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Existence, Uniqueness and Stability Solution of Non-linear System of Integro-Differential Equation of Volterra type

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Abstract

The aim of this work is to study the existence and uniqueness of solution of non-linear system of integro-differential equation of Volterra type by using Picard approximation method and Banach fixed point theorem .

The study of such integro-differential equations leads us to extend the results obtained by Butris for changing the system of non-linear integro-differential equations to another system of non-linear integro-differential equations of the Volterra type.

Keywords. Picard approximation method ,Banach fixed point theorem , Nonlinear system, Integro-Differential Equations, Existence,Uniqueness and stability Solution .

1. Introduction

The theory of integral equations has been an active research field for many years and is based on analysis, functional theory, and functional analysis.

An integral equation is a functional equation in which the unknown function appears under one or several integral signs; if, in addition, the equation contains a derivative of this function, we call the equations an integro-differential equation. In an integral or integro-differential equations of Volterra type, the integrals containing the unknown function are characterized by a variable upper limit of integration.

The functional equation (for the unknown function y) is of the form:

$$y'(t)=f(t,y(t),z(t)) \quad t\in I \quad ,$$

with

$$z(t)=\int_0^t K(t,s,y(t))ds \quad ,$$

is called a first order Volterra integro-differential equation .

Also, it should be noted that appropriate versions of the method considered can be applied in many situations for handling periodic in the case of systems of first or second order ordinary differential equations, integro-differential equations, equations with retarded arguments, systems containing unknown parameters, and countable systems of differential equations. A survey of the investigations on the subject can be found in the studies and researches [2,3,4,5,6,7,8,11,12,13] .

Butris [1] used both methods (Picard approximation method) and (Banach fixed point theorem) which were introduced by Rama [9]and[10] for studying some theorems in existence and uniqueness of system of non-linear integro-differential equations having the following form :

$$\frac{dx(t)}{dt}=(A+B(t))x(t)+f\left(t,x(t),\int_t^{t+T} g(s,x(s))ds\right) \quad ,$$

where $x\in D\subseteq R^n$, D is a closed and bounded domain.

In this work, we prove the existence , uniqueness and stability Solution .

for another system of non-linear integro-differential equations of Volterra type

Consider the following system of non-linear integro-differential equations which has the form :

$$\frac{dx(t)}{dt}=(A+B(t))x(t)+f\left(t,x(t),\sum_{i=1}^{\infty}\left(\int_{-\infty}^{t} G(t,s)\dot{x}(s)ds\right)^i\right) \qquad (1.1)$$

where $\dot{x}(s_i) = \dfrac{dx(s_i)}{ds_i}$, i= 1, 2, 3,…

and $x \in D \subseteq R^n$, D is a closed and bounded domain.

Let the vector function $f(t,x,y) = (f_1(t,x,y), f_2(t,x,y), ..., f_n(t,x,y))$,

is defined and continuous on the domain ,

$$(t,x,y) \in R^1 \times D \times D_1 = (-\infty, \infty) \times D \times D_1 \ , \qquad (1.2)$$

where D_1 is a bounded domain subset of Euclidean space R^m.

Suppose that the function f (*t*, x, y) satisfies the following inequalities:

$$\|f(t,x,y)\| \le M \qquad (1.3)$$

$$\|f(t,x_1,y_1) - f(t,x_2,y_2)\| \le L_1 \|x_1 - x_2\| + L_2 \|y_1 - y_2\| \qquad (1.4)$$

for all $t \in R^1$, $x, x_1, x_2 \in D$, $y, y_1, y_2 \in D_1$,where M and L_1, L_2 are positive constants.

Suppose that $A=[A_{ij}]$ and $B(t)=[B_{ij}(t)]$ are $n \times n$ positive matrices which are continuous in t , and satisfy the following inequalities:

$$\|e^{A(t-s)}\| \le Q < \infty \qquad (1.5)$$

$$\|B(t)\| \le H \quad , \ \|A\| = N \qquad (1.6)$$

$$\|x_0\| = \delta_0 \qquad (1.7)$$

where $-\infty < 0 \le s \le t \le T < \infty$ and Q , H , δ_0 , N are positive constants.

