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Analytical and Finite Element Methods for Simply Supported Beam Deflection and Slope Comparison under Various Loading Conditions

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Abstract—The finite element formulation for a simply supported beam is discussed in this study, along with the corresponding deflection and slope results. Numerical validation is carried out using the analytical solution based on Euler–Bernoulli beam theory. A uniformly varying load (UVL) is applied, and the slope and deflection are evaluated at different sections of the beam. For a smaller number of elements, the finite element method (FEM) produces results that differ from the analytical solution; however, as the number of elements increases, the FEM results gradually converge. The simply supported beam is also analyzed under a uniformly distributed load (UDL), and the results are compared with those obtained for the UVL case. For different loading conditions, a comparison is made between the analytical solution and FEM. The deflection and slope are computed using a MATLAB program. The MATLAB-based implementation is used to determine the deflection and slope of Euler–Bernoulli beams at various points under different loading conditions.

Index Terms: MATLAB, FEM, uniformly variable loads, Euler-Bernoulli beam, and simply supported

1. Introduction

Many structural issues that arise in engineering applications and practical challenges in applied science may be solved effectively with the finite element method (FEM). However, FEM findings are not as trustworthy as those from other numerical techniques. The number of elements, interpolation function, and discretization all affect how accurate the finite element approach is. Pre-processing, processing, and post-processing are the three crucial processes in the finite element technique. [4] Pre-processing involves providing input data prior to simulation, such as material characteristics, geometry, loads, and boundary conditions. MATLAB software is used to discretize the model into components and build meshes automatically. A stiffness matrix is created during processing and put together to create a global stiffness matrix.

Euler-Bernoulli Beam Theory: This theory is predicated on the idea that a plane section stays plane and perpendicular to the axis both before and after bending. Fig. (1) depicts a simply supported beam with uniformly variable load (UVL). In this case, loading q(x) varies linearly from q at the pined support to zero at the simply supported beam's roller support. The beam's governing equation is the fourth-order differential equation, where "w" stands for transverse deflection [2].

$$\frac{d^2}{dx^2}\left(EI\,\frac{d^2w}{dx^2}\right) = q(x) \text{-------(1)}$$

Here, "E" stands for the modulus of elasticity, "I" for the area moment of inertia of the beam about an axis perpendicular to the x and z axes, and q(x) = qo(1-(distance from left end/L)) for the loading variation, where L is the beam's entire length. Fig. (2) depicts a beam with uniformly distributed load (UDL).

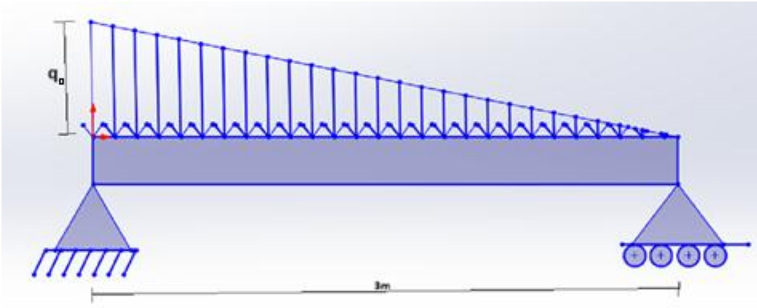

Fig (1) Simple supported beam with uniform varying load.

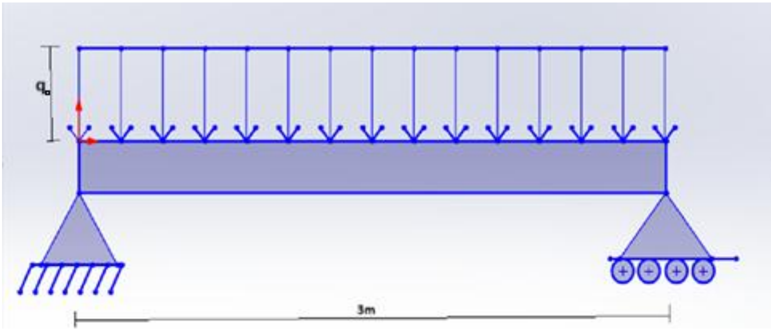

Fig (2) Simply supported beam with uniform distributed load.

In Fig. (3), the beam is made up of several elements and takes into account an nth element. The distance of the nth node is "c" from global coordinates.

