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Fixed Points of Generalised Nonexpansive Mappings In Banach Spaces

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ABSTRACT: In this paper we prove a fixed point theorem for the self mappings of a closed convex and bounded subset of the Banach space satisfying a generalized nonexpansive type condition. Some result concerning the approximations of fixed points with Krasnoselskii and Mann type iterations are also proved under suitable conditions. Our results include the well – known result ok kannan (1968) and Bose and Mukherjee (1981) and Include result of Approximating fixed points of generalized nonexpansive mappings in banach space(2014) as special cases with a different and constructive method.

INTRODUCTION

Let (X,d) be metric space. Then Banach contraction principle states that if X is complete and f : X→X satisfies the condition

(1.1) $d(f_x, f_y) \leq \propto d(x,y)$

for all x,y ∈ X and $0 \leq \propto < 1$, then f has a unique fixed point. The mapping f satisfying the condition (1.1) is called contraction and when $\propto = 1$, f is called nonexpansive . The nonexpansive mappings have been studied by Kirk and Goebel [6] for fixed points. Bogin[1] considered a class of generalized non expansive mappings characterized by the inquality

(1.2) $d(f_x, f_y) \leq a\, d(x,y) + b[d(x, f_x) + d(y, f_y)] + c[d(x, f_y) + d(y, f_x)]$

for all x,y ∈ X where a,b,c are nonnegative real numbers satisfying

(1.3) $a+2b+2c=1$

for the study of fixed points. Generalized the above class of mappings (1.2)-(1.3) to a wider class mappings characterized by the inequality

(1.4) $d(f_x, f_y) \leq a \max\{ d(x,y), d(x, f_x), d(y, f_y), \frac{1}{2}[d(x, f_y) + d(y, f_x)]\}$

+bmax { d(x, f_x),d (y,f_y)}+ c[d(x, f_y)+d (y,f_x)]

for all x,y ∈ X where the real numbers a,b,c ≥ 0 satisfy the condition

(1.5) a+b+2c=1

Similarly the study of nonexpansive mappings in Banach spaces has been made extensively by several authers. Bose and Mukherjee[2] studied the class of generalized nonexpansive mappings for the study of fixed points characterized by the inequlity

(1.6) $\|f_x - f_y\| \leq a + \|x-y\|$+b[$\|x - f_x\|$+$\|y - f_y\|$] +c [$\|x - f_x\|$+$\|y - f_y\|$]

For all x,y ∈ X where a,b,c are nonnegative real numbers a>0 satisfying the condition

(1.7) 3a+2b+4c=1

The aim of the present note is generalize the above class of mappings (1.6)-(1.7) and prove a couple of fixed point theorems under generalized contraction condition with a different method which in turn generalize fixed point theorem Bose and mukherjee[2] as the special cases.

GENERALIZED NONEXPANSIVE MAPPINGS

Given a non-empty, closed, convex and bounded subset C of the Banach space X, consider the class of nonexpansive type mappings f:C→C characterized the inequality

$$\|f_x - f_y\| \leq a \max\{\|x-y\|, \|fx - f_y\|, \|y - f_x\|, \tfrac{1}{2}[\|x - f_x\| + \|y - f_y\|]\} + b[\|x - f_y\| + \|y - f_x\|] + c \max\{\|x-y\|, \|x - f_x\|, \|y - f_y\|\} \quad (2.1)$$

$$\|f_x - f_y\| \leq a \max\{\|x-y\|, \|fx - f_y\|, \|y - f_x\|, \tfrac{1}{2}[\|x - f_x\| + \|y - f_y\|]\} + b[\|x - f_y\| + \|y - f_x\|] + c \max\{\|f_x - f_y\|, \|x - f_x\|\} \quad (2.2)$$

for all x, y ∈ X where the real numbers a,b,c ≥ 0 satisfy the inequality

$$a + b + c \leq \tfrac{1}{2} \quad (2.3)$$

The generalized nonexpansive mappings characterized by the inequalities (2.1),(2.2) and (2.3) in the setting of a metric space for fixed points and are different from the class of Ciric's mappings characterized by the inequality (1.6) and (1.7) . in this section we prove a couple of results concerning the existence of fixed point for the class of generalized nonexpansive mappings.

