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On the Enestrom-Kakeya Theorem

M. H. Gulzar

Department of Mathematics, University of Kashmir, Srinagar 190006

Abstract: In this paper we relax the hypotheses of some results concerning the Enestrom –Kakeya theorem and obtain results which considerably improve the bounds in certain cases.

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1. Introduction

A well-known result in the distribution of zeros of a polynomial is the following theorem known as the Enestrom –Kakeya theorem [3,4]:

Theorem A: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that

$$a_n \geq a_{n-1} \geq ...... \geq a_1 \geq a_0 > 0 .$$

Then all the zeros of P(z) lie in $|z| \leq 1$.

If we apply this result to the polynomial P(tz), t>0, we get the following result:

Theorem B: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that

$$a_n t^n \geq a_{n-1} t^{n-1} \geq ...... \geq a_1 t \geq a_0 > 0 .$$

Then all the zeros of P(z) lie in $|z| \leq t$.

Using Schwarz Lemma, Aziz and Mohammad [1] generalized Enestrom –Kakeya theorem in a different way and proved the following:

Theorem C: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with real coefficients.

If $t_1 > t_2 \geq 0$ can be found such that

$$\alpha_j t_1 t_2 + \alpha_{j-1}(t_1 - t_2) - \alpha_{j-2} \geq 0,\ j = 1,2,.....,n,n+1(a_{-1} = a_{n+1} = 0),$$

then all the zeros of P(z) lie in $|z| \leq t_1$.

Singh and Shah [5] gave a more general result by proving the following:

Theorem D: Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that $a_j = \alpha_j + i\beta_j$, where $\alpha_j, \beta_j, j = 0,1,2,......,n$ are real numbers. If $t_1 > t_2 \geq 0$ can be found such that forj=2,3,……,n,

$$\alpha_j t_1 t_2 + \alpha_{j-1}(t_1 - t_2) - \alpha_{j-2} \geq 0,$$
$$\beta_j t_1 t_2 + \beta_{j-1}(t_1 - t_2) - \beta_{j-2} \geq 0,$$

and for j=n+1, there exist some $\lambda = \lambda_1 + i\lambda_2$ such that

$$(\lambda_1 + \alpha_n)(t_1 - t_2) - \alpha_{n-1} \geq 0$$
$$(\lambda_2 + \beta_n)(t_1 - t_2) - \alpha_{n-1} \geq 0,$$

then all the zeros of P(z) lie in $\left| z + \frac{\lambda(t_1 - t_2)}{a_n} \right| \leq R$, where

$$R = \frac{1}{|a_n|}[\{(\lambda_1 + \alpha_n) + (\lambda_2 + \beta_n)\}(t_1 - t_2) + (\alpha_n + \beta_n)t_2 - \frac{(\alpha_1 + \beta_1)t_2}{t_1^{\,n-1}} - \frac{(\alpha_0 + \beta_0)}{t_1^{\,n-1}}$$
$$+ \frac{|\alpha_1 t_1 t_2 + \alpha_0(t_1 - t_2)|}{t_1^{\,n}} + \frac{|\beta_1 t_1 t_2 + \beta_0(t_1 - t_2)|}{t_1^{\,n}} + \frac{(|\alpha_0| + |\beta_0|)t_2}{t_1^{\,n}}].$$

Gulzar [2] proved the following result:

Theorem E: Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that $a_j = \alpha_j + i\beta_j$, where $\alpha_j, \beta_j, j = 0,1,2,......,n$ are real numbers. If $t_1 > t_2 \geq 0$ can be found such that for some positive integer k

$$\alpha_j t_1 t_2 + \alpha_{j-1}(t_1 - t_2) - \alpha_{j-2} \geq 0,\ j = 2,....., k, k+1,$$
$$\alpha_j t_1 t_2 + \alpha_{j-1}(t_1 - t_2) - \alpha_{j-2} \leq 0,\ j = k+2, k+3,......, n,$$
$$\beta_j t_1 t_2 + \beta_{j-1}(t_1 - t_2) - \beta_{j-2} \geq 0,\ j = 1,2,....., k, k+1,$$
$$\beta_j t_1 t_2 + \beta_{j-1}(t_1 - t_2) - \beta_{j-2} \leq 0,\ j = k+2, k+3,......, n,$$

where $\alpha_{-1} = \alpha_{-2} = 0 = \beta_{-1} = \beta_{-2}$, then all the zeros of P(z) lie in

$$\left| z + \frac{a_{n-1}}{a_n} - (t_1 - t_2) \right| \leq \frac{2}{|a_n| t_1^{n-k}}[t_2(\alpha_k + \beta_k) + (\alpha_{k-1} + \beta_{k-1})] - \frac{1}{|a_n|}[t_2(\alpha_n + \beta_n) + (\alpha_{n-1} + \beta_{n-1})].$$

