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Convergence in L_p Spaces and Vitali's Convergence Theorem

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Abstract: Convergence in L_p spaces does not imply almost everywhere convergence, it does imply another type of convergence that is often of interest. We prove a theorem that implies that if a sequence converges in measure to a function f, then some subsequence converges almost everywhere to f. We conclude this paper with a set of necessary and sufficient conditions for L_p convergence and observe that second and third conditions are automatically fulfilled when the sequence is dominated by a function in L_p .

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1.1 Definition: A sequence $\{f_n\}$ of measurable real-valued functions is said to converge in measure to a measurable real-valued function f in case

$\lim_{n\to\infty} \mu(\{x \in X : |f_n(x) - f(x)| \geq \alpha\}) = 0$ For each $\alpha > 0$. …………… (1)

The sequence $\{f_n\}$ said to be Cauchy in measure in case

$\lim_{n\to\infty} \mu(\{x \in X : |f_m(x) - f_n(x)|\}) \geq \alpha)\} = 0$ For each $\alpha > 0$. ……… (2)

If the sequence $\{f_n\}$ converges uniformly to f, then the set $\{x \in X : |f_n(x) - f(x)| \geq \alpha\}$ is empty for sufficiently large n. Hence uniform convergence implies the convergence of the sequences in measure. It is mentioned here that point wise convergence (and therefore almost everywhere convergence) need not imply convergence in measure, unless the space X has a finite measure However in L_p spaces convergence does not imply convergence in measure. But if we have $E_n(\alpha) = \{x \in X : |f(x)_n - f(x)| \geq \alpha\}$

then $\int |f_n - f|^p \, d\mu \geq \int_{E_n(\alpha)} |f_n - f|^p \, d\mu \geq \alpha^p \mu(E_n(\alpha))$.

Since $\alpha > 0$, it follows that $||f_n - f||_p \to 0$ implies that $\mu(E_n(\alpha)) \to 0 \quad as\, n \to \infty$.

Now we have to prove a theorem that implies that if a sequence converges in measure to a function f, then some sub sequence converges almost everywhere to f.

1.2 Theorem: Let $\{f_n\}$ be a sequence of measurable real-valued functions which is Cauchy in measure. Then there is a subsequence which converges almost everywhere and in measure to a measurable real-valued function f.

Proof: Let (g_k) be a sub sequence of $\{f_n\}$ such that the sub set $E_k = \{x \in X : |g_{k+1}(x) - g_k(x)| \geq 2^{-k}\}$be such that $\mu(E_k) < 2^{-k}$. Let $F_k = \bigcup_{j=k}^{\infty} E_j$ so that $F_k \in X$ and $\mu(F_k) < 2^{-(k-1)}$. If $I \geq j \geq k$ and $x \notin F_k$ then

$|g_i(x) - g_j(x)| \leq |g_i(x) - g_{i-1}(x)| + \dots \dots \dots + |g_{j+1}(x) - g_j(x)|$

$\leq \frac{1}{2^{i-1}} + \ldots\ldots + \frac{1}{2^j} < \frac{1}{2^{i-1}}$ ……….. (3)

Let F = $\bigcap_{k=1}^{\infty} F_k$ so that F ∈ X and $\mu(F) = 0$ as it is given follows that that (g_j) converges on X \ F. If we define a function f as that $f(x) = \begin{cases} \lim g_j(x), & x \notin F \\ 0, & x \in F \end{cases}$ then (g_i) converges almost everywhere to a measurable real-valued function f. Therefore for i → ∞ in (3) and for j ≥ k and $x \notin F_k$ then we can infer that $|f(x) - g_j(x)| \leq \frac{1}{2^{i-1}} \leq \frac{1}{2^{k-1}}$ which shows that the sequence (g_j) converges uniformly to f on the compliment of each set F_k.

To see that (g_j) converges in measure to f, let α, ε be positive real numbers and choose k so large that $\mu(F_k) < 2^{-(k-1)} < \inf(\alpha, \varepsilon)$. If j ≥ k , the above estimate shows that $\{ x \in X : |f(x) - g_j(x)| \geq \alpha\} \subseteq \{ x \in X : |f(x) - g_j(x)| \geq 2^{-(k-1)}\} \subseteq F_k$

Therefore $\mu(\{ x \in X : |f(x) - g_j(x)| \geq \alpha\}) \leq \mu(F_k) < \varepsilon$ for all j $\geq k$ so that (g_j) converges in measure to f.

