle^{-} f(k-1, s(k-1), x(k-1-\tau))$.

By applying fuzzy contractive property and completeness of fuzzy metric space, T maps a closed ball into itself and is continuous. By Schaduer's Fixed-Point Theorem, a fixed point exists ⇒ solution exists.

Theorem 2 (Uniqueness)

If further $\boldsymbol{\lambda} + \boldsymbol{\mu} < 1$ then the solution is unique.

Proof:

The condition implies T is a strict fuzzy contraction. Using **Banach fixed-point theorem**, the fixed point must be unique.

Illustrative Examples

1) Consider $\triangle^{0.5} x(k) = \frac{1}{2} x(k) + \frac{1}{3} x(k-1)$, $\phi(0) = 1, \phi(-1) = 0$.

   We use the shifted fractional sum convention (mild solution form)

   $$\triangle^{-\alpha} g(k) = \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{k-1} (k-J)^{\alpha-1} g(j), \quad k \geq 1$$

   For $\alpha = \frac{1}{2}$ $\triangle^{-\frac{1}{2}} g(k) = \frac{1}{\Gamma(\frac{1}{2})} \ {}_{j=0}^{k-1} (k-J)^{\frac{-1}{2}} g(j)$, $\Gamma\left(\frac{1}{2}\right. = \sqrt{\pi}$.

   The mild solution for our equation is

$$x(k) = x(0) + \triangle^{-1/2} \ \tfrac{1}{2}x(j) + \tfrac{1}{3}x(j-1)$$

So , for each $k \geq 1$

$$x(k) = 1 + \tfrac{1}{\sqrt{\pi}} \ {}_{j=0}^{k-1}(k-j)^{-1/2} \ \tfrac{1}{2}x(j) + \tfrac{1}{3}x(j-1)$$

We compute iteratively, since $x(k)$ on the right depends only on previously computed values $x(j)$ with $j < k$.

For $k = 1$,

$$\triangle^{-1/2} f(1) = \tfrac{1}{\sqrt{\pi}.} \ 1)^{-\frac{1}{2}} f(0) = \tfrac{1}{\sqrt{\pi}.}\tfrac{1}{2}$$

So $x(1) = 1 + \frac{1}{2\sqrt{\pi}} \approx 1.28209479177388$

For $k = 2$,

$$\triangle^{-1/2} f(2) = \tfrac{1}{\sqrt{\pi}.} \quad 2)^{-\frac{1}{2}} f(0) + (1)^{-\frac{1}{2}} f(1)$$

We computed $f(0) = \frac{1}{2}$ and $f(1) = \frac{1}{2}x(1) + \frac{1}{3}x(0) \approx 0.97438072922027$

Evaluating gives

$$x(2) \approx 1.74920659803646$$

For $k = 3$,

$$f(2) = \tfrac{1}{2}x(2) + \tfrac{1}{3}x(1) \approx 1.74920659803646. \text{ then}$$

$$x(3) = 1 + \tfrac{1}{\sqrt{\pi}.} \quad 3)^{-\frac{1}{2}} f(0) + (2)^{-\frac{1}{2}} f(1) + (1)^{-\frac{1}{2}} f(0)\Big) \approx 2.28614608731779$$

Fractional memory and delay amplify growth while preserving stability.
All assumptions from Theorem 1 to 2 are satisfied ⇒ unique fuzzy solution exists.

2) Consider the linear scalar fractional difference equation with finite delay $\tau = 1$

$$\triangle^{\alpha} x(k) = ax(k) + b(x-1) + g(k), \qquad k \in \mathbb{N}$$

With initial history $x(-1) = \psi, x(0) = \xi$ constants $a, b \in \mathbb{R}$ are known forcing $g: \mathbb{N} \to \mathbb{R}$ we seek an explicit mild solution using discrete fractional convolution.

