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Existence of Smooth Epimorphism from a Fuchsian Group to a Molecular Point Group 1991 Mathematics subject classification : 20H10,30F10.

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Abstract

It is observed that every finite group can be realized as a group of Automorphisms of compact Riemann surfaces of genus g(≥2).In this paper we have considered the molecule C_6H_6 (the benzene) and then constructed the group of symmetries of C_6H_6 which is a group of order 24 . We now find a set of necessary and sufficient conditions for existence of a smooth Epimorphism from a Fuchsian group to this point group formed by all the symmetries of the Benzene molecule C_6H_6.

Keywords: Point group, Fuchsian group, Smooth quotient, Riemann surface, Automorphism group and Genus.

1. Introduction

The study of symmetries is one of the most appealing applications of group theory. The set of all symmetries of F always forms a group under the operation of functions, called group of symmetries of F [2] .

It is found that every finite group is isometric to the Automorphism group of some compact Riemann surface of genus (≥ 2) [3] .

The set of automorphisms of a compact Riemann surface S of genus (≥ 2) forms a finite group whose order can't exceed $84(g-1)$. The maximum bound is called Hurwitz bound and it is attained for infinitely many values of g, the least being 3 [7,8] .

The problem of finding minimum genus for various sub-classes of finite groups has been the theme of many research papers during last few decades [4,5]

The theory of Fuchsian group is intimately related to the theory of Riemann surface Automorphism groups. A Fuchsian group Γ is an infinite group having presentation of the form:

$$\langle a_1, a_2, a_3, \dots\dots\dots\dots a_k; b_1, c_1, b_2, c_2 \dots\dots\dots b_\gamma, c_\gamma \; ; a_1^{m_1} = a_2^{m_2} = \cdots \dots\dots = a_k^{m_k} = \prod_{i=1}^{k} a_i \prod_{j=1}^{\gamma} [b_{j,} c_j] = 1 \rangle$$

Where $[b_{j,}c_j] = b_j^{-1}c_j^{-1}b_jc_j$ and

$$\delta(\Gamma) = 2\gamma - 2 + \sum_{i=1}^{k}\left(1 - \frac{1}{m_i}\right) > 0 \dots\dots\dots\dots (1)$$

It is known that if Γ_1 is a subgroup of Γ of finite index then Γ_1 is a Fuchsian group and

$$[\,\Gamma : \Gamma_1\,] = \frac{\delta(\Gamma_1)}{\delta(\Gamma)} \dots\dots\dots\dots (2) \quad [7,8].$$

A homomorphism φ from a Fuchsian group Γ to a finite group G is called smooth if the kernel is a surface subgroup of Γ.

Another remarkable result is that a finite group G is represented as an Automorphism group of compact Riemann surface of genus g if and only if there is a smooth epimorphism φ from a Fuchsian group Γ to G such that kerφ has genus g. [7,8]

Following these remarkable results in this paper we find a set of necessary and sufficient conditions on periods and genus of Fuchsian group Γ for which we have a smooth Epimorphism from Γ to the point group formed by all the symmetries of the Benzene molecule C_6H_6.

It is observed that the point group associated with C_6H_6 is of order 24 which can be represented by

$\mathcal{B} = \langle a, b \,/\, a^2 = b^2 = (ab)^{12} = 1 \rangle$ ....(3)

2. Theorem

The group of symmetries $\mathcal{B}$ of the molecule benzene C_6H_6 can be acted as a group of Automorphism of some Riemann surface S of genus g (≥ 2) if there is a smooth Epimorphism $\phi: \Gamma \to \mathcal{B}$ where Γ and $\mathcal{B}$ are defined as (1) and (3) respectively satisfying the conditions:

(1) When $k = 0$ i.e. $\Gamma = \Delta(\gamma; -)$, a surface group, then $\gamma \geq 2$

(2) When k ≠ 0 , $\phi(x_i) = m_i$ and m_i divides 24 .