1.3 Corollary: Let $\{f_n\}$ be a sequence of measurable real-valued functions which is Cauchy in measure. Then there is a measurable real-valued function f to which the sequence converges in measure. The limit function f is uniquely determined almost everywhere.

Proof: As proved above there is a subsequence $\{f_{n_k}\}$ which converges in measure to a function f. To see that the entire sequence converges in measure to f, observe that since

$|f(x) - f_n(x)| \leq |f(x) - f_{n_k}(x)| + |f_{n_k}(x) - f_n(x)|$,

it follows that

$\{x \in X : |f(x) - f_n(x)| \geq \alpha\} \subseteq \{x \in X : |f(x) - f_{n_k}(x)| \geq \frac{\alpha}{2}\} \cup \{x \in X : |f_{n_k}(x) - f_n(x)| \geq \frac{\alpha}{2}\}$.

The convergence in measure of the sequence (f_n) to f follows from this relation.

Suppose that the sequence (f_n) converges in measure to both f *and* g.

Since $|f(x) - g(x)| \leq |f(x) - f_n(x)| + |f_n(x) - g(x)|$.

It follows that $\{x \in X : |f(x) - g(x)| \geq \alpha\} \subseteq \{x \in X : |f(x) - f_n(x)| \geq \frac{\alpha}{2}\} \cup \{x \in X : |f_n(x) - g(x)| \geq \frac{\alpha}{2}\}$. So that $\mu(\{x \in X : |f(x) - g(x)| \geq \alpha\}) = 0$ for all $\alpha > 0$. Taking $\alpha = \frac{1}{n}$ where $n \in$ N, we infer that $f = g. a. e.$ Proved.

1.4 Theorem: Let (f_n) be a sequence of functions in L_p space, which converges in measure to f and let $g \in L_p$ be such that $|f_n(\mathrm{x})| \leq g(x)$ a.e. then $f \in L_p$, and the sequence (f_n) converges to f in L_p.

Proof: In contrary assume that (f_n) does not converge in L_p space to a function f, then there exist a subsequence $(\mathrm{g_k})$ of (f_n) and $\varepsilon > 0$ such that

$\mathrm{g_k} - f_\mathrm{p} > \varepsilon$ for $\mathrm{k} \in \mathrm{N}$. ……………(4)

Since $(\mathrm{g_k})$ be a subsequence of (f_n), it follows that it converges in measure to the function f. By theorem (1.2) there is a subsequence $(\mathrm{h_r})$ of $(\mathrm{g_k})$ which converges almost everywhere and in measure to a function h. From the uniqueness part of corollary (1.3) it follows that h = f a. e. Since $(\mathrm{h_r})$ converges almost everywhere to f and is dominated by g and implies that $||\,\mathrm{h_r} - \mathrm{f}\,||_{\,p} \rightarrow 0$. which contradicts our assumption hence theorem is proved.

1.5 Definition: Almost Uniform Convergence: In the proof of the theorem (1.2) proved above we construct a sequence $(\mathrm{g_k})$ of measurable real valued functions which was uniformly convergent on the compliments of the sets which have arbitrarily small measure, which sounds equivalent to uniform convergence outside a set of measure zero.

1.6 Definition: A sequence (f_n) of measurable functions is said to be almost uniformly convergent to a measurable function f if for each $\delta > 0$ there is a set E_δ in X with $\mu(E_\delta) < \delta$ such that (f_n) converges uniformly to f on $\mathrm{X} \setminus E_\delta$. The sequence $(f_\mathrm{n})$is said to be an almost uniformly Cauchy sequence if for every $\delta > 0$ there exist a set E_δ in X with $\mu(E_\delta) < \delta$ such that (f_n) is uniformly convergent on $\mathrm{X} \setminus E_\delta$.

1.7 Lemma: Let (f_n) be an almost uniformly Cauchy sequence, then there exist a measurable function f such that (f_n) converges almost uniformly and almost everywhere to f.

