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Convolution based generalized integral equations

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Abstract: - The aim of this paper is to establish Convolution based generalized integral equations format, which is the special cases of hyperbolic differential equation and extended classical Runge – Kutta method.

Key words: *convolution equation, Fourier transforms, Laplace transform, Taylor series.*

1. Introduction:

The Laplacetransform, become of its simple exponential kernel, is densed we consider nonlinear second kind integral equations of convolution type

$$y(x)=f(x)+\int_{x_0}^{x} k(x-s)\, g\left(s,\left(y(x)\right)\right) ds, \qquad x_0 \le x \le \bar{x}$$

The kernel k and the function f,g are assume to be sufficiently smooth on $[x_0,\bar{x}]$, so the solution y (x) is smooth and in the treatment of special hyperbolic differential equation [5] .Further application are given in [3].

Usually the starting point for numerical method is the more general equation

$$y(x)= f(x)+\int_{x_0}^{x} k(x-s)\, g\left(s,\left(y(x)\right)\right) ds, \quad x_0 \le x \le \bar{x}$$

If the integration interval is discredited with n grid point s, then algorithms for (2) need $0(n^2)$ evaluation of K. For the special equation (1), which appear most often in application, the number of function evaluations can be reduced to 0(n) K- and g- evaluation for suitably chosen method ,eg. Extended Runge-kutta method s and linear multistep method.

In 2 we describe the extended classical Runge –Kutta method. If this method is implementing straightforwardly, it requires still $0(n^2)$ addition and multiplications. it is shown in 3 the central part of this paper, that this is overhead can be reduced to $0(n(\log n)^2)$ .In 4 the asymptotic expantion of the global error is used improving the accuracy of the numerical solution and estimating the global error.

The ideas of this article are not restricted to Runge kutta methods , they are also application to multistep methods.

2. Nonlinear Convolution Equation

:The extended classical Runge-kutta method. Extended Runge-Kutta methods have been introduce by [11].The classical 4th order method, applied to (2), is given by

$$y_n = \tilde{F}_n(x_n)$$

$$\breve{F}_n = f(x) + \frac{h}{6}\sum_{j=0}^{n-1}\left\{K\left(x, x_j, Y^{(j)}{}_i\right) + 2k\left(x, x_j + \frac{h}{2}, Y^{(j)}{}_2\right) + j, Y^{(j)}{}_4\right)\right\}$$

,

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$$Y_I^{(j)} = \tilde{F}_j(x_j),$$

$$Y_I^{(j)} = \tilde{F}_j\left(x_j + \frac{h}{2}\right) + \frac{h}{2}K\left(x_j + \frac{h}{2}, x_j, Y^{(j)}{}_1\right)$$

,

$$Y_2^{(j)} = \tilde{F}_j\left(x_j + \frac{h}{2}\right) + \frac{h}{2}K\left(x_j + \frac{h}{2}, x_j + \frac{h}{2}, Y^{(j)}{}_1\right)$$

,

$$Y_3^{(j)} = \tilde{F}_j\left(x_j + \frac{h}{2}\right) + \frac{h}{2}K\left(x_j + \frac{h}{2}, x_j + \frac{h}{2}, Y^{(j)}{}_2\right)$$

,

,

$$Y_4^{(j)} = \tilde{F}_j\left(x_j + \frac{h}{2}\right) + \frac{h}{2}K\left(x_j + h, x_j + \frac{h}{2}, Y^{(j)}{}_3\right)$$

,

Here h denotes the stepsize, and y_n approximate the solution at $x_n = x_0 + nh$. This method is convergent with a global error of $0(h^4)$ for a proof see pouzet [11] or Hairer – Lubich –Norsett [6].The positions, where the kernel K (x,s,y) has to be evaluated, lie very regular in the (x, s)-plane.

Since these point lie on as parallel to the diagonal and the x-axis, the number of the function evaluations can be reduced for the convolution equation (1). Here the method (3) read

$$\left(\tilde{F}_0(x_n) = f(x)\right)$$

$$y_n = \tilde{F}_n(x_n)$$

$$\breve{F}_n = f(x) + \frac{h}{6}\sum_{j=0}^{n-1}\left\{K(x - x_0)g(x_0, Y_1{}^{(0)}) + \frac{h}{3}K\left(x - x_j - \frac{h}{2}\right)\left[g\left(x_j + \frac{h}{2}, Y_2{}^{(j)})\right) + g\left(x_j + \frac{h}{2}, Y_3{}^{(j)}\right)\right]\right\} + \frac{h}{6}\sum_{j=0}^{n-1}k(x - x_j)\left[g\left(x_j, Y^{(j-1)}{}_4 + g(x_j, Y_1{}^{(j)})\right)\right]$$

$$+\frac{h}{6}k(x - x_n)g(x_m Y^{(n-1)}{}_4),$$

$$Y_I^{(j)} = \tilde{F}_j(x_j),$$

$$Y_2^{(j)} = \tilde{F}_j\left(x_j + \frac{h}{2}\right) + \frac{h}{2}K\left(x_j + \frac{h}{2}, x_j + \frac{h}{2}, Y^{(j)}{}_1\right)$$

