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Analytical solution of a transverse magnetic field on UCM fluids flow and species transfer with porous medium passing through channel

Mostafa Ebrahimian[1], Mehran Khaki[1] and Morteza Abbasi[1]

[1] Department of Mechanical Engineering, Sari Branch, Islamic Azad University, Sari, Iran

Abstract

In the present work, an analysis of the nonlinear problem of two-dimensional steady, laminar flow of an upper convected Maxwell viscoelastic fluid with species diffusion in channel containing a homogeneous, isotropic porous medium under slip condition is considered. Analytical expressions for velocity profile and the species concentration profile are obtained by using Homotopy Analysis Method (HAM). In this works, the HAM has been used to solve nonlinear differential equations with mixed (Robin) boundary conditions. It has been attempted to show the capabilities and wide-range applications of the Homotopy Analysis Method in comparison with a type of numerical analysis as Boundary Value Problem (BVP) in solving this problem. The obtained solutions, in comparison with the numeric solutions admit a remarkable accuracy.

Keywords: *Homotopy Analysis Method; upper-convected Maxwell fluid; permeable channel; mass transfer; slip condition.*

1. Introduction

The flow problem in porous tubes or channels received much attention in recent years because of its various applications in biomedical engineering, for example in the dialysis of blood in artificial kidney, in the flow of blood in the capillaries, in the flow in blood oxygenators, as well as in many other engineering areas such as the design of filters, in transpiration cooling boundary layer control and gaseous diffusion. Because of its relevance to a variety of situations, convection in porous media is a well-developed field of investigation. Viscoelastic flows are encountered in numerous areas of petrochemical, biomedical and environmental engineering including polypropylene coalescence sintering[1], dynamically-loaded journal bearings[2], blood flow[3,4] and geological flows[5]. A wide range of mathematical models have been developed to simulate the nonlinear stress–strain characteristics of such fluids which exhibit both viscous and elastic properties. A detailed discussion of such models which include the upper convected Maxwell model, the Walters-B model and the Reiner–Rivlin second-order model is provided in Zahorski[6]. For highly elastic fluids such as polymer melts, the upper convected Maxwell (UCM) model has proved to be very reliable. Most of problems and scientific phenomena such as heat transfer are inherently of nonlinearity. We know that except a limited number of these problems, most of them do not have exact solutions. Therefore, these nonlinear equations should be solved approximately either numerically or analytically. In the numerical method, stability and convergence should be considered so as to avoid divergence or inappropriate results. Time consuming is another problem of numerical techniques. In analytical methods, the perturbation method [7] is widely used, but in most cases to find a suitable small parameter is difficult. Other many different methods have introduced to solve nonlinear equation such as the δ-expansion method [8], Adomian's decomposition method [9], Homotopy Perturbation Method (HPM) [10–13], Variational Iteration Method (VIM) [14–19], Homotopy analysis method [20-25], Optimal Homotopy Asymptopic Method (OHAM) [26, 27] and optimal Homotopy Perturbation Method (OHPM) [28].

In this Letter, the equations of the steady, laminar flow of an incompressible, viscoelastic fluid with species diffusion in a parallel plate channel are solved through HAM. The convergence of the series solution is also explicitly discussed. Obtaining the analytical solution of the models and comparing with numerical result reveal the capability, effectiveness and convenience of HAM

2. Problem statement and mathematical formulation

Let us consider the steady laminar flow of an incompressible and electrically conducting fluid which saturates the rigid, isotropic, homogenous porous material intercalated between the plates as shown in Fig.1.The slip boundary conditions are exerted on walls. The flow is symmetric about both axes. Following [29, 30] the flow is assumed to be symmetric about both axes. The steady state flow and species diffusion occur in the channel with the fluid extraction or influx taking place at both plates. The uniform magnetic field is imposed along the y-axis. It is assumed that the magnetic Reynolds Number is small and the induced magnetic field due to the motion of the electrically conducting fluid is negligible. It is also assumed that the electrical conductivity of fluid σ , is constant and the external electric field is zero. The constitutive equation for a Maxwell fluid is:

$$\tau + \lambda_1 \hat{\tau} = \mu_0 \gamma \quad (1)$$

Where τ is the extra stress tensor and the upper convected time derivative of the stress tensor $\hat{\tau}$ satisfies:

$$\hat{\tau} = \frac{\partial \tau}{\partial t} + v.\nabla\tau - (\nabla v)^T.\tau - \tau.\nabla v \quad (2)$$

In which μ_0 is the low-shear viscosity, λ_1 is the relaxation time, γ is the rate-of-strain tensor, t denotes time, v is the velocity vector, $(0)^T$ is the transpose of the tensor and ∇v represents the fluid velocity gradient tensor.

