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Connectedness via Quasi I-Open Sets

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Abstract. In this paper, we introduce the notion quasi I-separated sets and study some of their basic properties.

1 . INTRODUCTION AND PRELIMINARIES

The subject of ideals in topological spaces has been introduced and studied by Kuratowski [3] and Vaidyanathasamy [7]. An ideal I on a topological space (X, τ) is a nonempty collection of subsets of X which satisfies (i) $A \in I$ and $B \subset A$ implies $B \in I$ and (ii) $A \in I$ and $B \in I$ implies $A \cup B \in I$. Given a topological space (X, τ) with an ideal I on X and if P(X) is the set of all subsets of X, then the set operator $(.)^*$: $P(X) \to P(X)$, called the local function [7] of A with respect to τ and I, is defined as follows: for $A \subset X$, $A^*(\tau, I) = \{x \in X / U \cap A \notin I$, for every open set U containing x $\}$. A Kuratowski closure operator $Cl^*(.)$ for a topology $\tau^*(\tau, I)$ called the $*$-topology, finer than τ is defined by $Cl^*(A) = A \cup A^*(\tau, I)$. When there is no chance for confusion, $A^*(\tau, I)$ is denoted by A^*. If I is an ideal on X, then (X, τ, I) is called an ideal topological space. By a space, we always mean a topological space (X, τ) with no separation properties assumed. If $A \subset X$, Cl(A) and Int(A) will denote the closure and interior of A in (X, τ), respectively.A subset A of (X, τ) is said to be β-open [1] if $A \subset Cl(Int(Cl(A)))$. The complement of a β-open set is called a β-closed [1] set. In this paper, we introduce the notion quasi I-separated sets and study some of their basic properties.

2 . PREILIMINARIES

A subset S of an ideal topological space (X, τ, I) is quasi I-open [2] if $S \subset Cl(Int(S^*))$. The complement of a quasi I-open set is called a quasi I-closed [2] set. The intersection of all quasi I-closed (resp.β-closed) sets containing S is called the quasi I-closure (resp. quasi β-closure) of S and is denoted by qICl(S) (resp.$_{\beta}$Cl(S)). The family of all quasi I-open (resp. quasi I-closed) sets of (X, τ, I) is denoted by QIO(X) (resp.QIC(X)).

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The family of all quasi I-open (resp.quasi I-closed) sets of (X, τ, I) containing a point $x \in X$ is denoted by QIO(X,x) (resp.QIC(X,x)).

3. ON QUASI I –SEPARATED SETS

Definition 3.1.Two nonempty subsets A and B of a topological space(X, τ) are said to be quasi I-separated if $A \cap qICl(B) = qICl(A) \cap B = \emptyset$. If $X = A \cup B$ such that A and B are quasi I-separated sets, then we say that A and B form an quasi I-separation of X.

Remark 3.2.Each two quasi I-separated sets are always disjoint, since A∩B ⊂ A∩qICl(B) = ∅.

Theorem 3.3.For the subsets A and B of a topological space(X, τ),the following statements are equivalent:

(1)A and B are quasi I-separated.

(2)There exist quasi I-closed sets F1 and F2 satisfying A ⊂ F1⊂ (X\B) and B ⊂ F2 ⊂ (X\A).

(3)There exist quasi I-open sets G1 and G2 satisfying A ⊂ G1 ⊂ (X\B) and B ⊂ G2 ⊂ (X\A).

Proof.The proof is clear.

Proposition 3.4. Let A and B be subsets of a topological space(X, τ). If A and B are quasi I-separated, ∅ ≠ C ⊂ A and ∅ ≠ D ⊂ B, then C and D are quasi I-separated.

Proof. Since A and B are quasi I-separated sets, A ∩ qICl(B)= ∅and qICl(A)∩B = ∅. By hypothesis C⊂A, we have qICl(C) ∩ D = ∅.Similarly, we have C ∩ qICl(D) = ∅. Therefore, C and D are quasi I-separated sets.