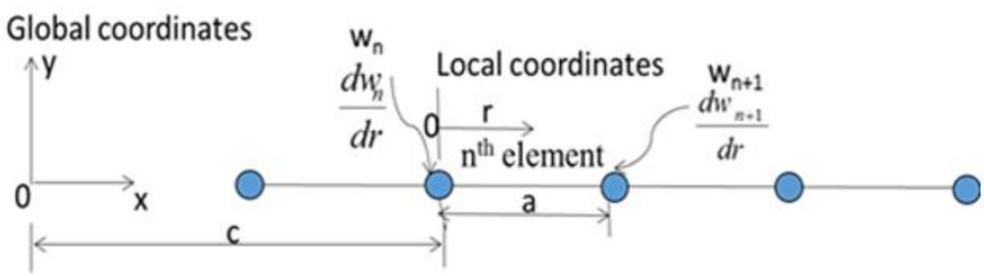

Fig (3) Degree of freedom for n^{th} element with local and global coordinates.

The four variables c1, c2, c3, and c4 are used to express the element's transverse displacement (w) in the z direction:

$$W = c_1 + c_2 r + c_3 r^2 + c_4 r^3 \quad \text{-------(2)}$$

Four form functions/interpolation functions (N1, N2, N3, and N4) are generated by applying boundary conditions at both ends of the element in Fig. 3. Hermite cubic interpolation functions are another name for these functions.

$$N_1 = (1 - 3\frac{r^2}{a^2} + 2\frac{r^3}{a^3}) \quad N_2 = \left(r - 2\frac{r^2}{a} + \frac{r^3}{a^2}\right) \quad N_3 = \left(3\frac{r^2}{a^2} - 2\frac{r^3}{a^3}\right) \quad N_4 = \left(\frac{r^3}{a^2} - \frac{r^2}{a}\right) \text{-------(3)}$$

From equation (2), the transverse displacement is represented using interpolation functions.

$$w = N_1 w_n + N_2 \frac{dw_n}{dx} + N_3 w_{n+1} + N_4 \frac{dw_{n+1}}{dx} \text{ ------------(4)}$$

Utilize weighted residual techniques and Galerkin's methodology to assess the load vector and stiffness matrix[1]. This approach equates the weighted error to zero by integrating it over the domain. In this case, "w" and the weight function are equivalent.

$$\int_0^a (ERROR)(w)dr = 0 \qquad \text{--------(5)}$$

The element stiffness matrix (Ke) is acquired.

$$Ke = \frac{EI}{a^3}\begin{bmatrix} 12 & 6a & -12 & 6a \\ 6a & 4a^2 & -6a & 2a^2 \\ -12 & -6a & 12 & -6a \\ 6a & 2a^2 & -6a & 4a^2 \end{bmatrix} \qquad \text{-------(6)}$$

By integrating the weighted load, the elemental force matrix is produced as follows:

$$\{f^e\} = \int_0^a q_o\left(1 - \frac{r+c}{L}\right)w(r)dr$$

Following integration, the final elemental load matrix takes the shape of a column vector.

$$f^e = -(L_e/20) \begin{bmatrix} 7w^1 + 3w^2 \\ (L_e/2)(3w^1 + 2w^2) \\ 3w^1 + 7w^2 \\ -(L_e/2)(2w^1 + 3w^2) \end{bmatrix} \text{ ------(7)}$$

Where $w_1 = \frac{w_0}{L}x_1$, $w_2 = \frac{w_0}{L}x_2$, $x_1 = (e-1)L_e$, $x_2 = eL_e$

The element number is shown here by the letter "e." Fig. (4) shows that there are 10 elements in total and eleven nodes. The elements are represented by the number within the circle, while nodes are represented by basic numbers on a black solid circle. It is evident from formula (7) above that the load vector for UVL varies from element to element and is not constant for every element. The global matrix [Ke] is created by assembling the elemental matrix for the whole beam. In a similar manner, the global load matrix is created by assembling the elemental load matrix {Fe}.

Fig (4) Number of elements (10) and number of nodes (11)

Since the loading q_0 for UDL is constant over the beam's length, an elemental force matrix is produced:

$$f^e = -\frac{w_0 L_e}{12}\begin{bmatrix} 6 \\ L_e \\ 6 \\ -L_e \end{bmatrix} \qquad \text{---------(8)}$$

According to the Galerkin approach, deflection, stiffness, and load are connected by relation (8) if global displacement is represented by {δ}.

$$[K]\{\delta\} = \{F\} + \{F_p\} \qquad \text{---------(9)}$$

The external point loads operating on the nodes in this case are denoted by Fp. Boundary constraints are applied at fixed nodes (node 1 and node 2). Equation (8) is used to find solutions for rotations and displacements. The answer was obtained by preparing and using MATLAB code. The beam has the following dimensions, loadings, and mechanical characteristics: The beam's length is L=3 m, its Young modulus is E=200 x 10^9 kN/m^2, its moment of inertia around the y-axis is I=8.33 x 10^{-6} m^4, its qo is 2000 kN, and its moment at the roller support end is Mo=0 kN-m [2]. The beam's length and mechanical characteristics are the same in diagram 2, but the loading circumstances change from those in Fig. 1.