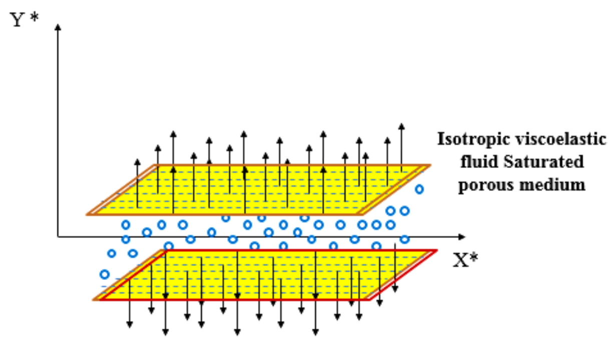

Fig. 1 Schematic diagram of the permeable channel

Implementing the shear-stress strain tensor for a UCM liquid from Esq. (1) and (2), in the absence of a pressure gradient, the steady two-dimensional boundary layer equations for this flow in usual notation are:

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0 \quad (3)$$

$$u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} + \lambda\left[u^{*2}\frac{\partial^2 u^*}{\partial x^{*2}} + v^{*2}\frac{\partial^2 u^*}{\partial y^{*2}} + 2u^* v^* \frac{\partial^2 u^*}{\partial x^* \partial y^*}\right] = \upsilon \frac{\partial^2 u^*}{\partial y^{*2}} - \upsilon \frac{u^*}{k'}, \quad (4)$$

$$u^* \frac{\partial C}{\partial x^*} + v^* \frac{\partial C}{\partial y^*} = D\frac{\partial^2 C}{\partial y^{*2}}, \quad (5)$$

Where (u^*, v^*) are the fluid velocity components along x^* and y^* directions, respectively. The flow is symmetric about the center line of the channel, $y^* = 0$ and we only focus our attention on the flow in the region $0 < y^* < H$. The appropriate boundary conditions for the velocity are symmetry about the x^* -axis and slip conditions at $y^* = H$ yield:

$$y^* = 0: \quad \frac{\partial u^*}{\partial y^*} = 0, \quad v^* = 0, \quad C = C_w \quad (6)$$

$$y^* = H: \quad -\beta u^* = \frac{\partial u^*}{\partial y^*} 0, \quad v^* = V_w,$$
$$y^* = H: \quad C = C_H. \quad (7)$$

Where C_w, β and C_H are the species concentration at the channel center-line, coefficients of sliding friction and C_H is the concentration at the upper plate respectively. As such only a semi-region of the channel is considered. The following dimensionless variables are introduced:

$$x = \frac{x^*}{H}; \quad y = \frac{y^*}{H}; \quad u^* = -V_w\, x\, f'(y);$$
$$v^* = V_w\, f(y); \quad \phi = \frac{C - C_H}{C_w - C_H} \quad (8)$$

Equation (3) is automatically satisfied and Eqs (4) – (7) may be written as:

$$f''' + \frac{1}{Da} f' + \mathrm{Re}_T\left(f'^2 - f\, f''\right) + De\left(2 f\, f'\, f'' - f^2\, f'''\right) = 0 \quad (9)$$

$$\phi'' - \mathrm{Re}_T\, S_c\, f\, \phi' = 0 \quad (10)$$

The boundary conditions become:

$$y = 0: \quad f'' = 0; \quad f = 0; \quad \phi = 1,$$
$$y = 1: \quad f' = -k\, f''; \quad f = 1; \quad \phi = 0. \quad (11)$$