,

$$Y_3^{(j)} = \tilde{F}_j\left(x_j + \frac{h}{2}\right) + \frac{h}{2}K\left(x_j + \frac{h}{2}, x_j + \frac{h}{2}, Y^{(j)}{}_2\right)$$

,

$$Y_4^{(j)} = \tilde{F}_j\left(x_j + \frac{h}{2}\right) + \frac{h}{2}K\left(x_j + h, x_j + \frac{h}{2}, Y^{(j)}{}_3\right)$$

It is seen that for the computation of y_n only 2n+1 k- and f-evaluation and 4n.

3. **For fast commutation terms.** We describe in the section, how the overhead for method (4) can be reduce using FFT-techniques.

Assume that $Y_i^{(j)}$ for i=1, 4 and j=0 …r-1 are commuted directly by

(4) Step 1 of the algorithm). With the notation.

$K_j = K\left((j+1)\frac{h}{2}\right)$, j=0,……………,4r-1,

(5) $y_0 = \frac{h}{6}g(x_0, Y^{(0)}{}_1)$,

(6)
$$y_{2j+1} = \frac{h}{3}\left[g\left(x_j + \frac{h}{2}, Y_2{}^{(j)}\right) + g\left(x_j + \frac{h}{2}, Y_3{}^{(j)}\right)\right]$$
, j=0,…………,r-1,

(7) $y_{2j} = \frac{h}{3}[g(x_j, Y_2{}^{(j-1)}) + g(x_j, Y_1{}^{(j)})]$, j=1,……………….r-1,

$y_{2j} = \frac{h}{6}$,$g(x_n, Y_4{}^{(r-1)})$

,$y_j = 0$ for j=2r+1,…….,4r-1,

The lag –term

$\tilde{F}_r(x) = f(x) + (k * y)_{2r+j}$ for $x = x_r + (j+1)\frac{h}{2}$ $and\ j = 0,1, \ldots, 2r-1$

Here the convolution of the two 4r-dimensional sequences K=(K_j) and $y = (y_j)$ is given by

$(k * y)_m = \sum_{i=0}^{4r-1} k_{m-1,} y_i$

This convolution can be computed efficiently using the fast Fourier transform (FFT).

For the computation of step III we observe that method (4) applied to

(8)
$y(x) = \tilde{F}_n(x) + \int_{x_n}^{x} k(x-s) g\big(s, y(s)\big)\, ds,$

(with n=r) yield the same numerical solution for $x \geq x_n$ as when applied to (1). This permits us to compute $Y^{(j)}{}_i$ for j=r ,……..2r-1 since the required $\tilde{F}_r(x)$. values are known .

In step iv we employ the same arguments as in step II with r replaced by 2r and compute

$\tilde{F}_{2r}(x)$ for $x = x_{2,} + (j+1)\frac{h}{2}$,j=0,1………………..,4r-1 .

For Error Estimation: It numerical solution by

$y(x,h) = y_n$ if $x = x_0 + nh$, in order to indicate its dependence on the step size.

It has been shown in [6] that global error has an asymptotic expansion of the form

(9) $y(x) - y(x,h) = e_4(x)h^4 + e_5(x)h^5 + \ldots\ldots.. + e_N(x)h^N + O(h^{N+1})$.

The numerical solution at x is compute for the step sizes h ,$\frac{h}{2}$,$\frac{h}{4}\frac{h}{8}$,…………

and is denoted by $T_{i0} = y\left(x, \frac{h}{2^i}\right)$. Then the extrapolation tableau [4].

Observe that the leading term in (9) equals that of $T_{i,k+1} - T_{ik}$, which is a numerically available estimate of (9)

This motivates the following strategy: For a For prescribed tolerance TOL, calculate the extrapolation tableau until we have for some indices i,k

(10)
$\left|T_{i,k+1} - T_{ik}\right| \leq TOL$

By the above consideration this difference estimates the error of T_{ik} . The more accurate value $T_{i,k+1}$ is then accepted as a numerical approximation to (x) .