2. Analytical Method

Equation (10) provides the beam deflection for UVL by directly double-integrating the moment at section "x" from the pinned support end. As in equation (11) [2], the slope of the beam is determined by directly integrating the moment at section "x" from the pinned support end. The diagrams (5 to 9) compare the FEM and analytical solution for displacement and slopes.

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$$w(x) = -\frac{w_0}{EI}\left(\frac{Lx^3}{36} - \frac{x^5}{120L} - \frac{7L^3x}{360}\right) \text{ ----------(10)}$$

$$\theta(x) = -\frac{w_0}{EI}\left(\frac{Lx^2}{12} - \frac{x^4}{24L} - \frac{7L^3}{360}\right) \text{ ------------(11)}$$

The moment at section "x" from the pinned support end is directly double integrated to determine the beam's deflection for UDL, as shown in equation 12. As in equation (13) [3], the slope of the beam determined by directly integrating the moment at section "x" from the pinned support end.

$$w(x) = -\frac{w_0}{24EI}(x^4 - 2Lx^3 + L^3x) \text{ ----------(12)}$$

$$\theta(x) = -\frac{w_0}{24EI}(4x^3 - 6Lx^2 + L^3) \text{ ----------(13)}$$

Table (1) displays the FEM and analytic solution for UVL loading over the whole length of a simply supported beam at every of 0.3 meters. Similar results are found for UDL loading displacement and slope, which are displayed in table 2.

Table (1) Comparison of finite element method (FEM) solution with analytical for UVL.

Distance_m	Deflection_FEM	Deflection_Analytical	Slope_FEM	Slope_Analytical
0	0	0	0.00069002	0.00060002
0.3	-0.00020206	-0.00017744	0.00064238	0.00057444
0.6	-0.00037808	-0.00033969	0.00052107	0.00049922
0.9	-0.00050997	-0.00047246	0.00035231	0.000379
1.2	-0.00058705	-0.00056335	0.00015925	0.00022149
1.5	-0.00060511	-0.0006027	-3.8046e-05	3.7501e-05
1.8	-0.00056551	-0.00058458	-0.00022262	-0.00015909
2.1	-0.00047422	-0.00050767	-0.00038057	-0.00035131
2.4	-0.00034092	-0.00037622	-0.00050111	-0.0005191
2.7	-0.00017808	-0.00020096	-0.00057652	-0.00063932
3	0	0	-0.00060218	-0.00068574

Table (2) Comparison of finite element method (FEM) solution with analytical for UDL.

Distance_m	Deflection_FEM	Deflection_Analytical	Slope_FEM	Slope_Analytical
0	0	0	0.0012858	0.0012858
0.3	-0.0003784	-0.0003784	0.0012138	0.0012138
0.6	-0.00071591	-0.00071591	0.0010183	0.0010183
0.9	-0.00098014	-0.00098014	0.00073031	0.00073031
1.2	-0.0011479	-0.0011479	0.00038059	0.00038059
1.5	-0.0012054	-0.0012054	-1.9059e-17	0
1.8	-0.0011479	-0.0011479	-0.00038059	-0.00038059
2.1	-0.00098014	-0.00098014	-0.00073031	-0.00073031
2.4	-0.00071591	-0.00071591	-0.0010183	-0.0010183
2.7	-0.0003784	-0.0003784	-0.0012138	-0.0012138
3	0	0	-0.0012858	-0.0012858

5. RESULT AND DISCUSSION

Fig. 5 & 6 show the deflection and slope of the beam caused by UVL using the FEM and analytical approaches, respectively. The displacement and slope of the simply supported beam resulting from an analytical match with a finite element solution for ten elements due to UVL loading are displayed in Fig. 5 & 6. The data obtained from the analytical solution and FEM solution using MATLAB code are shown in Table (1) for UVL loading [5]. The nodes distances from beam roller end to its pinned support end are displayed in the first column at intervals of 0.3 meters. The second and third columns display the displacement values for UVL loading as estimated by FEM and analytical techniques, respectively. The slope that was calculated both analytically and using FEM is shown in the fourth and fifth columns of table (1).