Theorem 3.5. If for an quasi I-closed subset S of a topological space (X, τ), A and B are quasi I-separated sets such that S = A∪B, then A and B are quasi I-closed sets.

Proof. Let S = A∪B, where qICl(A) ∩ B = ∅ = A ∩ qICl(B). Now, S ∩ qICl(A) = (A∪B) ∩ qICl(A) = A. As the intersection of quasi I-closed sets is quasi I-closed, A is quasi I-closed. Similarly B is quasi I-closed.

Theorem 3.6. Let A and B be nonempty subsets in a topological space (X, τ). The following statements hold:

(1)If A and B are quasi I-separated and A1 ⊂ A, B1 ⊂ B, then A1 and B1 are so.

(2)If A∩B = ∅such that A and B are quasiI-closed (quasiI-open), then A and B are quasi I-separated.

(3)If A and B are quasiI-closed (quasiI-open) and H = A∩(X\B) and G= B∩(X\A), then H and G are quasi I-separated.

Pr oof.(1). Since A1 ⊂ A, qICl(A1) ⊂ qICl(A). Then B ∩ qICl(A) = ∅ implies B1 ∩ qICl(A) = ∅ and B1 ∩ qICl(A1) =∅. Similarly A1∩ qICl(B1)=∅. Hence A1 and B1 are quasiI-separated.

(2). Since A= qICl(A), B = qICl(B) and A∩B=∅, qICl(A)∩B=∅ and qICl(B)∩A = ∅. Hence A and B are quasi I-separated sets. If A and B are quasiI-open, then their complements are quasiI-closed.

(3). If A and B are quasiI-open, then X\A and X\B are quasiI-closed. Since H⊂X\B, qICl(H) ⊂ qICl(X\B) =X\B and so qICl(H) ∩B=∅. Thus G ∩ qI Cl(H)= ∅. Similarly, H ∩ qICl(G) =∅.Hence H and G are quasi I-separated sets.

Theorem 3.7. The sets A and B of a topological space(X, τ) are quasi I-separated if, and only if there exist U,V∈ QIO(X, τ) such that A⊂U, B⊂V, A∩V=∅ and B∩U=∅.

Proof. Let A and B be quasi I-separated sets. Set V = X\qICl(A) and U = X\qICl(B). ThenU,V∈ QIO(X, τ) such that A⊂U, B⊂V, A∩V=∅ and B∩V=∅. On the other hand, let U,V∈ QIO(X, τ) such thatA ⊂U, B⊂V, A∩V=∅ and B∩U=∅. Since X\V and X\U are quasi I-closed sets, qICl(A) ⊂ X\V ⊂ X\B and qICl(B) ⊂ X\U ⊂ X\A. Thus qICl(A) ∩ B=∅ and qICl(B) ∩ A=∅.

Theorem 3.8.Let A and B be nonempty disjoint subsets of a topolog-ical space(X, τ) and E = A∪B. Then A and B are quasiI-separated if and only if each of A and B are quasi I-closed (quasiI-open) in E.

Proof. Let A and B are quasi I-separated sets. By Definition 3.1,A contains no quasiI-limit points of B. Then B contains all quasi I-limit points of B which are in A∪B and B is quasi I-closed in A∪B.Therefore Bis quasiI-closed in E. Similarly A is quasiI-closed in E.

Theorem 3.9. Let (X, τ) be a topological space. If A and B are quasiI-separation of X itself, then A and B are quasiI-closed sets of(X, τ).

Proof. Since A and B are quasiI-separated, A ∩ qICl(B) = qICl(A) ∩ B=∅. Then A ∩ qICl(B) =∅ if, and only if B is quasiI-closed inA∪B=X. Similarly, we can show that A is quasi I-closed in X.

4. PROPERTIES OF QUASI I – CONNECTED SPACES

In this section, we introduce and study quasi I-connected spaces and also investigate some of their basic properties.

Denition 4.1. A subset A of a topological space (X, τ) is said to be Quasi I-connected if it cannot be expressed as the union of two quasi I -separated sets. Otherwise, the set A is called quasi I-disconnected.