Proof: If $\mathrm{k} \in \mathrm{N}$, and let $E_k \in \mathrm{X}$ be such that $(E_k) < 2^{-k}$ and (f_n) is uniformly convergent on $\mathrm{X} \setminus E_k$. Let $F_k = \bigcup_{j=k}^{\infty} E_j$, so that $F_k \in \mathrm{X}$ and $\mu(F_k) < 2^{-(k-1)}$. It is here to note that (f_n) is uniformly convergent on $\mathrm{X} \setminus F_k \subseteq \mathrm{X} \setminus E_k$ and define g_k by $g_k(\mathrm{x}) = \begin{cases} \lim f_\mathrm{n}(\mathrm{x}) \text{ if } \mathrm{x} \notin F_k \\ 0 \text{ if } \mathrm{x} \in F_k \end{cases}$

we observe here that the sequence (F_k) is decreasing and $\mathrm{F} = \cap F_k$, then $\mathrm{F} \in \mathrm{X}$ and $\mu(F) = 0$. If $\mathrm{h} \leq k$, then $g_h(\mathrm{x}) = g_k(\mathrm{x})$ for all $\mathrm{x} \in F_h$. Therefore the sequence (g_k) coverges on all of X to a measurable limit function f (say). If x $\notin F_k$, then $f(x) = g_k$(x) = lim f_n(x). It follows that (f_n) converges to f on X\F. So that (f_n) converges to f almost everywhere on X.

Here to see that the convergence is almost uniform, let $\varepsilon > 0$, and K be so large that $2^{-(k-1)} < \varepsilon$. Then $\mu(F_k) < \varepsilon$ and (f_n) converges uniformly to $g_k = f$ on X\ F_k.

Hence proved.

1.8 Theorem: If a sequence (f_n) be an almost uniformly convergent to a function f, then it converges in measure. Conversely, if a sequence (h_n) converges in measure to a function h, then some subsequence converges almost uniformly to h.

Proof: Suppose that the sequence (f_n) convrges almost uniformly to a function f, and let α *and* ε be two positive numbers. Then there exist a set E_ε in X with (E_ε) < ε such that (f_n) convrges to f uniformly on X\ E_ε. Therefore, if n is sufficiently large, then the set $\{x \in X : |f_n(x) - f(x)| \geq \alpha\}$ must be contained in E_ε.This shows that (f_n) converges in measure to f.

Conversely, suppose that (h_n) converges in measure to h. It follows from the theorem (1.2) that there is a subsequence (g_k) of (h_n) which converges in measure to a function g and the proof of the theorem (1.2) shows that the convergence is almost uniform. Since (g_k) converges in measure to both to h and g, it follows from the corollary (1.3) that h = g a.e. Therefore the subsequence (g_k) of (h_n) converges almost uniformly to h. Proved.

1.9 Remark: From the theorem (1.8) that if a sequence converges in L_p, then it has a subsequence which converges almost uniformly but conversely it may be seen that almost uniform convergence does not imply convergence in L_p.

1.10 Theorem: (Egoroff's): Suppose that $\mu(X) <+\infty$ and that (f_n) is a sequence of measurable real valued functions which converges almost everywhere on X to a measurable function f. Then the sequence (f_n) converges almost uniformly and in measure to f.

Proof: Without any loss of generality suppose that (f_n) converges at every point of X to a function f. If m, n ∈ N then let $E_n(\mathrm{m}) = \bigcup_{k=1}^{\infty}\{ x \in X: |f_k(x) - f(x)| \geq \frac{1}{m}\}$,

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So that $E_n(\mathrm{m})$ is a member of X and $E_{n+1}(\mathrm{m}) \subseteq E_n(\mathrm{m})$. Since $f_{\mathrm{n}}(\mathrm{x}) \to f(x)$ for x ∈ X, it follows that $\bigcap_{n=1}^{\infty} E_n(\mathrm{m}) = \phi$. Since $\mu(X) <+\infty$ we infer that $\mu\big(E_n(\mathrm{m})\big) \to 0$ as $\mathrm{n} \to \infty$.