2. Main Results

In this paper we prove the following result:

Theorem 1: Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that $a_j = \alpha_j + i\beta_j$, where $\alpha_j, \beta_j, j = 0,1,2,......, n$ are real numbers. If $t_1 > t_2 \geq 0$ can be found such that for some positive integer k

$$\alpha_j t_1 t_2 + \alpha_{j-1}(t_1 - t_2) - \alpha_{j-2} \geq 0,\ j = 2,....., k, k+1,$$
$$\alpha_j t_1 t_2 + \alpha_{j-1}(t_1 - t_2) - \alpha_{j-2} \leq 0,\ j = k+2, k+3,......, n,$$
$$\beta_j t_1 t_2 + \beta_{j-1}(t_1 - t_2) - \beta_{j-2} \geq 0,\ j = 1,2,....., k, k+1,$$
$$\beta_j t_1 t_2 + \beta_{j-1}(t_1 - t_2) - \beta_{j-2} \leq 0,\ j = k+2, k+3,......, n,$$

and for j=n+1, there exist some $\lambda_1 < 1, \lambda_2 < 1$ such that for $\lambda = \lambda_1 + i\lambda_2$,

$$\lambda_1 \alpha_n (t_1 - t_2) - \alpha_{n-1} \leq 0$$
$$\lambda_2 \beta_n (t_1 - t_2) - \alpha_{n-1} \leq 0,$$

then all the zeros of P(z) lie in

$$|z - (1-\lambda)(t_1 - t_2)| \leq \frac{1}{|a_n|}[\frac{2(\alpha_k + \beta_k)}{t_1^{n-k-1}} + \frac{2(\alpha_{k+1} + \beta_{k+1})t_2}{t_1^{n-k-1}} - (\lambda_1 \alpha_n + \lambda_2 \beta_n)(t_1 - t_2) - \frac{(\alpha_0 + \beta_0)}{t_1^{n-1}}$$
$$+ \frac{|\alpha_1 t_1 t_2 + \alpha_0(t_1 - t_2)|}{t_1^n} + \frac{|\beta_1 t_1 t_2 + \beta_0(t_1 - t_2)|}{t_1^n} + \frac{(|\alpha_0| + |\beta_0|)t_2}{t_1^n}].$$

Taking $t_1 = t, t_2 = 0$ in Theorem 1, we get the following result:

Corollary 1: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that $a_j=\alpha_j+i\beta_j$, where $\alpha_j, \beta_j, j=0,1,2,......,n$ are real numbers. If $t_1>t_2\geq 0$ can be found such that for some positive integer k

$$\lambda_1 t^n \alpha_n \leq t^{n-1}\alpha_{n-1} \leq ......\leq t^{n-k+1}\alpha_{k+1} \leq t^{n-k}\alpha_k \geq t^{n-k-1}\alpha_{k-1} \geq ......\geq t\alpha_1 \geq \alpha_0,$$
$$\lambda_2 t^n \beta_n \leq t^{n-1}\beta_{n-1} \leq ......\leq t^{n-k+1}\beta_{k+1} \leq t^{n-k}\beta_k \geq t^{n-k-1}\beta_{k-1} \geq ......\geq t\beta_1 \geq \beta_0,$$

then all the zeros of P(z) lie in

$$|z-(1-\lambda)t| \leq \frac{1}{|a_n|}[\frac{2(\alpha_k+\beta_k)}{t^{n-k-1}}-(\lambda_1\alpha_n+\lambda_2\beta_n)t-\frac{(\alpha_0+\beta_0)}{t^{n-1}}+\frac{|\alpha_0|+|\beta_0|}{t^{n-1}}].$$

Taking t=1 in Cor.1, we get the following result:

