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On the Cantor intersection theorem

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Abstract

In this paper we propose an activity to assist in the teaching of the Cantor intersection theorem. This activity is based on a construction of Pappus of Alexandria concerning the construction of the arithmetic, geometric and harmonic means in a semicircle.

Keywords: *Cantor intersection theorem; arithmetic mean; geometric mean; harmonic mean.*

1. Introduction

A way to get numbers is to produce convergent sequences Thus, we can regard a convergent sequence as defining that point to which it converges. Here is another way of producing numbers. Any decreasing sequence of closed bounded intervals with length decreasing to zero would all have exactly one common point. Thus, such a sequence of intervals can be regarded as defining that unique number, which is common to all intervals. This is the Cantor intersection theorem.

Cantor Intersection Theorem: Let $I_n = [a_n, b_n]$ *for* $n=1,2\ldots$ *be a sequence of intervals where* a_n, b_n *are real numbers. Suppose that the sequence of intervals is decreasing, that is* $I_{n+1} \subset I_n$ *for* $n = 1,2,\ldots$.
If $b_n - a_n \to 0$ *then there is exactly one point which is common to all intervals.* [4]

Cantor intersection theorem is an important topological property of the real number system, a property that depends strongly upon the completeness of the real number system. This theorem is used to prove important facts about sequences and continuous functions. [4]

In this note we propose an activity to assist in the teaching of this theorem. The activity is based on a construction of Pappus of Alexandria. [1]. In book III of the Collection, Pappus describes also the theory of means and gives an attractive construction that includes the arithmetic, the geometric and the harmonic means within a single semicircle. Pappus shows that if in the semicircle ADC with centre O (figure 1) one has $DB \perp OD$ and $BF \perp OD$, then DO is the arithmetic mean, DB the geometric mean and DF the harmonic mean of the magnitudes AB and BC.

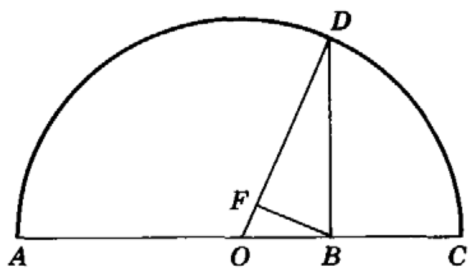

Figure1: Pappus construction

2. The activity

In figure 1, if AO = r, AB= a, BC= b, DB = g, DF = h,

$$r = \frac{a+b}{2},\ g^2 = ab,\ g^2 = rh \qquad (1)$$

Then

$$h = \frac{2ab}{a+b} \qquad (2)$$

with

$$b < h < g < r < a \qquad (3)$$

(1) Observing that g is also the geometric mean of r and h we ask to students to construct a decreasing sequence of closed intervals I_n such that

$$\{g\} = \bigcap_{n=1}^{\infty} I_n \qquad (4)$$

To solve this problem, we set

$$r_1 = \frac{r+h}{2},\ h_1 = \frac{2rh}{r+h}$$

r_1 is the arithmetic mean, h_1 is the harmonic mean of r and h such that

$$h < h_1 < g < r_1 < r$$

In this way we define two sequences

$$r_n = \frac{r_{n-1} + h_{n-1}}{2},\ h_n = \frac{2r_{n-1}h_{n-1}}{r_{n-1} + h_{n-1}},\ n = 2,3,... \qquad (5)$$

satisfying

$$r_n h_n = g^2 \qquad (6)$$

and

$$b < h < h_1 < h_2 < ... < h_n < g < r_n < ... \\ ... < r_2 < r_1 < r < a \qquad (7)$$

with

$$g \in \bigcap_{n=1}^{\infty} [h_n, r_n] \qquad (8)$$

Let x be any other point of this intersection, say g < x. Since $r_n - h_n \to 0$, fix k large so $r_k - h_k < x-g$. Since $g \in [h_k, r_k]$, if $x \in [h_k, r_k]$ then we would have $x-g \le r_k - h_k$, contradiction. Then

$$\{g\} = \bigcap_{n=1}^{\infty} [h_n, r_n] \qquad (9)$$

(2) We ask to students to prove that g is the common limit of the two sequences

Proof: Any r_k is an upper bound for the increasing sequence (h_n). So $h_n \to s$ where s is the supremum of the sequence (h_n). In particular $h_n \le s$ for each n. Moreover, each r_n being an upper bound for the sequence (h_n) we see that $s \le r_n$ for each n. In other words $h_n \le s \le r_n$ for each n, showing that the point s is in all intervals.
Then s = g.

3. Numerical approach

Table 1: The sequences h_n, r_n for a = 9, b = 4

h (decimal form)	***r (decimal form)***	**Length of [h, r] less than**
5.538461538	6.500000000	1
5.980830670	6.019230770	0.1
5.99996928	6,000030720	0.0001
6,00000000	6.00000000	0.000000001

In this case

$$\{6\} = \bigcap_{n=1}^{\infty} [h_n, r_n] \qquad (10)$$

Table 2: The sequences h_n, r_n for a = 2, b = 1

h (decimal form)	***r (decimal form)***	**Length of [h, r] less than**
1.33333333	1.500000000	0.2
1.411764706	1.416666667	0.01
1.414211438	1.414215687	0.00001
1.414213562	1.414213563	0.000000001

In this case

$$\{\sqrt{2}\} = \bigcap_{n=1}^{\infty} [h_n, r_n] \qquad (11)$$

4. Conclusions

In this paper we proposed an introductory activity to assist in the teaching of the Cantor intersection theorem. This theorem is an important topological property of the real number system. It is used to prove important facts about sequences and continuous functions.

References

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