Also $G(t,s)$ is $(n \times n)$ continuous positive matrix such that:

$$\left.\begin{aligned} &\int_{-\infty}^{t} \|G(t,s)\| ds \le K \ , \ K > 0 \\ &S_1 = \sum_{i=1}^{\infty} K^i M_2^{i-1} \\ &S_2 = \sum_{i=1}^{\infty} i K^i M_2^{i-1} \ . \end{aligned}\right\} \qquad (1.8)$$

where S_1 and S_2 are convergent series

We define the non-empty sets as follows:

$$\left.\begin{aligned} D_f &= D - M_1 \\ D_{1f} &= D_1 - M_2 \end{aligned}\right\} \qquad (1.9)$$

where $M_1 = TQ(H\delta_0 + M)$, $M_2 = QN\delta_0 + Q(H\delta_0 + M)$, $\|.\| = \max_{t\in[0,T]}|.|$

and $W = [TQ(H + L_1) + QL_2S_2] < 1$ (1.10)

2. **Existence and Uniqueness solution of the equation** (1.1).

In this section, we study the existence and uniqueness solution for the equation (1.1).

Theorem 1. If the system (1 .1) satisfies the inequalities (1.3), (1 .4) and conditions (1.5), (1.6) ,(1.7) ,(1.8) has a solution $x = x(t,x_0)$, then the sequence of functions:

$$x_{m+1}(t,x_0) = x_0e^{At} + \int_0^t e^{A(t-s)}[B(s)x_m(s,x_0) + f(s,x_m(s,x_0),\sum_{i=1}^{\infty}(\int_{-\infty}^{s} G(s,\tau)\dot{x}_m(\tau,x_0)d\tau)^i)]ds$$

(2.1)

with

$x_0(t,x_0) = x_0$, m= 0 , 1, 2 ,…

converges, when $m \to \infty$, uniformly with respect to:

$$(t,x_0) \in R^1 \times D_f \qquad (2.2)$$

to the function $x_\infty(t,x_0)$ defined in domain (2.2),and satisfying the following integral equations:

$$x(t,x_0) = x_0e^{At} + \int_0^t e^{A(t-s)}[B(s)x(s,x_0) + f(s,x(s,x_0),\sum_{i=1}^{\infty}(\int_{-\infty}^{s} G(s,\tau)\dot{x}(\tau,x_0)d\tau)^i)]ds$$

(2.3)

provided that :

$$\|x_\infty(t,x_0)-x_0\| \le M_1 \tag{2.4}$$

and

$$\|x_\infty(t,x_0)-x_m(t,x_0)\| \le TQW_1(1-W)^{-1}W^{m-1} \tag{2.5}$$

for all $m \ge 1$ and $t \in R^1$.

Proof. Consider the sequence of functions $x_1(t,x_0), x_2(t,x_0), \ldots, x_m(t,x_0), \ldots$ defined by the recurrence relation (2.1). Each of these functions of sequence is defined and continuous in the domain (1.2).

Now, by using (2.1), when m= 0, we get:

$$\|x_1(t,x_0)-x_0\| = \left\| x_0e^{At} + \int_0^t e^{A(t-s)}[B(s)x_0 + f(s,x_0,0)]ds - x_0e^{At} \right\|$$

$$\le \int_0^t \|e^{A(t-s)}\| [\|B(s)\|\|x_0\| + \|f(s,x_0,0)\|]ds$$

$$\le TQ[H\delta_0 + M] = M_1 \ ,$$

and hence,

$$\|x_1(t,x_0)-x_0\| \le M_1 \tag{2.6}$$

i.e. $x_1(t,x_0) \in D$, for all $t \in R^1$, $x_0 \in D_f$.

Suppose that $x_p(t,x_0) \in D$, for all $x_0 \in D_f$, $p \in Z^+$,

when m= p+1 ,we find that:

$$\|x_{p+1}(t,x_0)-x_0\| \le M_1$$

i.e. $x_{p+1}(t,x_0) \in D$, for all $t \in R^1$, $x_0 \in D_f$, $p \in Z^+$.

Therefore, by mathematical induction, we obtain:

$$\|x_m(t,x_0)-x_0\| \le M_1 \tag{2.7}$$

for all m= 0,1,2,…

In addition to that , we have:

$$\|\dot{x}_1(t,x_0)\| = \|x_0 A e^{At} + e^{A(t-s)}[B(t)x_0 + f(t,x_0,0)]\|$$

$$\leq \|x_0\|\|A\|\|e^{At}\| + \|e^{A(t-s)}\|[\|B(t)\|\|x_0\| + \|f(t,x_0,0)\|]$$

$$\leq \delta_0 NQ + Q[H\delta_0 + M] = M_2$$

and hence ,

$$\|\dot{x}_1(t,x_0)\| \leq M_2 \tag{2.8}$$

i.e. $\dot{x}_1(t,x_0) \in D_1$, for all $x_0 \in D_f$.