Lemma 4.2. Let A ⊂ B∪C such that A be a nonempty quasi I-connected set in a topological space (X, τ) and B , C be quasi I-separated sets. Then only one of the following conditions holds:

(1)A⊂B and A∩C = ∅.

(2)A⊂C and A∩B = ∅.

Proof. Since A∩C = ∅, A⊂B. Also, if A∩B = ∅, then A ⊂ C. Since A ⊂ B∩C, then both A∩B = ∅ and A∩C = ∅ cannot hold simultaneously. Similarly, suppose that A∩B = ∅ and A∩C = ∅,then by Theorem 3.6 (1), A∩B and A∩C are quasiI- separated sets such that A = (A∩B) ∪ (A∩C) which contradicts with the quasi I-connectedness of A. Hence one of the conditions (1) and (2) must be hold.

Theorem 4.3. If an quasiI-connected set S of a topological space (X, τ) is contained in A∪B, where A and B are quasi I-separated sets, then either

S ⊂ A or S ⊂ B.

Proof. We have S = (S∩A) ∪ (S∪B) where S∩A and S∩B are quasi I-separated sets. So either S∩A=∅ or S∩B=∅ and hence either S⊂B or S⊂A.

Theorem 4.4. A subset M of a topological space(X , τ) is an quasi I-connected set if there exists an quasi I-connected set C satisfying C ⊂ M ⊂ qICl(C).

Proof. Let M = A∪B, where A and B are quasi I-separated sets. . Then either C⊂A and C⊂B and hence either M ⊂ qICl(C) ⊂qICl(A) ⊂ (X\B) or M⊂(X\A). Therefore either B = ∅ or A=∅.

Corollary 4.5. If C is an quasi I-connected set of a topological space(X, τ), then qICl(C) is so.

Proof . Follows from Theorem 4.4.

Theorem 4.6. If $\{M_\alpha: \alpha \in\Delta\}$ is a family of quasi I-connected sets of a topological space(X,τ) satisfying the property that any two of which are not quasi I-separated, then $M=\cup\alpha\in\Delta M_\alpha$ is quasi I-connected

Proof. Let $M = A\cup B$, where A and B are quasi I-separated sets. Then for each $\alpha \in\Delta$ either $M_\alpha\subset A$ or $M_\alpha\subset B$. Since any two members of the family $\{M_\alpha:\alpha \in\Delta\}$ are not quasiI-separated, either $M_\alpha\subset A$ for each $\alpha \in\Delta$ or $M_\alpha\subset B$ for each $\alpha\in\Delta$. So either $B=\emptyset$ or $A=\emptyset$.

Corollary 4.7. If $M = \cup\alpha\in\Delta\ M$, where each M_α is quasi I-connected set in a topological space(X, τ) and also $M_\alpha\cap M\alpha' = \emptyset$ for ; $\alpha, \alpha' \in\Delta$, then M is quasi I-connected.

Proof. Follows from Theorem 4.6.

Corollary 4.8. If $M = \cup\alpha\in\Delta\ M\alpha$, where each M_α is quasi I connected in a topological space (X, τ) and $\cap_\alpha\in\Delta M \neq \emptyset$,for each $\alpha \in \Delta$,then M is quasi I connected

Proof. Suppose that $\cup\alpha\in\Delta M_\alpha$ is not quasi I-connected. Then we have $\cup_\alpha\in\Delta M_\alpha = H\cup G$, where H and G are quasi I-separated sets in X. Since $\cap_\alpha\in\Delta M_\alpha \neq \emptyset$, we have a point x in $\cap_\alpha\in\Delta M_\alpha$. Since $x \in \cup_\alpha\in\Delta M_\alpha$, either $x \in G$ or $x\in H$. Suppose that $x \in H$. Since $x \in M_\alpha$ foreach $\alpha \in\Delta$, then M_α and H intersect for each $\alpha\in\Delta$. By Theorem 4.3,$M_\alpha\subset H$ or $M_\alpha\subset G$. Since H and G are disjoint, $M_\alpha \subset H$ for all $\alpha \in \Delta$ and hence $\cup_\alpha\in\Delta M_\alpha \subset H$. This implies that G is empty. This is contradiction. Suppose that $x\in G$. By similar way, we have that H is empty. This is a contradiction. Thus $\cup_\alpha\in\Delta M_\alpha$ is quasi I-connected

Theorem 4.9. For a topological space(X, τ), the following statements are equivalent:

(1)X is quasi I-connected.