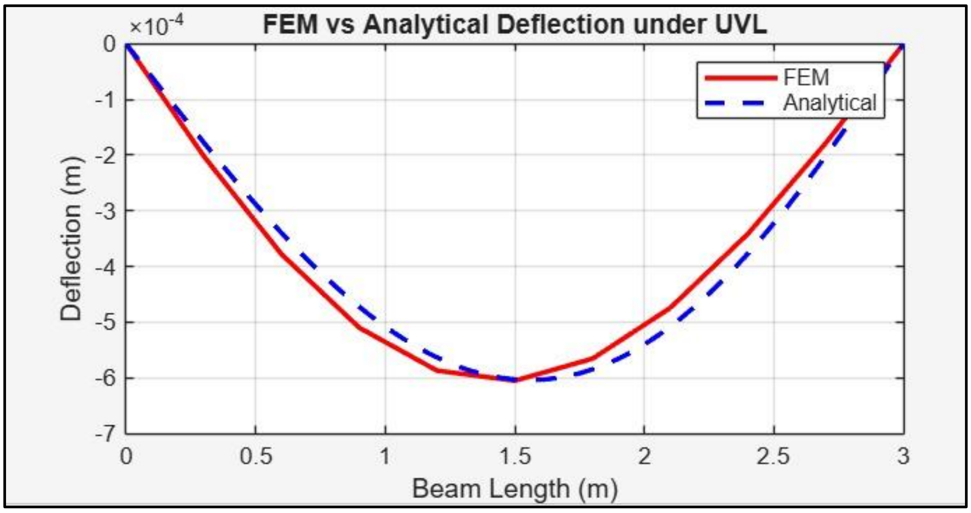

Fig. 5: Analytical and FEM-derived deflection for UVL are compared.

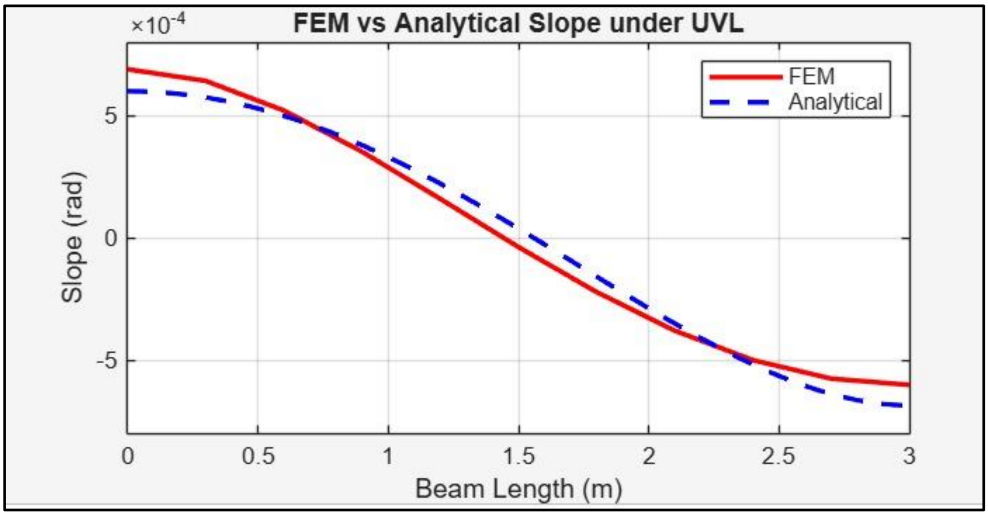

Fig. 6: Slopes determined analytically and via FEM for UVL are compared.

The deflection and slope of the beam caused by UDL are depicted in Fig. (7) and (8), respectively, using analytical and FEM techniques with MATLAB code [6]. Due to UVL loading, the analytically determined displacement and slope of the simply supported beam deviate somewhat from a finite element solution for ten elements (Fig. (7) and (8)). For beams under distributed loads that are represented by a cubic displacement function, this is always the case. Beam theory yields a quartic (fourth order) polynomial solution, while FEM always assumes a cubical polynomial displacement [3]. Therefore, the structure predicted by the finite element solution is stiffer than the real structure. Table (2) for UDL loading includes the data from the analytical solution and the FEM solution. The distance of nodes from the pinned support end to the free end of a simply supported beam is shown in the first column at intervals of 0.3 meters. The displacement values for UDL loading are shown in the second and third columns, respectively, based on FEM and analytical methods. The fourth and fifth columns of table (2) display the slope that was determined both analytically and using FEM.