Theorem 2.1. Let C be a non empty , closed ,convex and bounded subset of the normed linesr space X and let f: C→ C be a mapping satisfying the inequality (2.1) and (2.3) with a>0. If the sequence $\{x_n\}$ defined by x_{n+1} = (1-t) x_n +t f x_n n= 0,1,2,3,4,……(2.4)

For some t ∈ (0,1) and for some x=x_0 ∈ C converges to u, then u is a unique fixed point of f.

Proof: By (2.1) one gets

$\|x_{n+1} - f_u\|$ = (1-t) $\|x_n - f_u\| + t\|fx_n - f_u\|$

$\leq$(1-t) $\|x_n - f_u\|$ + t[a$\{\|x_n - u\|\|fx_n - f_u\|, \|u - fx_n\|,\frac{1}{2}[\|x_n - fx_n\|+\|u - f_u\|]\}$+ b $[\|x_n - f_u\|+\|u - fx_n\|]$ +c max $\{\|x_n - u\|, \|x_n - fx_n\|, \|u - fu\|\}$ (2.5)

Now x_{n+1} = (1-t) x_n +t f x_n

x_{n+1} - x_n = -t $(x_n - fx_n)$

$\|x_{n+1} - x_n\|$ = t $\|(x_n - fx_n)\| \to 0 \; as \; n \to \infty$.

Taking the limit as $n \to \infty$.we obtain

$\|u - f_u\|$ $\leq$(1-t) $\|u - f_u\|$ + t[a$\{\|u - u\|, \|u - f_u\|, \|u - u\|,\frac{1}{2}[\|u - u\|+\|u - f_u\|]\}$

+ b $[\|u - f_u\|+\|u - u\|]$ +c max $\{\|u - u\|, \|u - u\|, \|u - f_u\|\}$

$\|u - f_u\|$ $\leq$(1-t) $\|u - f_u\|$ + t[a$\{0, \|u - f_u\|, 0,\frac{1}{2}[0+\|u - f_u\|]\}$ + b $[\|u - f_u\|+0]$

+c max $\{ \|u - f_u\|, 0,0\}$

$\|u - f_u\|$ $\leq$(1-t) $\|u - f_u\|$ + t[a$\{\|u - f_u\|,\frac{1}{2}[\|u - f_u\|]\}$ + b $[\|u - f_u\|]$ +c max$\{ \|u - f_u\|\}$

$\|u - f_u\|$ $\leq$[(1-t)+ta+ tb+ tc] $\|u - f_u\|$

$\|u - f_u\|$ ≤[1-t+ta+ tb+ tc] $\|u - f_u\|$

$\|u - f_u\|$ ≤[(1+t(a+ b+ c-1)] $\|u - f_u\|$

Since a+b+c <1 we may choose t ∈ (0,1) such that t> a+b+c . then inequality , we obtain u= fu.

To Prove uniqueness , let v(≠u) be another fixed point of f. then by (2.1).

$\|u - v\| \le \|f_u - f_v\|$

$\|u - v\| \le$a max$\{\|u - v\|, \|f_u - f_v\|, \|v - f_u\|, \frac{1}{2}[\|u - f_u\| + \|v - f_v\|]\}$

$+$ b $[\|u - f_v\| + \|v - f_u\|]$ +c max $\{\|u - v\|, \|u - f_u\|, \|v - f_v\|\}$

=(a+c)$\|u - v\|$

Which is a contradiction .Hence u=v and the proof of the theorem is complete.

Theorem 2.2: Let C be a non empty , closed , convex and bounded subset of a banach space X. If f: C→C satisfies the inequalities (2.2) and (2.3) with a>0,b>0 then f has a unique fixed point.

Proof: Let x = x_0 ∈ C be arbitrary and consider the sequence $\{x_n\}$ defined by (2.4). Then, we have

$x_1 - x_2$ = (1-t) $(x_0 - x_1)$ + t$(fx_0 - fx_1)$.