If $\delta > 0$, Let k_m be such that $\mu\left(E_{k_m}(\mathrm{m})\right) < \frac{\delta}{2^m}$ and let $E_\delta = \bigcup_{m=1}^{\infty} \{E_{k_m}(\mathrm{m})\}$, so that $E_\delta \in$ X and $\mu(E_\delta) < \delta$. Observe that if $\mathrm{x} \notin E_\delta$,then $\mathrm{x} \notin E_{k_m}(\mathrm{m})$,

so that $| f_{\mathrm{k}}(\mathrm{x}) - f(x)| < \frac{1}{m}$ for all $\mathrm{k} \geq k_m$. Therefore (f_{k}) is uniformly convergent on the complement of E_δ. Hence the proof.

1.11 Theorem: (Vitali's Convergence Theorem): Let (f_{n}) be a sequence in $L_p(\mathrm{X}, \mathfrak{X}, \mu)$, $1 \leq p < \infty$. Then the following three conditions are necessary and sufficient for the L_p space for convergence of (f_{n}) to f:

(1) (f_{n}) Converges to f in measure.

(2) For each $\varepsilon > 0$ there is a set $E_\varepsilon \in \mathfrak{X}$ with $\mu(E_\varepsilon) < \infty$ such that if $\mathrm{F} \in \mathfrak{X}$ and $\mathrm{F} \cap E_\varepsilon = \phi$, then $\int_F |f_n|^p \, d\mu < \varepsilon^p$ for all n ∈ N.

(3) For each $\varepsilon > 0$ there is a $\delta(\varepsilon) > 0$, such that if $\mathrm{E} \in \mathfrak{X}$ an d $\mu(\mathrm{E}) < \delta(\varepsilon)$, then $\int_E |f_n|^p \, d\mu < \varepsilon^p$ for all n ∈ N.

Proof: In the definition (1.1) it was observed that convergence in L_p space implies the convergence in measure. The fact that convergence in L_p space of the sequence (f_{n}) implies (2) and same is (3) obviously.

Now it is to show that above said three conditions implies that the sequence (f_{n}) converges in L_p to f. Let $\varepsilon > 0$, and E_ε defined in (2) and let $\mathrm{F} = \mathrm{X} \backslash E_\varepsilon$. By applying the Minkowski Inequality to $f_n - f_m = (f_n - f_m)\chi_{E_\varepsilon} + f_n\chi_{\mathrm{F}} - f_{\mathrm{m}}\chi_{\mathrm{F}}$, we get

$||f_n - f_m||_p \leq \{\int_{E_\varepsilon} |f_n - f_m|^{\mathrm{p}} d\mu\}^{\frac{1}{p}} + 2\varepsilon$ for m, n ∈ N. Now let $\alpha = \varepsilon[\mu(E_\varepsilon)]^{-1/p}$ and let $H_{nm} = \{\mathrm{x} \in E_\varepsilon : |\, |f_n(\mathrm{x}) - f_m(\mathrm{x})| \geq \alpha\}$. In view of (1) there exist a $\mathrm{K}(\varepsilon)$ such that if n, m ≥ $\mathrm{K}(\varepsilon)$, then $\mu(H_{nm}) < \delta(\varepsilon)$. Also by another application of Minkowski Inequality along with (3) we get

$\{\int_{E_\varepsilon} |f_n - f_m|^{\mathrm{p}} d\mu\}^{\frac{1}{p}} \leq \{\int_{E_\varepsilon \backslash H_{nm}} |f_n - f_m|^{\mathrm{p}} d\mu\}^{\frac{1}{p}} + \{\int_{H_{nm}} |f_n|^{\mathrm{p}} d\mu\}^{\frac{1}{p}} + \{\int_{H_{nm}} |f_m|^{\mathrm{p}} d\mu\}^{\frac{1}{p}} \leq \alpha[\mu(E_\varepsilon)]^{\frac{1}{p}} + \varepsilon + \varepsilon = 3\varepsilon$, when n, m ≥ $\mathrm{K}(\varepsilon)$. On combining this with the earlier inequality, we infer that the sequence (f_n) is Cauchy and hence convergent in L_p. As we already know

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that the sequence (f_n) is convergent in measure to f. It follows from the corollary (1.3) that (f_n) converges in L_p space to f, Hence the proof.

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