Using fractional sum operator we have

$$x(k) = x(0) + \triangle^{-\alpha} \left(ax(.) + b(.-1) + g(. \quad (k), \qquad k \geq 1$$

Now the discrete fractional kernel $h_\alpha(k) = \frac{k^{(\alpha-1)}}{\Gamma(\alpha)} = \frac{\Gamma(k+\alpha-1)}{\Gamma(k)\Gamma(\alpha)} \quad k \geq 1$

So that $\triangle^{-\alpha} u(k) = \ {}^{k-1}_{j=0} h_\alpha (k-j)u(j)$

Therefore $x(k) = \xi + \ {}^{k-1}_{j=0} h_\alpha (k-j)(ax(j) + bx(j-1) + g(j))$

Rearrange $x(k) - \ {}^{k-1}_{j=0} h_\alpha (k-j)\big(ax(j) + bx(j-1)\big) = \xi + \ {}^{k-1}_{j=0} h_\alpha (k-j)g(j)$

Convolution form and iteration / resolvent kernel

$$r(k) = \delta_{k,0} + \ {}^{k-1}_{j=0} h_\alpha (k-j)(ar(j) + br(j-1)) \qquad k \geq 0$$

With $r(-1) \equiv 0$ and $\delta_{k,0}$, the Kroneker delta then the mild solution can be written formally as

$$x(k) = \sum_{j=0}^{k-1} r(k-j)\big(\xi.\delta_{j,0} + \ {}^{j-1}_{m=0} \delta_{j,0}(j-m)g(m)\big)$$

In the homogenous case $g = 0$, this reduces to a discrete convolution evolution

$$x(k) = (r * \eta)(k)$$

Where η contains initial data the resolvent kernel $r(k)$ may be computed recursively and for small k we can produce closed form values using gamma identities.

Applications:-

The fractional summation difference equations in fuzzy metric spaces are relevant to many contemporary applied problems. Below we list detailed modeling sketches, explicit equations showing where fractional summation/delay/fuzziness enters, and computational implementation .

1) Population Dynamics with Uncertain Environment and Time-lagged Interactions

Model idea: population at discrete times with reproduction depending on present and

population under environmental uncertainty

Model :- $\triangle^\alpha N(k) = rN(k)\left(1 - \frac{N(k-\tau)}{K} \right. + \eta(k)$

Where

- $N(k)$ population at generation k,
- r intrinsic growth rate,
- K carrying capacity,
- τ maturation delay,
- $\eta(k)$ fuzzy perturbation representing environmental uncertainty.

Implement fuzziness by treating initial $N(k)$ for $k \in [-\tau, 0]$ as fuzzy numbers or replacing additive perturbation $\eta(k)$ with fuzzy-valued noise $\tilde{\eta}(k)$Work in a fuzzy metric space $(\mathcal{F}, M,*)$ of fuzzy numbers with a suitable distance (e.g L^p- type metric on alpha cut) and apply the existence framework above.

Numerical method: apply Picard iteration with fractional kernel; simulate many alpha-cuts of fuzzy initial profile to obtain band of possible solutions; aggregate via level sets.

2) Fuzzy Control Systems with Fractional Discrete Controllers

Model idea: digital controller uses past samples (finite delay) and fractional difference dynamics to tune response; sensors and actuators have uncertain calibration -> fuzzy state.
Control law sketch:

$$\triangle^{\alpha} x(k) = Ax(k) + Bu(k-\tau) + \tilde{v}(k$$

$$u(k) = K\varepsilon\big(\tilde{y}(k), r(k)\big)$$

where $\tilde{y}, \tilde{v}$ are fuzzy measurements/noise, ε is a fuzzy error mapping, and K is gain. Stability and controller design use fractional closed-loop resolvent and fuzzy-norm-based gains ensuring contraction in fuzzy metric.

Implementation: discretize and implement fractional-sum via convolution with pre computed kernel; tune K using robust/fuzzy optimization

Conclusion:-

In this paper, we established the existence and uniqueness of mild solutions for fractional summation difference equations of finite delay in a fuzzy metric space. Using fixed-point theory and fractional difference operators, we transformed the problem into a contraction mapping framework ensuring a unique solution. Illustrative examples demonstrated the applicability of the theoretical results. The study extends classical results on fractional difference equations to fuzzy environments, incorporating uncertainty and delay. These findings provide a foundation for future research on impulsive, stochastic, or hybrid fuzzy fractional systems with potential applications in engineering, biology, and control theory.

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