Also by mathematical induction, we find that:

$$\|\dot{x}_m(t,x_0)\| \leq M_2 \tag{2.9}$$

where

$$\dot{x}_{m+1}(t,x_0) = x_0 A e^{At} + e^{A(t-s)}[B(s)x_m(s,x_0) + f(s,x_m(s,x_0),\sum_{i=1}^{\infty}(\int_{-\infty}^{s} G(s,\tau)\dot{x}_m(\tau,x_0)d\tau)^i)]$$

for all m= 0,1,2,… (2.10)

We claim that the sequence of functions (2.1) is uniformly convergent on the domain (2.2).

Now, when m=1 in (2.1) , we get:

$$\|x_2(t,x_0) - x_1(t,x_0)\| = \|x_0 e^{At} + \int_0^t e^{A(t-s)}[B(s)x_1(s,x_0)$$

$$+ f(s,x_1(s,x_0),\sum_{i=1}^{\infty}(\int_{-\infty}^{s} G(s,\tau)\dot{x}_1(\tau,x_0)d\tau)^i)]ds$$

$$-x_0 e^{At} - \int_0^t e^{A(t-s)}[B(s)x_0 + f(s,x_0,0)]ds\|$$

$$\leq \int_0^t \|e^{A(t-s)}\|[\|B(s)\|\|x_1(s,x_0) - x_0\|$$

$$+\| f(s,x_1(s,x_0),\sum_{i=1}^{\infty}(\int_{-\infty}^{s} G(s,\tau)\dot{x}_1(\tau,x_0)d\tau)^i) - f(s,x_0,0) \|$$

By using (2. 6), we have

$$\|x_2(t,x_0)-x_1(t,x_0)\| \le \int_0^t \|e^{A(t-s)}\|[\|B(s)\|\|x_1(s,x_0)-x_0\|+L_1\|x_1(s,x_0)-x_0\|$$

$$+L_2\left\|\sum_{i=1}^{\infty}(\int_{-\infty}^{s} G(s,\tau)\dot{x}_1(\tau,x_0)d\tau)^i\right\|]ds$$

where

$$\left\|\sum_{i=1}^{\infty}(\int_{-\infty}^{s} G(s,\tau)\dot{x}_1(\tau,x_0)d\tau)^i\right\| \le \sum_{i=1}^{\infty}(\int_{-\infty}^{s}\|G(s,\tau)\|\|\dot{x}_1(\tau,x_0)\|d\tau)^i$$

$$\le (\sum_{i=1}^{\infty} K^i M_2^{i-1})\|\dot{x}_1(\tau,x_0)\|$$

$$\le S_1\|\dot{x}_1(\tau,x_0)\| = S_1 M_2 \ . \qquad (2.11)$$

Then, by using equation (2.6) and (2.11), we get:

$$\|x_2(t,x_0)-x_1(t,x_0)\| \le \int_0^t Q[HM_1+L_1M_1+L_2S_1M_2]ds$$

$$\le TQ[(H+L_1)M_1+L_2S_1M_2]$$

Let

$$W_1=[(H+L_1)M_1+L_2S_1M_2]$$

So we have:

$$\|x_2(t,x_0)-x_1(t,x_0)\| \le TQW_1 \qquad (2.12)$$

Also when m=1 in (2.10) ,we get:

$$\|\dot{x}_2(t,x_0)-\dot{x}_1(t,x_0)\| = \|x_0Ae^{At}+e^{A(t-s)}[B(t)x_1(t,x_0)$$

$$+f(t,x_1(t,x_0),\sum_{i=1}^{\infty}(\int_{-\infty}^{t} G(t,s)\dot{x}_1(s,x_0)ds)^i)]-x_0Ae^{At}$$

$$-e^{A(t-s)}[B(t)x_0+f(t,x_0,0)]\|$$

By using (2.6), we find

$$\|\dot{x}_2(t,x_0)-\dot{x}_1(t,x_0)\| \le \|e^{A(t-s)}\|[\|B(t)\|\|x_1(t,x_0)-x_0\|+L_1\|x_1(t,x_0)-x_0\|$$