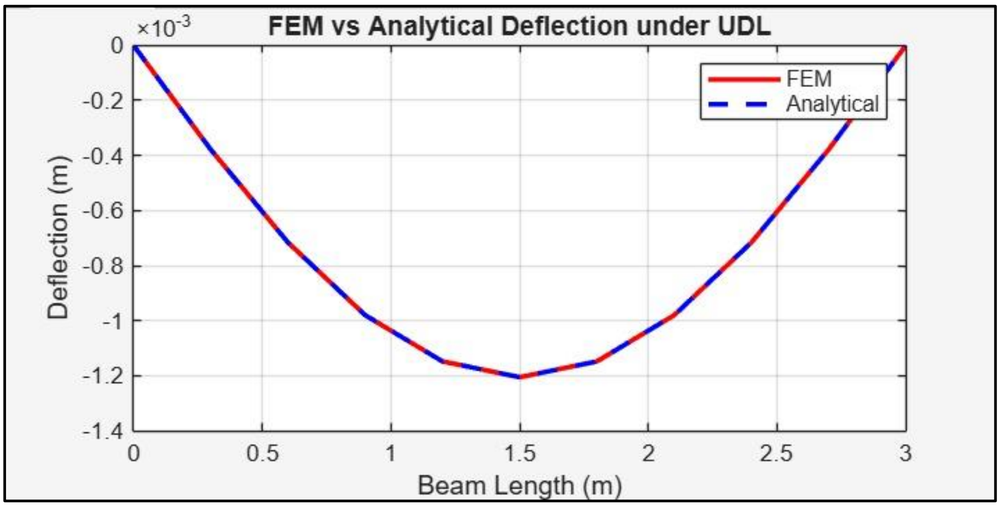

Fig. 7: Analytical and FEM-derived deflection for UDL are compared.

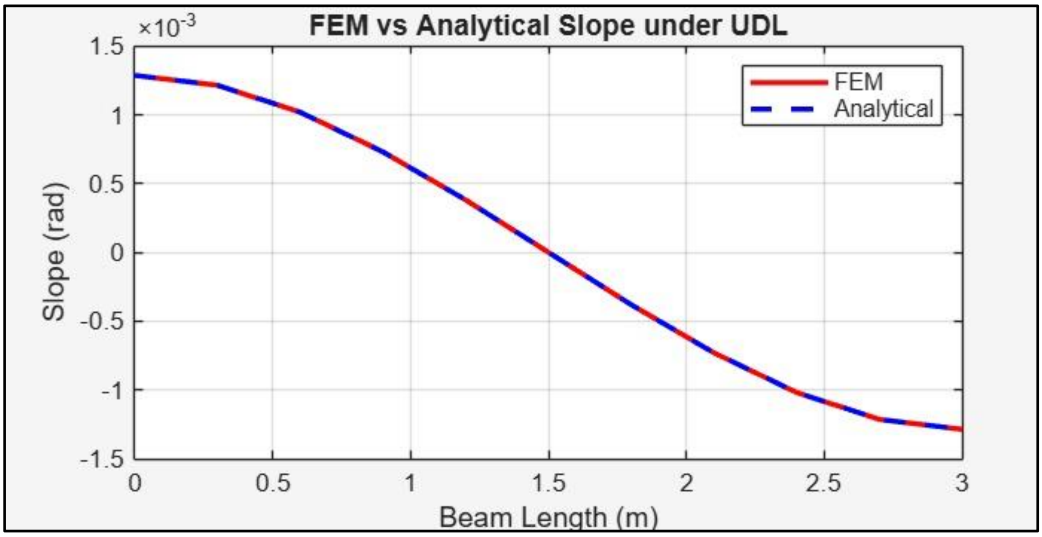

Fig. 8: Slopes derived analytically and using FEM for UDL are compared.

The beam's deflection caused by UDL and UVL is compared using the FEM approach for the same loading in Fig. (9). The simply supported beam's displacement as a result of UDL is greater than UVL. This is accurate as UDL is constant over the whole length of the beam, but in UDL, it peaks at pinned support end and drops to zero at the roller support end.