Observe that the k – and f- values, which are needed for the computation of $y(x,h)$ can be used again for the computation of $y\left(x, \frac{h}{2}\right)$.

Furthermore, the Fourier transforms of the kernel – values, which were were computed before, can be used to reduce the effort for the computation of the kernel Fourier transforms for the step size $\frac{h}{2}$.

In this section we solved integral equation by the Laplace transform and Taylor series. First, the Laplace transform is applied to both sides of Equation

$$L[u(x)] = L[f(x)] + L\left[\int_0^x k(x-t)u(t)\,dt\right]$$

Using the Laplace transform property, the equation below can be obtained

$$L[u] = L[f] + L[k]L[u]$$

Thus, the given equation is equivalent to

$$L[u] = \frac{L[f]}{1 - L[k]} = F(s)$$

Applying above theorem, $F(s)$ can be expanded as an absolutely convergent series, which is given by

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$$L[u] = \frac{c_1}{s} + \frac{c_2}{s^2} + \frac{c_3}{s^3} + \ldots\ldots\ldots\ldots\ldots..$$

where $c_1, c_2, c_3 \ldots\ldots\ldots$ are the known constants. Considering the inverse Laplace transform on both sides of the above equation, we can then obtain

$$u(x) = c_1 + \frac{c_2}{\tau(2)} x + \frac{c_2}{\tau(3)} x^2 + \frac{c_4}{\tau(4)} x^3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(A)}$$

which is uniformly convergent to the exact solution. We approximate the solution $u(x)$ by using

$$u_n(x) = c_1 + \frac{c_2}{\tau(2)} x + \frac{c_2}{\tau(3)} x^2 + \ldots\ldots\ldots\ldots\ldots + \frac{c_n}{\tau(n)} x^{n-1}$$

Let $e_n(x) = u(x) - u_n(x)$ be the error function, where $u_n(x)$ is the estimation of the true solution $u(x)$. Using Taylor's theorem and assuming that $ju^{(n)}(x)j \cdot M$, the equation below can be obtained

$$|e_n| = |u(x) - u_n(x)| \leq M \frac{c_{n+1}}{\tau(n+1)} |x^n|.$$

References

[1]H. J. Teriele, [1982]"Collocation method for weakly singular second kind integral equations with non-smooth solution," *IMA Journal of Numerical Analysis*, vol. 2, no. 4, pp. 437–449.

[2]P. K. Lamm and L. Elden,´ [1997]"Numerical solution of first-kind equations by sequential Tikhonov regularization," *SIAM Journal on Numerical Analysis*, vol. 34, no. 4, pp. 1432–1450.

[3]L. Huang, Y. Huang, and X. Li,[1998] "Approximate solution of Abel integral equation," *Computers* & *Mathematics with Applications*, vol. 56, no. 7, pp. 1748–1757.

[4] C. Yang, "Chebyshev polynomial solution of nonlinear integral equa-tions,[2002] " *Journal of the Franklin Institute*, vol. 349, no. 3, pp. 947–956, 2012.

[5]S. A. Yousefi, [2006]"Numerical solution of Abel's integral equation by using Legendre wavelets," *Applied Mathematics and Computation*, vol. 175, no. 1, pp. 574–580.

[6]) M. Khodabin, K. Maleknejad, and Gupta, [2007]"Application of triangular functions to numerical solution of stochastic inte-gral equations," *IAENG International Journal of Applied Mathematics*, vol. 43, no. 1, pp. 1–9.

[7] A. V. Kamyad, M. Mehrabinezhad, and J. Saberi-Nadjafi,[2010] "A numerical approach for solving linear and nonlinear integral equations with controlled error," IAENG International Journal of Applied Math-ematics, vol. 40, no. 2, pp. 27–32.

[8] G. Adomian, *Frontier Problem of Physics: The Decomposition Method*. Kluwer Academic Publish, [2010]. L. Bougoffa, R. C. Rach, and A. Mennouni, [2011]"A convenient technique for solving linear and nonlinear Abel integral equations by the Adomian decomposition method," *Applied Mathematics and Computation*, vol. 218, no. 5, pp. 1785–1793, 201.

[9]R. K. Pandey, O. P. Singh, and V. K. Singh,[2010] "Efficient algorithms to solve singular integral equations of Abel type," *Computers* & *Mathematics with Applications*, vol. 57, no. 4, pp. 664–676.

[10] J. H. He, "Homotopy perturbation technique," *Computer Methods in Applied Mechanics and Engineering*, vol. 178, no. 3-4, pp. 257–262, 1999.