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$$+L_2 \left\| \sum_{i=1}^{\infty} \left(\int_{-\infty}^{s} G(s,\tau)\dot{x}_1(\tau,x_0)d\tau \right)^i \right\|]$$

From the equations (2.6) and (2.11), we obtain that

$$\|\dot{x}_2(t,x_0)-\dot{x}_1(t,x_0)\| \le Q[HM_1+L_1M_1+L_2S_1M_2]$$
$$\le QW_1$$

and hence,

$$\|\dot{x}_2(t,x_0)-\dot{x}_1(t,x_0)\| \le QW_1 \tag{2.13}$$

Suppose that the inequalities

$$\|x_{p+1}(t,x_0)-x_p(t,x_0)\| \le TQW_1W^{p-1}, \tag{2.14}$$

and

$$\|\dot{x}_{p+1}(t,x_0)-\dot{x}_p(t,x_0)\| \le QW_1W^{p-1}, \tag{2.15}$$

are holds for m= p , then we can prove the following inequality :

$$\|x_{p+2}(t,x_0)-x_{p+1}(t,x_0)\| \le TQW_1W^{p} \tag{2.16}$$

where m=p +1 in (2.2.1) ,we find that:

$$\|x_{p+2}(t,x_0)-x_{p+1}(t,x_0)\| \le \int_0^t \|e^{A(t-s)}\| [\|B(s)\| \|x_{p+1}(s,x_0)-x_p(s,x_0)\|$$
$$+\|f(s,x_{p+1}(s,x_0),\sum_{i=1}^{\infty}(\int_{-\infty}^{s} G(s,\tau)\dot{x}_{p+1}(\tau,x_0)d\tau)^i)$$
$$-f(s,x_p(s,x_0),\sum_{i=1}^{\infty}(\int_{-\infty}^{s} G(s,\tau)\dot{x}_p(\tau,x_0)d\tau)^i)\|]ds$$

By (2.14) and (2.15) we get

$$\|x_{p+2}(t,x_0)-x_{p+1}(t,x_0)\| \le \int_0^t Q[H\|x_{p+1}(s,x_0)-x_p(s,x_0)\| + L_1\|x_{p+1}(s,x_0)-x_p(s,x_0)\|$$
$$+L_2(\sum_{i=1}^{\infty} iK^iM_2^{i-1})\|\dot{x}_{p+1}(s,x_0)-\dot{x}_p(s,x_0)\|]ds$$

And hence

$$\|x_{p+2}(t,x_0)-x_{p+1}(t,x_0)\|\le TQ[HTQW_1W^{P-1}+L_1TQW_1W^{P-1}+L_2S_2QW_1W^{P-1}]$$

$$\le TQW_1W^{P-1}[TQ(H+L_1)+L_2S_2Q]$$

$$\le TQW_1W^{P}$$

Thus, by mathematical induction, the following inequality holds

$$\|x_{m+1}(t,x_0)-x_m(t,x_0)\|\le TQW_1W^{m-1} \quad (2.17)$$

for m=1, 2, 3,… where $W=[TQ(H+L_1)+L_2S_2Q]$.

Also from the inequality (2.17), we conclude that for $k>1$, the following inequality holds

$$\|x_{m+k}(t,x_0)-x_m(t,x_0)\|=\sum_{j=0}^{\infty}\|x_{m+1+j}(t,x_0)-x_{m+j}(t,x_0)\|$$

for all m.

And hence

$$\|x_{m+k}(t,x_0)-x_m(t,x_0)\|\le\sum_{j=0}^{\infty}\|x_{m+1+j}(t,x_0)-x_{m+j}(t,x_0)\|$$

$$\le\sum_{j=0}^{\infty}TQW_1W^{m-1+j}$$

$$\le TQW_1W^{m-1}\sum_{j=0}^{\infty}W^{j}$$

$$\le TQW_1(1-W)^{-1}W^{m-1}$$

So

$$\|x_{m+k}(t,x_0)-x_m(t,x_0)\|\le TQW_1(1-W)^{-1}W^{m-1} \quad (2.18)$$

Now by using (2.18)and the condition(1.10), we have:

$$\lim_{m\to\infty}W^{m-1}=0 \quad (2.19)$$

where m = 1,2,3,…

Relations (2.18) and (2.19) prove the uniform convergence of the sequence of functions (2.1) in the domain (2.2) as $m \to \infty$.