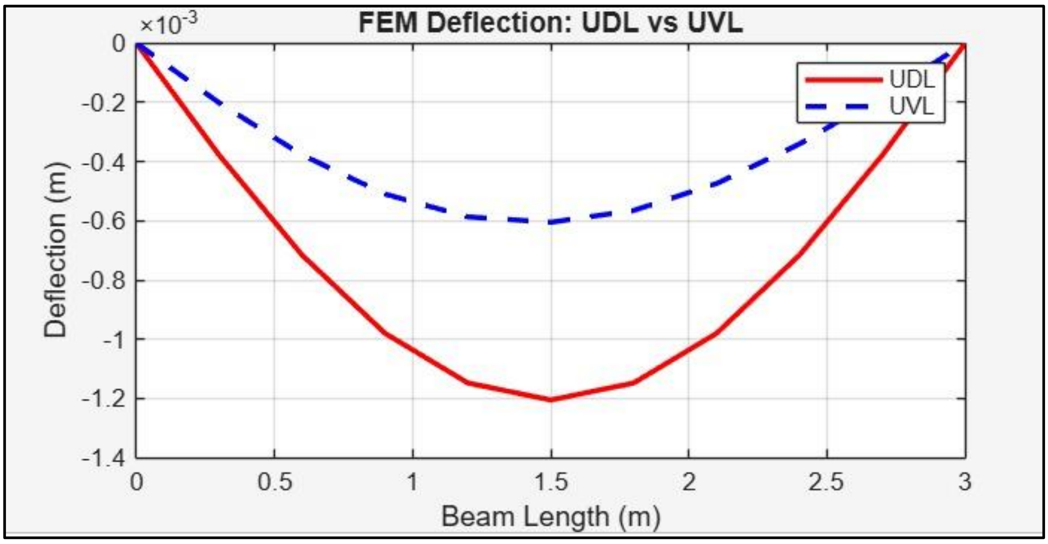

Fig. 9: Comparison of deflection under various loadings.

The slope of the beam caused by UDL and UVL using the FEM approach for the same loading is shown in Fig. 10. Because of UDL, the beam slope is greater than UVL. This is true because the loading in UDL is consistent throughout the whole length of the beam, but in UVL it peaks at the pinned support end and drops to zero at the roller support end. For the same loading, the deflection for UDL is shown to be larger than that of UVL.

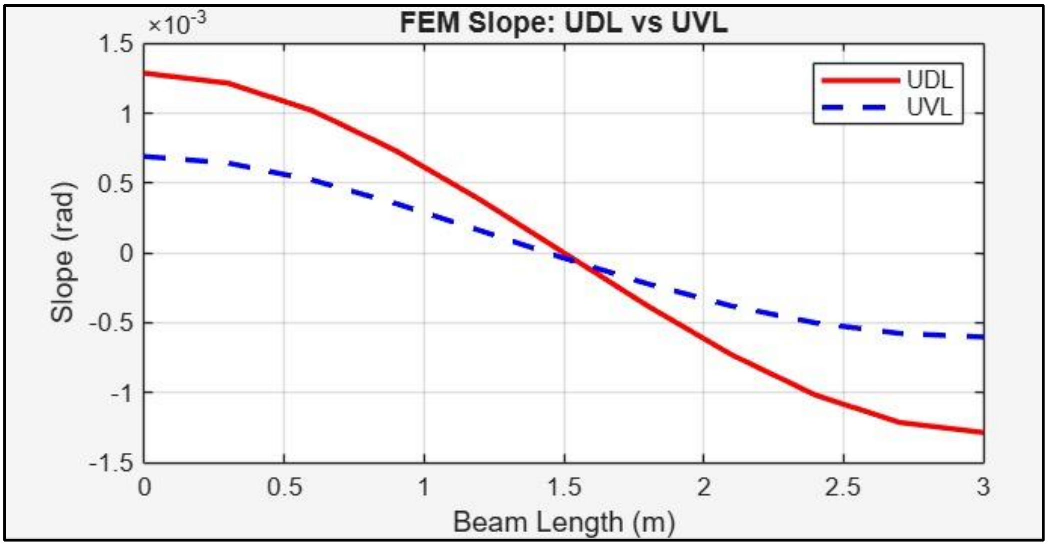

Fig. 10: Slopes for various loadings are compared.

6. CONCLUSION

The research yielded the following conclusions.

- The deflection and slope solutions obtained from FEM and beam theory are slightly distorted for 10 elements with UVL loading.
- The deflection and slope solutions obtained from FEM and beam theory agree exactly for ten elements with UDL loading.
- The structure predicted by the finite element solution is more rigid than the real structure.
- For a simply supported beam, the deflection and slope increase from the pinned support end to the roller support end and are greater for UDL than UVL.

7. REFERENCES

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