Then ,by(2.2), we obtain

$\|x_1 - x_2\|$ = (1-t) $\|x_0 - x_1\|$ + t $\|fx_0 - fx_1\|$

≤ (1-t) $\|x_0 - x_1\|$ + t[a max$\{\|x_0 - x_1\|, \|fx_0 - fx_1\|, \|x_1 - fx_0\|, \frac{1}{2}[\|x_0 - fx_0\| + \|x_1 - fx_1\|]\}$ + b$[\|x_0 - fx_1\| + \|x_1 - fx_0\|]$ + c max $\{\|fx_0 - fx_1\|, \|x_0 - fx_0\|\}$]

(2.6)

Now x_1 = (1-t) x_0 + t x_0

$x_1 - x_0$ = -t$(x_0 - fx_0)$

t$\|x_0 - fx_0\| = \|x_0 - x_1\|$

Again x_2 = (1-t) x_1 + t x_1

$x_2 - x_1$ = -t($x_1 - fx_1$)

t$\|x_1 - fx_1\|$ = $\|x_2 - x_1\|$

t$\|x_1 - fx_1\|$ = $\|x_1 - x_2\|$

Similarly $(x_0 - fx_1)$ = $(x_0 - x_1) + (x_1 - fx_1)$

t $(x_0 - fx_1)$ = t$(x_0 - x_1) + t(x_1 - fx_1)$

t $(x_0 - fx_1)$ = t$(x_0 - x_1) + t(x_1 - x_2)$

t$\|x_0 - fx_1\| \leq$ t $\|x_0 - x_1\|$ + t$\|x_1 - x_2\|$

Again $x_1 - fx_0$ = $(x_1 - x_0)$ + $(x_1 - fx_0)$

Which gives t $(x_1 - fx_0)$ = t $(x_1 - x_0)$ + t $(x_1 - fx_0)$

= t $(x_1 - x_0)$ + $(x_0 - x_1)$

= (1-t) $(x_0 - x_1)$

$$t\ \|x_1 - fx_0\| = (1 - \text{t})\ \|x_0 - x_1\|$$

Substituting the above values in(2.6)

$\|x_1 - x_2\|$ $\leq$ (1-t) $\|x_0 - x_1\|$ + t[a max{$\|x_0 - x_1\|, \|x_1 - x_2\|, \|x_1 - x_1\|, \frac{1}{2}[\|x_0 - x_1\| +$
$\|x_1 - x_2\|]\}$ + b[$\|x_0 - x_1\| + \|x_1 - x_1\|$] + c max {$\|x_1 - x_2\|, \|x_0 - x_1\|$ }]

= (1-t) $\|x_0 - x_1\|$ + t[a max{$\|x_0 - x_1\|, \|x_1 - x_2\|, \frac{1}{2}[\|x_0 - x_1\| + \|x_1 - x_2\|]\}$ +

b[$\|x_0 - x_1\|$] + c max {$\|x_1 - x_2\|, \|x_0 - x_1\|$ }]

= (1-t) $\|x_0 - x_1\|$ + t[a max{$\|x_0 - x_1\|, \|x_1 - x_2\|, \frac{1}{2}[\|x_0 - x_1\| + \|x_1 - x_2\|]\}$

+ b[$\|x_0 - x_1\|$] + c max {$\|x_1 - x_2\|, (1-t)\|x_0 - x_1\|$ +t$\|x_0 - x_1\|$}] (2.7)

Now these are these Cases:

Case I : Suppose that

max $\{\|x_0 - x_1\|, \|x_1 - x_2\|, \frac{1}{2}[\|x_0 - x_1\| + \|x_1 - x_2\|\} = \|x_0 - x_1\|$

and max $\{\|x_1 - x_2\|, (1-t)\|x_0 - x_1\| + t\|x_0 - x_1\|\} = \|x_0 - x_1\|$

for $t > \frac{1}{2}$. Then from (2.7)

$\|x_1 - x_2\| \leq (1\text{-}t)\ \|x_0 - x_1\| + ta\ \|x_0 - x_1\| + tb\ \|x_0 - x_1\| + tc\ \|x_0 - x_1\|$

$\|x_1 - x_2\| \leq [(1\text{-}t) + ta + tb + tc]\ \|x_0 - x_1\|$

$\|x_1 - x_2\| \leq \alpha_1\ \|x_0 - x_1\|$.