Let

$$\lim_{m\to\infty} x_m(t,x_0) = x_\infty(t,x_0) \tag{2.20}$$

Since the sequence of functions (2.1) is defined and continuous in the domain (2.2) , then the limiting function $x_\infty(t,x_0)$ is also defined and continuous in the domain (2.2).

Moreover, by the hypotheses and conditions of the theorem, the inequality (2.4) and (2.5) are satisfied for all $m \geq 1$.

Using relation (2.20) and proceeding in (2.1) to the limit , when $m \to \infty$, this shows that the limiting function $x_\infty(t,x_0)$ is the solution of the equation (1.1).

Theorem 2. If all assumptions and conditions of theorem 1 are satisfied, then the function $x(t,x_0)$ is a unique solution of the equation (1.1).

Proof. Suppose that:

$$x(t,x_0) = x_0 e^{At} + \int_0^t e^{A(t-s)}[B(s)x(s,x_0) + f(s,x(s,x_0)), \sum_{i=1}^{\infty}(\int_{-\infty}^{s} G(s,\tau)\dot{x}(\tau,x_0)d\tau)^i)]ds$$

and

$$z(t,x_0) = x_0 e^{At} + \int_0^t e^{A(t-s)}[B(s)z(s,x_0) + f(s,z(s,x_0)), \sum_{i=1}^{\infty}(\int_{-\infty}^{s} G(s,\tau)\dot{z}(\tau,x_0)d\tau)^i)]ds$$

are two solutions of the equation (1.1) .

Now, for their difference, we obtain :

$$\|x(t,x_0) - z(t,x_0)\| = \|x_0 e^{At} + \int_0^t e^{A(t-s)}[B(s)x(s,x_0)$$

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$$+f\ (s,x(s,x_0),\sum_{i=1}^{\infty}(\int_{-\infty}^{s}G(s,\tau)\dot{x}(\tau,x_0)d\tau)^i)]ds - x_0e^{At}$$

$$-\int_0^t e^{A(t-s)}[B(s)z(s,x_0)+f\ (s,z(s,x_0),\sum_{i=1}^{\infty}(\int_{-\infty}^{s}G(s,\tau)\dot{z}(\tau,x_0)d\tau)^i)]ds\|$$

$$\|x(t,x_0)-z(t,x_0)\| \le \int_0^t \|e^{A(t-s)}\|[\|B(s)\|\|x(s,x_0)-z(s,x_0)\|$$

$$+\|f\ (s,x(s,x_0),\sum_{i=1}^{\infty}(\int_{-\infty}^{s}G(s,\tau)\dot{x}(\tau,x_0)d\tau)^i)$$

$$-f\ (s,z(s,x_0),\sum_{i=1}^{\infty}(\int_{-\infty}^{s}G(s,\tau)\dot{z}(\tau,x_0)d\tau)^i)\|]ds$$

By using all assumptions and conditions of theorem 1,we get

$$\|x(t,x_0)-z(t,x_0)\| \le \int_0^t Q[H\ \|x(s,x_0)-z(s,x_0)\| + L_1\|x(s,x_0)-z(s,x_0)\|$$

$$+L_2(\sum_{i=1}^{\infty} iK^iM_2^{\,i-1})\|\dot{x}(s,x_0)-\dot{z}(s,x_0)\|]ds$$

and hence,

$$\|x(t,x_0)-z(t,x_0)\| \le TQ[(H+L_1)\|x(t,x_0)-z(t,x_0)\| + L_2S_2\|\dot{x}(t,x_0)-\dot{z}(t,x_0)\|] \tag{2.21}$$

Also, we find:

$$\|\dot{x}(t,x_0)-\dot{z}(t,x_0)\| \le \|e^{A(t-s)}\|[\|B(t)\|\|x(t,x_0)-z(t,x_0)\|$$

$$+\|f\ (t,x(t,x_0),\sum_{i=1}^{\infty}(\int_{-\infty}^{t}G(t,s)\dot{x}(s,x_0)ds)^i)$$

$$-f\ (t,z(t,x_0),\sum_{i=1}^{\infty}(\int_{-\infty}^{t}G(t,s)\dot{z}(s,x_0)ds)^i)]\|$$