Moreover,

(i) if all $\phi(x_i) \in < a >$ then $m_i = 2$, $for\ all\ i, k$ is even and $\gamma \geq 1$

(ii) if all $\phi(x_i) \in < b >$ then $m_i = 2$, $for\ all\ i$, k is even and $\gamma \geq 1$

(iii) if all $\phi(x_i) \in < ab >$ then m_i divides 12, $for\ all\ i$ and $\gamma \geq 1$

(iv) if $\phi(x_i) \in < ab >$ for $i = 1,2,3 \ldots \ldots \ldots s$ and either $\phi(x_{s+j}) \in < a >$ or $< b >$ not both, $j = 1,2, \ldots \ldots \ldots t$ then $s + t = k$, t is even and also $st \equiv 0 (mod 12)$ and $\gamma \geq 0$

(v) if $\phi(x_i) \in < ab >$, $i = 1,2,3 \ldots \ldots \ldots s$ and $\phi(x_{s+j}) \in < a >$, j = $1,2,3 \ldots \ldots \ldots t$ and $\phi(x_{s+t+j}) \in < b >$ for $j = 1,2,3 \ldots \ldots \ldots p$ such that $s + t + p = k$, then $(t + p)$ is even. Moreover,

If both t and p are even then $st \equiv 0(mod 12)$ and

If both t and p are odd then $st \equiv 1(mod 12)$ and $\gamma \geq 0$

3.Proof

Let $\phi: \Gamma \to B$ be a smooth Epimorphism then we observe the followings:

(i) all $\phi(x_i) \in < a >$

(ii) all $\phi(x_i) \in < b >$

(iii) all $\phi(x_i) \in < ab >$

(iv) some $\phi(x_i) \in < ab >$ and others $\phi(x_i) \in < a >$ or $< b >$ but not both

(v) some $\phi(x_i) \in < ab >$ and some others $\phi(x_i) \in < a >$ and remaining $\phi(x_i) \in < b >$

We consider the cases separately :

(i) if all $\phi(x_i) \in < a >$ then $m_i = 2$,for all i and k is even, $\gamma \geq 1$

As ϕ is smooth Epimorphism.

(ii) if all $\phi(x_i) \in < b >$ then $m_i = 2$,for all i.Also k is even and $\gamma \geq 1$ as in case (i)

(iii) if all $\phi(x_i) \in < ab >$ for all i,

then m_i divides 12.Also, ϕ is onto only when $\gamma \geq 1$

(iv) if $\phi(x_i) \in < ab >$ for some $i = 1,2,3 \ldots \ldots \ldots s$ and either $\phi(x_{s+j}) \in < a >$ or $< b >$ not both for $j = 1,2, \ldots \ldots \ldots t$ then $s + t = k$, also t is even.

Moreover,$sl \equiv 0(mod 12), 1 \leq l \leq 11$ and $\gamma \geq 0$ as

$$\prod_{i=1}^{k} \phi(x_i) \prod_{j=1}^{\gamma} [\phi(b_j), \phi(c_j)] = 1$$

and ϕ is onto.

(v) if $\phi(x_i) \in < ab >$ for $i = 1,2,3 \ldots \ldots \ldots s$ and $\phi(x_{s+j}) \in < a >$ for j = $1,2,3 \ldots \ldots \ldots t$ and $\phi(x_{s+t+j}) \in < b >$, $j = 1,2,3 \ldots \ldots \ldots p$ then $s + t + p = k$ and $(t + p)$ is even. Moreover, ϕ is a homomorphism and hence

$$\prod_{i=1}^{k} \phi(x_i) \prod_{j=1}^{\gamma} [\phi(b_j), \phi(c_j)] = 1$$

implies that if both p and t are even then

$sq \equiv 0(mod 12)$ and $1 \leq q \leq 11$ and

if both p and t are odd then

$sl \equiv -1(mod 12)$, $1 \leq l \leq 11$.

Hence the conditions are necessary.

Next we shall show that the conditions are sufficient for existence of smooth Epimorphism $\phi: \Gamma \to \mathcal{B}$

Let us consider the cases one by one:

(1) If $k = 0$ and $\gamma \geq 2$ then we define $\phi: \Gamma \to B$ by

$\phi(b_1) = a = \phi(c_1)$

$\phi(b_2) = b = \phi(c_2)$

$\phi(b_\gamma) = a = \phi(c_\gamma)$, $\gamma \geq 3$ (if any)

Then already $\prod_{j=1}^{\gamma} [\phi(b_j), \phi(c_j)] = 1$ and also $a, b \in \phi(\Gamma)$ gives ϕ is onto.

Hence ϕ is smooth Epimorphism

(2) (i) when all $\phi(x_i) \in < a >$ and $\gamma \geq 1$, k is even then we construct ϕ as follows:

$\phi(x_i) = a \;\; \forall\, i = 1,2, \ldots \ldots . k$

$\phi(b_1) = b = \phi(c_1)$

$\phi(b_\gamma) = 1 = \phi(c_\gamma)$ for $\gamma \geq 2$ (if any)

Then $\prod_{i=1}^{k} \phi(x_i) \prod_{j=1}^{\gamma} [\phi(b_j), \phi(c_j)] = 1$ and ϕ is onto.