Corollary 2: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that $a_j=\alpha_j+i\beta_j$, where $\alpha_j, \beta_j, j=0,1,2,......,n$ are real numbers. If for some positive integer k

$$\lambda_1 \alpha_n \leq \alpha_{n-1} \leq ......\leq \alpha_{k+1} \leq \alpha_k \geq \alpha_{k-1} \geq ......\geq \alpha_1 \geq \alpha_0,$$
$$\lambda_2 \beta_n \leq \beta_{n-1} \leq ......\leq \beta_{k+1} \leq \beta_k \geq \beta_{k-1} \geq ......\geq \beta_1 \geq \beta_0,$$

then all the zeros of P(z) lie in

$$|z-(1-\lambda))| \leq \frac{1}{|a_n|}[2(\alpha_k+\beta_k)+(\lambda_1\alpha_n+\lambda_2\beta_n)-(\alpha_0+\beta_0)+|\alpha_0|+|\beta_0|].$$

If a_j is real i.e. $\beta_j=0, \forall j$, we get the following result from Theorem 1:

Corollary 3: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with real coefficients. If $t_1>t_2\geq 0$ can be found such that for some positive integer k

$$a_j t_1 t_2 + a_{j-1}(t_1-t_2)-a_{j-2}\geq 0,\ j=2,.....,k,k+1,$$
$$a_j t_1 t_2 + a_{j-1}(t_1-t_2)-a_{j-2}\leq 0,\ j=k+2,k+3,......,n,$$

and for j=n+1, there exists some real number λ <1 such that

$$\lambda a_n(t_1-t_2)-a_{n-1}\leq 0,$$

then all the zeros of P(z) lie in

$$|z-(1-\lambda)(t_1-t_2)|\le \frac{1}{|a_n|}[\frac{2a_k}{t_1^{\,n-k-1}}+\frac{2a_{k+1}t_2}{t_1^{\,n-k-1}}-\lambda a_n(t_1-t_2)-\frac{a_0}{t_1^{\,n-1}}+\frac{|a_1t_1t_2+a_0(t_1-t_2)|}{t_1^{\,n}}+\frac{|a_0|t_2}{t_1^{\,n}}].$$

If the coefficients of P(z) in Cor. 3 are positive, we have the following result:

Corollary 4: Let $P(z)=\sum_{j=0}^{n}a_jz^j$ be a polynomial of degree n with real and positive coefficients. If $t_1>t_2\ge 0$ can be found such that for some positive integer k

$$a_jt_1t_2+a_{j-1}(t_1-t_2)-a_{j-2}\ge 0,\ j=2,.....,k,k+1,$$
$$a_jt_1t_2+a_{j-1}(t_1-t_2)-a_{j-2}\le 0,\ j=k+2,k+3,......,n,$$

and for j=n+1, there exists some real number $\lambda<1$ such that

$$\lambda a_n(t_1-t_2)-a_{n-1}\le 0,$$

then all the zeros of P(z) lie in

$$|z-(1-\lambda)(t_1-t_2)|\le \frac{1}{|a_n|}[\frac{2a_k}{t_1^{\,n-k-1}}+\frac{2a_{k+1}t_2}{t_1^{\,n-k-1}}-\lambda a_n(t_1-t_2)+\frac{a_1t_2}{t_1^{\,n}}].$$

Many other known results and generalizations similarly follow from Theorem 1 by suitable choices of the parameters .

2. Proof of Theorem 1

Consider the polynomial

$$F(z)=(t_2+z)(t_1-z)P(z)$$

$$=\{t_1t_2+(t_1-t_2)z-z^2\}\{a_nz^n+a_{n-1}z^{n-1}+......+a_1z+a_0\}$$
$$=-a_nz^{n+2}+\{a_n(t_1-t_2)-a_{n-1}\}z^{n+1}+\{a_nt_1t_2+a_{n-1}(t_1-t_2)-a_{n-2}\}z^n+......$$
$$+\{a_{k+2}t_1t_2+a_{k+1}(t_1-t_2)-a_k\}z^{k+2}+\{a_{k+1}t_1t_2+a_k(t_1-t_2)-a_{k-1}\}z^{k+1}$$
$$+\{a_kt_1t_2+a_{k-1}(t_1-t_2)-a_{k-2}\}z^k+......+\{a_2t_1t_2+a_1(t_1-t_2)-a_0\}z^2$$
$$+\{a_1t_1t_2+a_0(t_1-t_2)\}z+a_0t_1t_2$$

$$=-a_nz^{n+2}+(1-\lambda)a_n(t_1-t_2)z^{n+1}+\{\lambda a_n(t_1-t_2)-a_{n-1}\}z^{n+1}$$

$$+\{a_n t_1 t_2 + a_{n-1}(t_1 - t_2) - a_{n-2}\}z^n + ...\{a_{k+2}t_1t_2 + a_{k+1}(t_1 - t_2) - a_k\}z^{k+2}$$