Against the differential equation of the model is in third order but there are four boundary conditions for problem. Some authors satisfy boundary conditions in the initial guess function. In this work creatively with derivation of Eq. (9) and introduce fourth order differential equation we can satisfy all of boundary condition in main equation. Then we have,

$$f'''' + \frac{1}{Da} f'' + \mathrm{Re}_T\left(f'\, f'' - f\, f'''\right) + De\left(2 f'^2\, f'' - 2 f\, f''^2 + f^2\, f''''\right) = 0 \quad (12)$$

$$\phi'' - \mathrm{Re}_T \; S_c \; f \, \phi' = 0 \tag{13}$$

Here, $\mathrm{Re}\,w = \frac{V_w H}{\upsilon}$ is the transpiration Reynolds number, $De = \frac{\lambda V_w^{\,2}}{\upsilon}$ is the Deborah number, $Da = \frac{k'}{H^2}$ is the Darcy number and $S_c = \frac{\upsilon}{D}$ is the Schmidt number, where $\mathrm{Re}\,w > 0$ corresponds to suction and $\mathrm{Re}\,w < 0$ for injection

3. Application of Homotopy Analysis Method

For HAM solutions, we choose the initial guess and auxiliary linear operator in the following form:

$$f_0(\mathrm{y}) = -\frac{1}{2}y^3 + \frac{3}{2}y, \qquad \phi_0(y) = 1 - y, \tag{14}$$

$$L_1(f) = f'''', \qquad L_2(\phi) = \phi'', \tag{15}$$

$$L_1\left(\frac{1}{6}c_1 y^3 + \frac{1}{2}c_2 y^2 + c_3 y + c_4\right) = 0, \quad L_2(c_5 y + c_6) = 0, \tag{16}$$

where $c_i(i = 1-6)$ are constants. Let $P \in [0,1]$ denotes the embedding parameter and $\hbar_1$, $\hbar_2$ indicates non –zero auxiliary parameters. We then construct the following equations:

Zeroth –order deformation equations

$$(1-P)L_1\left[F(y;p) - f_0(y)\right] = p\hbar_1 H(y) N\left[F(y;p)\right] \tag{17}$$

$$F(0;p) = 0; \quad F''(0;p) = 0, \quad F(1;p) = 1, \quad F'(1;p) = 0 \tag{18}$$

$$(1-P)L_2\left[\phi(y;p) - \phi_0(y)\right] = p\hbar_2 H(y) N\left[\phi(y;p)\right] \tag{19}$$

$$\phi(0;p) = 1; \quad \phi'(1;p) = 0 \tag{20}$$

$$N[F(y;p)] = \frac{d^4F(y;p)}{dy^4} + \mathrm{Re}\,w\left[\frac{dF(y;p)}{dy}\frac{d^2F(y;p)}{dy^2} - F(y;p)\frac{d^3F(y;p)}{dy^3}\right] + \frac{1}{Da}\frac{d^2F(y;p}{dy^2} + De\left[\begin{array}{l} 2\left(\frac{dF(y;p)}{dy}\right)^2\frac{d^2F(y;p)}{dy^2} - 2F(y;p)\left(\frac{d^2F(y;p)}{dy^2}\right)^2 \\ +\left(F(y;p)\right)^2\frac{d^4F(y;p)}{dy^4}\end{array}\right] \tag{21}$$

$$N[\phi(y;p)] = \frac{d^2\phi(y;p)}{dy^2} + \mathrm{Re}_T\, S_c\, F(y;p)\frac{d\phi(y;p)}{dy} = 0 \tag{22}$$