Thus ϕ is smooth Epimorphism.

(ii) when all $\phi(x_i) \in < b >$ and $\gamma \geq 1$, k is even, we can construct ϕ as

$\phi(x_i) = b \;\; \forall\, i = 1,2, \ldots \ldots . k$

$\phi(b_j) = a = \phi(c_j)$

$\phi(b_\gamma) = 1 = \phi(c_\gamma)$ for $\gamma \geq 2$ (if any)

Which fulfill our objective.

(iii) when $\phi(x_i) \in < ab >$, $\gamma \geq 1$, then we construct ϕ as follows:

$\phi(x_i) = (ab)^{l_i} \quad l_i = \frac{12}{m_i} s_i \ (mod\ 12), \quad (m_i, s_i) = 1$
$\phi(b_1) = a = \phi(c_1)$
$\phi(b_\gamma) = 1 = \phi(c_\gamma)$ for $\gamma \geq 2$ (if any)
Then $\prod_{i=1}^{k} \phi(x_i) \prod_{j=1}^{\gamma} [\phi(b_j), \phi(c_j)] = (ab)^l = 1$ where $l = \sum l_i$ is congruent $(mod\ 12)$
Which is always possible by taking suitable s_i .
(iv) when $\phi(x_i) \in <ab>, i = 1,2, \ldots, s$ and $\phi(x_{s+j}) \in <a>\ or\ <b>$
for t is even and $ls = 0(mod 12), j = 1,2, \ldots, t$ where $s + t = k$,
then we construct as follows:
$\phi(x_i) = (ab)^l \quad , i = 1,2, \ldots, s$
$\phi(x_{s+j}) = a\ or\ b$ for $j = 1,2, \ldots, t$
$\phi(b_\gamma) = 1 = \phi(c_\gamma)$ (if any)
Then $\prod_{i=1}^{k} \phi(x_i) = \prod_{i=1}^{s} \phi(x_i) \prod_{j=1}^{t} \phi(x_{s+j}) = (ab)^{ls} a^t = 1$
as $ls = 0(mod 12)$ and t is even.
Similarly for $\phi(x_{s+j}) = b$ also we have ϕ is smooth epimorphism.
(v) when $\phi(x_i) \in <ab>\ for\ i = 1,2, \ldots, s$ and $\phi(x_{s+j}) \in <a>$ for $j = 1,2, \ldots, t$ and $\phi(x_{s+t+j}) \in <b>$ for $j = 1,2, \ldots, p$ such that $s + t + p = k$ and $t + p$ is even
then we construct ϕ as follows:
$\phi(x_i) = (ab)^l \quad , 1 \leq l \leq 11, 1 \leq i \leq s$
$\phi(x_{s+j}) = a\ , \ 1 \leq j \leq t$
$\phi(x_{s+t+j}) = b\ , \ 1 \leq j \leq p$
$\phi(b_\gamma) = 1 = \phi(c_\gamma)$ (if any)
then $\prod_{i=1}^{k} \phi(x_i) \prod_{j=1}^{\gamma} [\phi(b_j), \phi(c_j)]$

$$= \prod_{i=1}^{s} \phi(x_i) \prod_{j=1}^{t} \phi(x_{s+j}) \prod_{j=1}^{p} \phi(x_{s+t+j})$$

$= \ (ab)^{ls} (a)^t (b)^p$
$= (ab)^{ls}$ if t and p are even
$= 1$ when $ls \equiv 0(mod 12)$ and
$= (ab)^{ls+1}$ if t and p are odd
$= 1$ when $ls \equiv -1(mod 12)$

Hence ϕ is a smooth Epimorphism.
Consequently the conditions are sufficient.
This completes the proof of the theorem.

3.Conclusion

From this above discussion we can establish a set of necessary and sufficient conditions that ϕ is a smooth Epimorphism from $\Gamma \rightarrow B(D_{6h})$. Hence we have the smooth quotient

$$\frac{\Gamma}{Ker\phi} \cong D_{6h}.$$

Clearly $Ker\phi$ is a surface group genus g(≥2),consequently we can conclude that the molecular symmetry group can be realized as group of Automorphisms of compact Riemann surface of genus g(≥2) with the above mentioned conditions in the theorem.

4.References

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