$$+\{a_{k+1}t_1t_2 + a_k(t_1 - t_2) - a_{k-1}\}z^{k+1} + \{a_k t_1 t_2 + a_{k-1}(t_1 - t_2) - a_{k-2}\}z^k$$
$$+ ...... + \{a_2 t_1 t_2 + a_1(t_1 - t_2) - a_0\}z^2 + \{a_1 t_1 t_2 + a_0(t_1 - t_2)\}z + a_0 t_1 t_2$$

$$= -a_n z^{n+2} + (1-\lambda)a_n(t_1 - t_2)z^{n+1} + \{\lambda_1 \alpha_n(t_1 - t_2) - \alpha_{n-1}\}z^{n+1}$$

$$+\{\alpha_n t_1 t_2 + \alpha_{n-1}(t_1 - t_2) - \alpha_{n-2}\}z^n + ...\{\alpha_{k+2}t_1t_2 + \alpha_{k+1}(t_1 - t_2) - \alpha_k\}z^{k+2}$$

$$+\{\alpha_{k+1}t_1t_2 + \alpha_k(t_1 - t_2) - \alpha_{k-1}\}z^{k+1} + \{\alpha_k t_1 t_2 + \alpha_{k-1}(t_1 - t_2) - \alpha_{k-2}\}z^k$$
$$+ ...... + \{\alpha_2 t_1 t_2 + \alpha_1(t_1 - t_2) - \alpha_0\}z^2 + \{\alpha_1 t_1 t_2 + \alpha_0(t_1 - t_2)\}z + \alpha_0 t_1 t_2$$
$$+ i\{(\lambda_2 \beta_n(t_1 - t_2) - \beta_{n-1})z^{n+1} + (\beta_n t_1 t_2 + \beta_{n-1}(t_1 - t_2) - \beta_{n-2})z^n$$

$$+ ...... + (\beta_{k+2}t_1t_2 + \beta_{k+1}(t_1 - t_2) - \beta_k)z^{k+2} + (\beta_{k+1}t_1t_2 + \beta_k(t_1 - t_2) - \beta_{k-1})z^{k+1}$$

$$+(\beta_k t_1 t_2 + \beta_{k-1}(t_1 - t_2) - \beta_{k-2})z^k + ... + (\beta_2 t_1 t_2 + \beta_1(t_1 - t_2) - \beta_0)z^2$$
$$+(\beta_1 t_1 t_2 + \beta_0(t_1 - t_2))z + \beta_0 t_1 t_2\}].$$

Therefore, for $|z| > t_1$, we have by using the hypothesis

$$|F(z)| \geq |z|^{n+1}[|a_n||z - (1-\lambda)(t_1 - t_2)| - \{|\lambda_1 \alpha_n(t_1 - t_2) - \alpha_{n-1}| + |\alpha_n t_1 t_2 + \alpha_{n-1}(t_1 - t_2) - \alpha_{n-2}|.\frac{1}{|z|}$$

$$+ ...... + |\alpha_{k+2}t_1t_2 + \alpha_{k+1}(t_1 - t_2) - \alpha_k|.\frac{1}{|z|^{n-k-1}} + |\alpha_{k+1}t_1t_2 + \alpha_k(t_1 - t_2) - \alpha_{k-1}|.\frac{1}{|z|^{n-k}}$$

$$+ |\alpha_k t_1 t_2 + \alpha_{k-1}(t_1 - t_2) - \alpha_{k-2}|.\frac{1}{|z|^{n-k+1}} + ... + |\alpha_2 t_1 t_2 + \alpha_1(t_1 - t_2) - \alpha_0|.\frac{1}{|z|^{n-1}}$$

$$+ |\alpha_1 t_1 t_2 + \alpha_0(t_1 - t_2)|.\frac{1}{|z|^n} + |\alpha_0 t_1 t_2|.\frac{1}{|z|^{n+1}}$$

$$+ |\lambda_2 \beta_n(t_1 - t_2) - \beta_{n-1}| + |\beta_n t_1 t_2 + \beta_{n-1}(t_1 - t_2) - \beta_{n-2}|.\frac{1}{|z|}$$

$$+ ...... + |\beta_{k+2}t_1t_2 + \beta_{k+1}(t_1 - t_2) - v_k|.\frac{1}{|z|^{n-k-1}} + |\beta_{k+1}t_1t_2 + \beta_k(t_1 - t_2) - \beta_{k-1}|.\frac{1}{|z|^{n-k}}$$