For $p = 0$ and $p = 1$ we have

$$F(y;0) = f_0(y)\;; \qquad F(y;1) = f(y) \tag{23}$$

$$\phi(y;0) = \phi_0(y)\;; \qquad \phi(y;1) = \phi(y) \tag{24}$$

When p increases from 0 to 1 then $F(y;p)$ and $\phi(y;p)$ varies from $f_0(y)$ and $\phi_0(y)$ to $f(y)$ and $\phi(y)$. By Taylor's theorem and using Eq. (21) and Eq.(22), $F(y;p)$ and $\phi(y;p)$ can be expanded in a power series of p as follows:

$$F(y;0) = f_0(y)\;; \qquad F(y;1) = f(y) \tag{25}$$

$$\phi(y;0) = \phi_0(y)\;; \qquad \phi(y;1) = \phi(y) \tag{26}$$

In which $\hbar$ is chosen in such a way that this series is convergent at $p = 1$, therefore we have through Eq. (25) and Eq. (26) that

$$F(y;p) = f_0(y) + \sum_{m-1}^{\infty} f_m(y)p^m, \quad f_m(y) = \frac{1}{m!}\left.\frac{\partial^m(F(y;p))}{\partial p^m}\right|_{p=0} \tag{27}$$

$$\phi(y;p) = \phi_0(y) + \sum_{m-1}^{\infty} \phi_m(y)p^m, \quad \phi_m(y) = \frac{1}{m!}\left.\frac{\partial^m(\phi(y;p))}{\partial p^m}\right|_{p=0} \tag{28}$$

mth –order deformation equations

$$L\left[f_m(y) - \chi_m f_{m-1}(y)\right] = \hbar H(y) R_m(y) \tag{29}$$

$$F(0;p) = 0; \quad F''(0;p) = 0, \quad F(1;p) = 0, \quad F'(1;p) = 0 \tag{30}$$

$$L\left[\phi_m(y) - \chi_m \phi_{m-1}(y)\right] = \hbar H(y) R_m(y) \tag{31}$$

$$\phi(0;p) = 0; \quad \phi'(1;p) = 0 \tag{32}$$

$$R_m(\mathrm{y})$$
$$= f'''_{m-1} + \frac{1}{Da} f''_{m-1} + \sum_{k=0}^{m-1}\Big[\operatorname{Re} w\left(f'_{m-1-k}\ f''_k - f_{m-1-k}\ f'''_k\right)$$
$$+De\ f'_{m-1-k}\left(\sum_{l=0}^{k}\left(2\ f'_{k-l}\ f''_l\right)\right) \tag{33}$$
$$-De\ f_{m-1-k}\left(\sum_{l=0}^{k}\left(2\ f''_{k-l}\ f''_l - f_{k-l}\ f'''_l\right)\right)\Big]$$

$$R_m(\mathrm{y}) = \phi''_{m-1} + \sum_{k=0}^{m-1}\left[\operatorname{Re} w S_c\ \ f_{m-1-k}\ \phi'_k\right] \tag{34}$$

Now we determine the convergency of the result, the differential equation, and the auxiliary function according to the solution expression. So let us assume:

$$H(y) = 1 \tag{35}$$

We have found the answer by maple analytic solution device. For first deformation of the solution are presented below

$$f_1(\mathrm{y}) = -\frac{5}{672}\hbar De \mathrm{y}^9 - \frac{\hbar_1}{3360}\frac{(-12\operatorname{Re}_T Da - 216 De\, Da)\,\mathrm{y}^7}{Da}$$
$$-\frac{\hbar_1}{3360}\frac{(378 De\, Da+84)\mathrm{y}^5}{Da} + \frac{\hbar_1}{840}\frac{(52 De\, Da - 9\operatorname{Re}_W Da + 42)\mathrm{y}^3}{Da} \tag{36}$$
$$-\frac{\hbar_1}{1120}\frac{(7 De\, Da - 8\operatorname{Re}_W Da + 28)\mathrm{y}}{Da}$$

$$\phi_1(\mathrm{y}) = \frac{1}{40}\hbar_2 \operatorname{Re}_w S_c \mathrm{y}^5 - \frac{1}{4}\hbar_2 \operatorname{Re}_w S_c \mathrm{y}^3$$
$$+\frac{9}{40}\hbar_2 \operatorname{Re}_w S_c \mathrm{y} \tag{37}$$

The solutions and were too long to be mentioned here, therefore, they are shown graphically

4. Convergence of the HAM solution

As pointed out by Liao, the convergence and rate of approximation for the HAM solution strongly depends on the value of auxiliary parameter $\hbar$. The auxiliary parameter $\hbar$ provides us with a convenient way to adjust and control the convergency.