(2)X can not be expressed as the union of two nonempty disjoint Quasi I-open sets.

(3)X contains no nonempty proper subset which is both quasi I-open and quasi I-closed.

Proof.(1)⇒(2): Suppose that X is quasi I-connected and if X can be expressed as the union of two nonempty disjoint sets A and B such that A and B are quasi I-open sets. Consequently $A\subset X\backslash B$. Then $qICl(A) \subset qICl(X\backslash B) =X\backslash B$. Therefore, $qICl(A) \cap B = \emptyset$. Sim-ilarly we can prove $A \cap qICl(B) =\emptyset$. This is a contradiction to thefact that X is quasi I-connected. Therefore, Xcannot be expressed as the union of two nonempty disjoint quasi I-open sets.

(2)⇒(3): Suppose that X cannot be expressed as the union of two nonempty disjoint sets A and B such that A and B are quasi I-open sets. If X contains a nonempty proper subset A which is both quasiI-open and quasiI-closed. Then $X = A\cup(X\backslash A)$. Hence A and $X\backslash A$ are disjoint quasi I-open sets whose union is X. This is the contradic-tion to our assumption. Hence ,X contains no nonempty proper subset which is both quasi I-open and quasi I-closed.

(3)⇒(1): Suppose that X contains no nonempty proper subset which is both quasi I-open and quasi I-closed and X is not quasi I-connected. Then X can be expressed as the union of two nonempty disjoint sets A and B such that (A∩qICl(B))∪(qICl(A)∩B) =∅. Since A∩B = ∅, A = X\B and B = X\A. Since qICl(A)∩B=∅, qICl(A)⊂X\B. Hence qICl(A) ⊂ A. Therefore, A is quasi I-closed. Similarly, B is quasiI-closed. Since A = X\B, Ais quasi I-open. Therefore, there exists a nonempty proper set a A which is both quasi I-open and quasi I- closed This is a contradiction to our assumption. Therefore, X is quasi I-connected.

Thereorem 4.10. A topological space (X, τ) is quasi I-connected if, and only if X is not the union of any two quasi I-separated sets

Proof. Let A and B be any two quasi I-separated sets such that X=A∪B. Therefore qICl(A)∩B = A∩qICl(B) =∅. Since A⊂qICl(A) and B⊂qICl(B), then A∩B=∅. Now qICl(A) ⊂ X\B =.Hence qICl(A) =A. Then A is quasi I-closed. By the same way we can show that B is quasiI-closed which contradicts with Theorem4.9 (2). Conversely, let A and B be any two disjoint non empty and quasiI-closed sets of X such that X=A∪B. Then qICl(A) ∩ B = A∩qIC(B) = A∩B=∅ , which contradicts with the hypothesis

Theorem 4.11. A topological space(X, τ) is quasi I-connected if, and only if for every pair of points x , y in X , there is an quasi I-connected subset of X which contains both x and y.

Proof. The necessity is immediate since the quasi I-connected space itself contains these two points. For the sufficiency, suppose that for any two points x and y, there is an quasiI-connected subset Cx,y of X such that x , y ∈ Cx,y. Let a ∈ X be a fixed point and{Ca,x:x∈X}be a class of all Quasi I-connected subsets of X which contain the points a and x. Then X=$\bigcup_{x\in X} C_{a,x}$ and $\bigcap_{x\in X} C_{a,x} \neq \emptyset$. Therfore by Corollary 4.8, X is quasi I – connected

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