$$+ |\beta_k t_1 t_2 + \beta_{k-1}(t_1 - t_2) - \beta_{k-2}|.\frac{1}{|z|^{n-k+1}} + ... + |\beta_2 t_1 t_2 + \beta_1(t_1 - t_2) - \beta_0|.\frac{1}{|z|^{n-1}}$$

$$+ |\beta_1 t_1 t_2 + \beta_0(t_1 - t_2)|.\frac{1}{|z|^n} + |\beta_0 t_1 t_2|.\frac{1}{|z|^{n+1}}\}]$$

$$> |z|^{n+1}[|a_n||z-(1-\lambda)(t_1-t_2)| - \{|\lambda_1\alpha_n(t_1-t_2)-\alpha_{n-1}| + |\alpha_n t_1 t_2 + \alpha_{n-1}(t_1-t_2)-\alpha_{n-2}|.\frac{1}{t_1}$$

$$+ ......+ |\alpha_{k+2}t_1t_2 + \alpha_{k+1}(t_1-t_2)-\alpha_k|.\frac{1}{t_1^{n-k-1}} + |\alpha_{k+1}t_1t_2 + \alpha_k(t_1-t_2)-\alpha_{k-1}|.\frac{1}{t_1^{n-k}}$$

$$+ |\alpha_k t_1t_2 + \alpha_{k-1}(t_1-t_2)-\alpha_{k-2}|.\frac{1}{t_1^{n-k+1}} + ...+ |\alpha_2 t_1t_2 + \alpha_1(t_1-t_2)-\alpha_0|.\frac{1}{t_1^{n-1}}$$

$$+ |\alpha_1 t_1t_2 + \alpha_0(t_1-t_2)|.\frac{1}{t_1^{n}} + |\alpha_0 t_1t_2|.\frac{1}{t_1^{n+1}}$$

$$+ |\lambda_2\beta_n(t_1-t_2)-\beta_{n-1}| + |\beta_n t_1t_2 + \beta_{n-1}(t_1-t_2)-\beta_{n-2}|.\frac{1}{t_1}$$

$$+ ......+ |\beta_{k+2}t_1t_2 + \beta_{k+1}(t_1-t_2)-\beta_k|.\frac{1}{t_1^{n-k-1}} + |\beta_{k+1}t_1t_2 + \beta_k(t_1-t_2)-\beta_{k-1}|.\frac{1}{t_1^{n-k}}$$

$$+ |\beta_k t_1t_2 + \beta_{k-1}(t_1-t_2)-\beta_{k-2}|.\frac{1}{t_1^{n-k+1}} + ...+ |\beta_2 t_1t_2 + \beta_1(t_1-t_2)-\beta_0|.\frac{1}{t_1^{n-1}}$$

$$+ |\beta_1 t_1t_2 + \beta_0(t_1-t_2)|.\frac{1}{t_1^{n}} + |\beta_0 t_1t_2|.\frac{1}{t_1^{n+1}}\}]$$

$$> |z|^{n+1}[|a_n||z-(1-\lambda)(t_1-t_2)| - \{\alpha_{n-1} - \lambda_1\alpha_n(t_1-t_2) + (\alpha_{n-2} - \alpha_n t_1t_2 - \alpha_{n-1}(t_1-t_2)).\frac{1}{t_1}$$

$$+ ......+ (\alpha_k - \alpha_{k+2}t_1t_2 - \alpha_{k+1}(t_1-t_2)).\frac{1}{t_1^{n-k-1}} + (\alpha_{k+1}t_1t_2 + \alpha_k(t_1-t_2)-\alpha_{k-1}).\frac{1}{t_1^{n-k}}$$

$$+ (\alpha_k t_1t_2 + \alpha_{k-1}(t_1-t_2)-\alpha_{k-2}).\frac{1}{t_1^{n-k+1}} + ...+ (\alpha_2 t_1t_2 + \alpha_1(t_1-t_2)-\alpha_0).\frac{1}{t_1^{n-1}}$$

$$+ |\alpha_1 t_1t_2 + \alpha_0(t_1-t_2)|.\frac{1}{t_1^{n}} + |\alpha_0|.\frac{t_2}{t_1^{n}}$$

$$+ \beta_{n-1} - \lambda_2\beta_n(t_1-t_2) + |\beta_{n-2} - \beta_n t_1t_2 - \beta_{n-1}(t_1-t_2)|.\frac{1}{t_1}$$