Case II: Suppose that

max $\{\|x_0 - x_1\|, \|x_1 - x_2\|, \frac{1}{2}[\|x_0 - x_1\| + \|x_1 - x_2\|\} = \|x_1 - x_2\|$

and max $\{\|x_1 - x_2\|, (1-t)\|x_0 - x_1\| + t\|x_0 - x_1\|\} = \|x_1 - x_2\|$

Then,

$\|x_1 - x_2\| \leq (1\text{-}t)\ \|x_0 - x_1\| + ta\ \|x_1 - x_2\| + tb\ \|x_0 - x_1\| + tc\ \|x_1 - x_2\|$

$(1\text{-}ta\text{-}tc)\ \|x_1 - x_2\| \leq (1\text{-}t)\ \|x_0 - x_1\| + tb\ \|x_0 - x_1\|$

$\|x_1 - x_2\| \leq \frac{(1-t+tb)}{(1-ta-tc)}\ \|x_0 - x_1\|$

$\|x_1 - x_2\| \leq \alpha_2\ \|x_0 - x_1\|$

Case III: Suppose that

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max $\{\|x_0 - x_1\|, \|x_1 - x_2\|, \frac{1}{2}[\|x_0 - x_1\| + \|x_1 - x_2\|\} = \frac{1}{2}[\|x_0 - x_1\| + \|x_1 - x_2\|]$

$\|x_1 - x_2\| \leq$ (1-t) $\|x_0 - x_1\|$ + ta$\{\frac{1}{2}[\|x_0 - x_1\| + \|x_1 - x_2\|]\}$ + tb $\|x_0 - x_1\|$+tc $\|x_0 - x_1\|$

$\|x_1 - x_2\| \leq$ (1-t) $\|x_0 - x_1\|$ + t $\frac{a}{2}\|x_0 - x_1\|$ + t $\frac{a}{2}\|x_1 - x_2\|$ + tb $\|x_0 - x_1\|$+tc $\|x_0 - x_1\|$

$\|x_1 - x_2\|$ - t $\frac{a}{2}\|x_1 - x_2\| \leq$ (1-t) $\|x_0 - x_1\|$ + t $\frac{a}{2}\|x_0 - x_1\|$ + tb $\|x_0 - x_1\|$+tc $\|x_0 - x_1\|$

$(1-\frac{a}{2})\ \|x_1 - x_2\| \leq (1\text{-t}+ \text{t}\frac{a}{2} + \text{tb} + \text{tc})\ \|x_0 - x_1\|$

$$\|x_1 - x_2\| \leq \frac{(1-\text{t}+\frac{ta}{2}+\text{tb}+\text{tc})}{1-\frac{a}{2}}\ \|x_0 - x_1\|$$

$$\|x_1 - x_2\| \leq \alpha_3\ \ \|x_0 - x_1\|$$

$Let\ \ \alpha = \max\{\alpha_1, \alpha_2, \alpha_3\}$, then all above three cases we obtain

$$\|x_1 - x_2\| \leq \alpha\ \ \|x_0 - x_1\|$$

, Therefore

$$\|x_1 - x_2\| \leq \sum_{i=n}^{n+p}\|x_i - x_{i+1}\|$$

$$\frac{\alpha^n}{1-\alpha}\ \ \|x_0 - x_1\|$$

$$\to 0 \quad as \quad n \to \infty$$

This shows that $\{x_n\}$ is a Cauchy is a sequence in C . Since C is a closed subset of a complete space,it is complete. Hence $\{x_n\}$ is convergent and converse to a point u ∈ C.

REFERENCES

[1] Bapurao C Dhage, Approximating Fixed points of generalized nonexpansive mappings in Banach Spaces.,volume 6. Number 2 (2014) 144-153.

[2] J.Bogin, A generalization of a fixed theorem of Goebel Krik and shimi, Canad Math.Bull.19(1976),7-12

[3]R.K .Bose, R.N. Mukherjee, Approximating fixed points if some mappings,proc. Amer.Math. Soc. 82(1981),603-606.

[4] Lj. B. Ciric,On some nonexpansive type mappings and fixed points,Indian J. Pure Appl. Math,24(1993),145-149.

[5] B.C. Dhage,Generalized nonexpansive mappings and fixed points in metric spaces,(Submitted)

[6] W.C. Dotson Jr., Fixed points of Quasi-nonexpansive mappings, J. Austral. Math.Soc.13(1972),167-170

[7] K. Goebel, W. A. Kirk, Topics in Metric Fixed Point Theory, Cambridge Univ. Press 1990.

[8] R. Kannan , Some results on fixed points, Bull. Cal. Math. Soc. 60 (1968),71-76.

[9] W. R. Mann, Mean value methods in iterations, Proc. Amer. Math. Soc. 4(1953),506-510.

[10] H.F. Senter, W.G. Dotson, Approximatting fixed points if nonexpansive mappings,Proc.Amer.Math.

Soc. 44 (1974),375-379