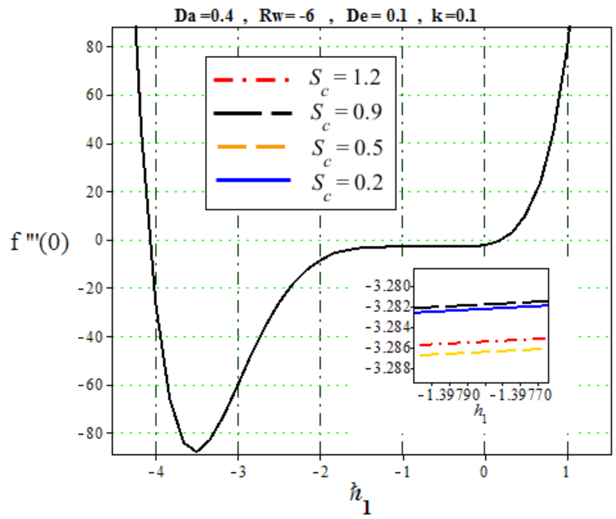

Fig. 2 The $\hbar_1$ - validity for $Da = 0.4, \operatorname{Re}_w = -6, De = 0.1$, $k = 0.1$ and different value of S_c

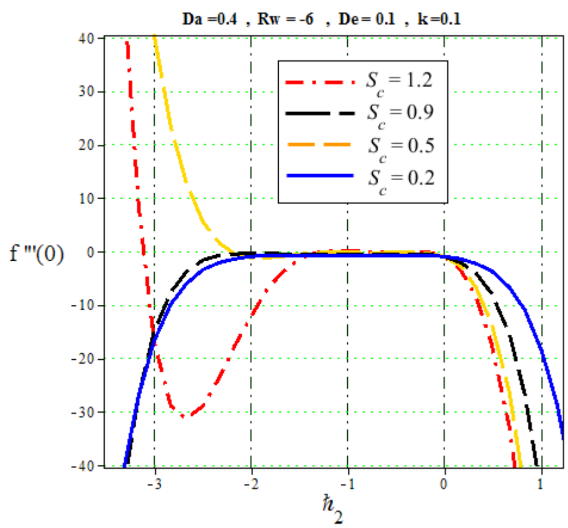

Fig. 3 The $\hbar_2$ - validity for $Da = 0.4, \operatorname{Re}_w = -6, De = 0.1$, $k = 0.1$ and different value of S_c .

As shown in Figs 2 and 3, For $De = 0.1, \operatorname{Re}_w = -6$, $Da = 0.4, k = 0.1$ and $0.2 < S_c < 1.2$ the ranges $-1.5 < \hbar_1 < -0.2$, and for $De = 0.1,\ \operatorname{Re}_w = -6$, $k = 0.1,\ Da = 0.4$ and $0.2 < S_c < 1.2$ the ranges $-1.4 < \hbar_2 < 0$, give suitable value of $\hbar$ for convergency. Then $\hbar_1 = -1$ and $\hbar_2 = -1$ is suitable value which is used for solution.

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5. Results and discussion

In this manuscript, the Homotopy Analysis Method such as analytical technique is employed to find an analytical solution of the Viscoelastic flow and species transfer in a Darcian high-permeability channel. Also, The above system of non-linear ordinary differential equations (12) and (13) along with the boundary conditions (11) is solved numerically using the algebra package Maple 16.0. The package uses a boundary value (B-V) problem procedure [31, 32].