$$+ ......+ (\beta_k - \beta_{k+2}t_1t_2 + \beta_{k+1}(t_1-t_2)).\frac{1}{t_1^{n-k-1}} + (\beta_{k+1}t_1t_2 + \beta_k(t_1-t_2)-\beta_{k-1}).\frac{1}{t_1^{n-k}}$$

$$+ (\beta_k t_1t_2 + \beta_{k-1}(t_1-t_2)-\beta_{k-2}).\frac{1}{t_1^{n-k+1}} + ...+ (\beta_2 t_1t_2 + \beta_1(t_1-t_2)-\beta_0).\frac{1}{t_1^{n-1}}$$

$$+ |\beta_1 t_1t_2 + \beta_0(t_1-t_2)|.\frac{1}{t_1^{n}} + |\beta_0|.\frac{t_2}{t_1^{n}}\}]$$

$$=|z|^{n+1}[|a_n||z-(1-\lambda)(t_1-t_2)|-\{\frac{2(\alpha_k+\beta_k)}{t_1^{\,n-k-1}}+\frac{2(\alpha_{k+}+\beta_{k+1})}{t_1^{\,n-k-1}}-(\lambda_1\alpha_n+\lambda_2\beta_n)(t_1-t_2)$$

$$-(\alpha_0+\beta_0).\frac{1}{t_1^{\,n-1}}+|\alpha_1 t_1 t_2+\alpha_0(t_1-t_2)|.\frac{1}{t_1^{\,n}}+|\alpha_0|.\frac{t_2}{t_1^{\,n}}$$

$$+|\beta_1 t_1 t_2+\beta_0(t_1-t_2)|.\frac{1}{t_1^{\,n}}+|\beta_0|.\frac{t_2}{t_1^{\,n}}\}]$$

$$>0$$

if

$$|z-(1-\lambda)(t_1-t_2)|>\frac{1}{|a_n|}[\frac{2(\alpha_k+\beta_k)}{t_1^{\,n-k-1}}+\frac{2(\alpha_{k+1}+\beta_{k+1})t_2}{t_1^{\,n-k-1}}-(\lambda_1\alpha_n+\lambda_2\beta_n)(t_1-t_2)-\frac{(\alpha_0+\beta_0)}{t_1^{\,n-1}}$$

$$+\frac{|\alpha_1 t_1 t_2+\alpha_0(t_1-t_2)|}{t_1^{\,n}}+\frac{|\beta_1 t_1 t_2+\beta_0(t_1-t_2)|}{t_1^{\,n}}+\frac{(|\alpha_0|+|\beta_0|)t_2}{t_1^{\,n}}].$$

This shows that those zeros of F(z) whose modulus is greater than t_1 lie in

$$|z-(1-\lambda)(t_1-t_2)|\le\frac{1}{|a_n|}[\frac{2(\alpha_k+\beta_k)}{t_1^{\,n-k-1}}+\frac{2(\alpha_{k+1}+\beta_{k+1})t_2}{t_1^{\,n-k-1}}-(\lambda_1\alpha_n+\lambda_2\beta_n)(t_1-t_2)-\frac{(\alpha_0+\beta_0)}{t_1^{\,n-1}}$$

$$+\frac{|\alpha_1 t_1 t_2+\alpha_0(t_1-t_2)|}{t_1^{\,n}}+\frac{|\beta_1 t_1 t_2+\beta_0(t_1-t_2)|}{t_1^{\,n}}+\frac{(|\alpha_0|+|\beta_0|)t_2}{t_1^{\,n}}].$$

But the zeros of F(z) whose modulus is less than or equal to t_1 already satisfy the above inequality. Hence it follows that all the zeros of F(z) lie in

$$|z-(1-\lambda)(t_1-t_2)|\le\frac{1}{|a_n|}[\frac{2(\alpha_k+\beta_k)}{t_1^{\,n-k-1}}+\frac{2(\alpha_{k+1}+\beta_{k+1})t_2}{t_1^{\,n-k-1}}-(\lambda_1\alpha_n+\lambda_2\beta_n)(t_1-t_2)-\frac{(\alpha_0+\beta_0)}{t_1^{\,n-1}}$$

$$+\frac{|\alpha_1 t_1 t_2+\alpha_0(t_1-t_2)|}{t_1^{\,n}}+\frac{|\beta_1 t_1 t_2+\beta_0(t_1-t_2)|}{t_1^{\,n}}+\frac{(|\alpha_0|+|\beta_0|)t_2}{t_1^{\,n}}].$$

Since the zeros of P(z) are also the zeros of F(z), the result follows.

References

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