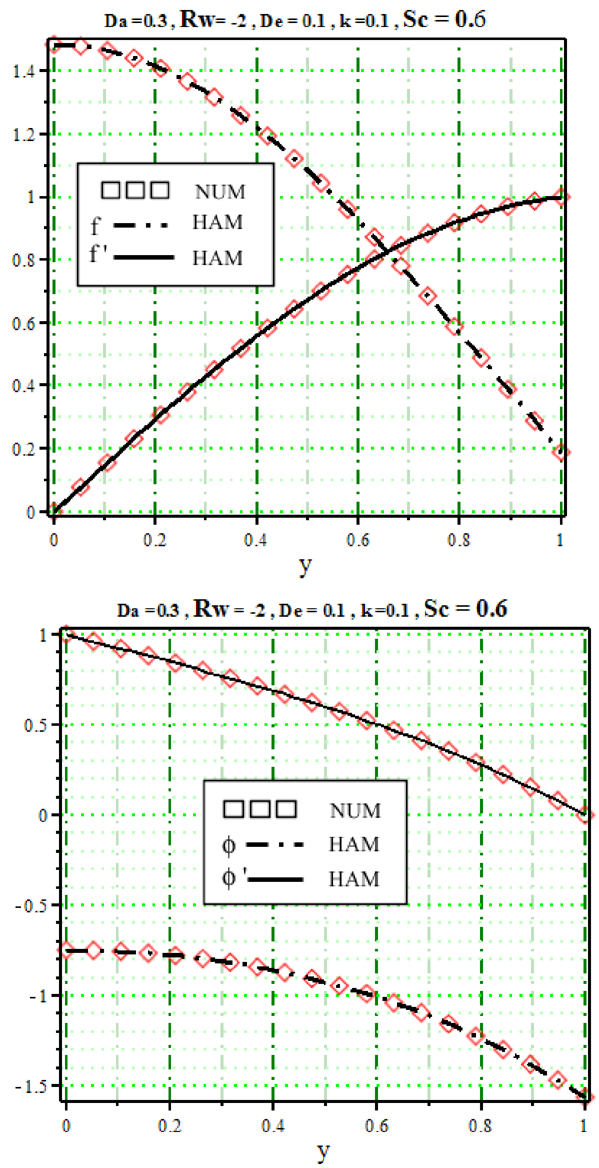

Fig. 4 The comparison between the numerical, HAM solution for $f(y), f'(y)$ and $\phi(y), \phi'(y)$ when $S_c = 0.6$ $\mathrm{Re}_w = -2,\ De = 0.4, k = 0.1$ and $Da = 0.3$.

The method uses either Richardson extrapolation or deferred corrections with a base method of either the trapezoid or midpoint method. The trapezoid method is generally efficient for typical problems, while the midpoint method is a powerful method for solving harmless end-point singularities that the trapezoid method cannot. The midpoint method, also known as the fourth-order Runge–Kutta–Fehlberg method, improves the Euler method by adding a midpoint in the step which increases the accuracy by one order.

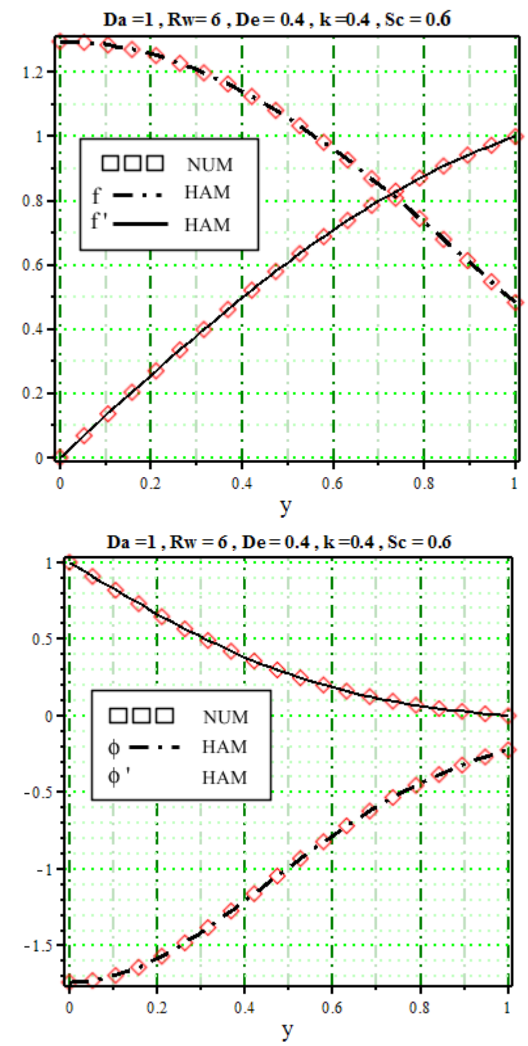

Fig. 5 The comparison between the numerical, HAM solution for $f(y), f'(y)$ and $\phi(y), \phi'(y)$ when $\mathrm{Re}_T = 6$, $De = 0.4, k = 0.4$ $S_c = 0.6$ and $Da = 1$.

Figs. 4 show comparison between the numerical solution and HAM solution for $f(y), f'(y)$ and $\phi(y),\ \phi'(y)$ when $\mathrm{Re}_w = 6,\ De = 0.4, k = 0.4$

Table 1: The results of HAM and NUM methods for $\phi(y)$ for $De = 0.4, k = 0.1,\ Da = 0.3\ \ ,S_c = 0.2$ and $\mathrm{Re}_w = -7$.

y	HAM	NUM	Error of HAM
0.00	1.000000000	1.000000002	0.000000000
0.10	0.928824213	0.928819727	0.000004486
0.20	0.856169409	0.856160635	0.000008774
0.30	0.780499931	0.780487334	0.000012596
0.40	0.700166608	0.700151025	0.000015583
0.45	0.657690594	0.657673970	0.000016624
0.50	0.613349913	0.613332622	0.000017290
0.55	0.566881391	0.566863859	0.000017531
0.6	0.518004470	0.517987164	0.000017306
0.65	0.466419783	0.466403193	0.000016590
0.70	0.411807926	0.411792549	0.000015376
0.75	0.353828503	0.353814821	0.000013681
0.80	0.292119519	0.292107977	0.000010764
0.85	0.226297231	0.226288212	0.000009018
0.90	0.155956540	0.155950349	0.000006191
0.95	0.080672065	0.080668913	0.000003151
1.00	0.000000000	0.000000000	0.000000000

,

Table 2: The results of HAM and NUM methods for $f(y)$ for $De = 0.4, k = 0.1,\ Da = 0.3\ \ ,S_c = 0.2$ and $\mathrm{Re}_w = -7$.

y	HAM	NUM	Error of HAM
0.00	0.000000000	0.000000000	0.0000000000
0.10	0.146913772	0.146913634	0.0000001374
0.20	0.290416635	0.290416367	0.0000002683
0.30	0.427261527	0.427261142	0.0000003846
0.40	0.554498307	0.554497831	0.0000004766
0.45	0.613694392	0.613693882	0.0000005092
0.50	0.669556407	0.669555876	0.0000005307
0.55	0.721827183	0.721826644	0.0000005391
0.60	0.770274690	0.770274159	0.0000005313
0.65	0.814691170	0.814690663	0.0000005072
0.70	0.854892173	0.854891708	0.0000004652
0.75	0.890715674	0.890715268	0.0000004065
0.80	0.922021354	0.922021022	0.0000001316
0.85	0.948690134	0.948689887	0.0000002462
0.90	0.970623956	0.970623800	0.0000001562
0.95	0.987745817	0.987745749	0.0000000684
1.00	1.000000000	1.000000000	0.0000000000

$, Da = 0.3$ and $S_c = 0.6$. Figs. 5 illustrate the accuracy of HAM solution compare to numerical solution when, $\mathrm{Re}_w = -2,\ De = 0.1, k = 0.1, Da = 0.3$ and $S_c = 0.6$, respectively. According to Tables 1, 2 and Figs 4, 5, clearly show that the results by HAM are in excellent agreement with the exact solutions. Also, the auxiliary parameter $\hbar$ provides us with a convenient way to adjust and control the convergence and its rate for the solutions series.

6. Conclusion

In this Letter, we studied the application of the Homotopy Analysis Method (HAM) to nonlinear equation arising in the upper convected Maxwell fluid and species transfer in a Darcian high- permeability channel under slip condition. Furthermore, the obtained solutions by proposed method have been compared with the direct numerical solutions using the Runge-Kutta-Fehlberg technique.

The comparison shows that the HAM solutions is highly accurate and provide the rapid achievement to compute the flow characteristics. According to the previous publications this method is a powerful technique for finding analytical solutions in science and engineering problems.

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