And hence

$$\|\dot{x}(t,x_0)-\dot{z}(t,x_0)\| \le Q[H\ \|x(t,x_0)-z(t,x_0)\| + L_1\|x(t,x_0)-z(t,x_0)\|$$

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$$+L_2(\sum_{i=1}^{\infty} iK^i M_2^{i-1})\|\dot{x}(s,x_0)-\dot{z}(s,x_0)\|]$$

$$\leq Q[(H+L_1)\|x(t,x_0)-z(t,x_0)\|+L_2S_2\|\dot{x}(t,x_0)-\dot{z}(t,x_0)\|]$$

So that

$$\|\dot{x}(t,x_0)-\dot{z}(t,x_0)\|\leq Q[(H+L_1)\|x(t,x_0)-z(t,x_0)\|+L_2S_2\|\dot{x}(t,x_0)-\dot{z}(t,x_0)\|] \quad (2.22)$$

From the inequalities (2.21) and (2.22), we have

$$\|x(t,x_0)-z(t,x_0)\|\leq TQ[(H+L_1)\|x(t,x_0)-z(t,x_0)\|+L_2S_2\|\dot{x}(t,x_0)-\dot{z}(t,x_0)\|]$$

$$\leq TQ(H+L_1)[TQ(H+L_1)\|x(t,x_0)-z(t,x_0)\|+TQL_2S_2\|\dot{x}(t,x_0)-\dot{z}(t,x_0)\|]$$

$$+TQL_2S_2[Q(H+L_1)\|x(t,x_0)-z(t,x_0)\|+QL_2S_2\|\dot{x}(t,x_0)-\dot{z}(t,x_0)\|]$$

Thus

$$\|x(t,x_0)-z(t,x_0)\|\leq (TQ(H+L_1)+L_2S_2Q)[TQ(H+L_1)\|x(t,x_0)-z(t,x_0)\|$$

$$+TQL_2S_2\|\dot{x}(t,x_0)-\dot{z}(t,x_0)\|] ,$$

Therefore,

$$\|x(t,x_0)-z(t,x_0)\|\leq CW ,$$

where $C=[TQ(H+L_1)\|x(t,x_0)-z(t,x_0)\|+TQL_2S_2\|\dot{x}(t,x_0)-\dot{z}(t,x_0)\|]$.

By mathematical induction , we get:

$$\|x(t,x_0)-z(t,x_0)\|\leq CW^m \quad (2.23)$$

From the inequality (2.23) and by using (1.10),when $m\to\infty$, $W^m\to 0$. we find that $x(t,x_0)=z(t,x_0)$. Hence, $x(t,x_0)$ is a unique solution of the equation (1.1), for all $t\in R^1$, $x_0\in D_f$.

3. **Stability solution of the equation** (1.1).

In this section, we study the stability solution of the equation (1.1).

Theorem 3. Let

$$x(t,x_0)=x_0e^{At}+\int_0^t e^{A(t-s)}[B(s)x(s,x_0)+f(s,x(s,x_0),\sum_{i=1}^{\infty}(\int_{-\infty}^{s}G(s,\tau)\dot{x}(\tau,x_0)d\tau)^i)]ds$$

is a solution of the equation (1.1) , then the following inequality :

$$\left\|x(t,x_0^1)-x(t,x_0^2)\right\|\le(F_1F_3Q+F_1F_2F_3TE_2QN)\left\|x_0^1-x_0^2\right\|$$

is satisfied for all $t\in R^1$, $x_0^1,x_0^2\in D_f$,

where $E_1=Q(H+L_1)$ and $E_2=QL_2S_2$.

Proof. Let $x(t,x_0^1)$ and $x(t,x_0^2)$ be two solutions of integro-differential equation (1.1) having the form:

$$x(t,x_0^1)=x_0^1e^{At}+\int_0^t e^{A(t-s)}[B(s)x(s,x_0^1)+f(s,x(s,x_0^1),\sum_{i=1}^{\infty}(\int_{-\infty}^{s}G(s,\tau)\dot{x}(\tau,x_0^1)d\tau)^i)]ds$$

and

$$x(t,x_0^2)=x_0^2e^{At}+\int_0^t e^{A(t-s)}[B(s)x(s,x_0^2)+f(s,x(s,x_0^2),\sum_{i=1}^{\infty}(\int_{-\infty}^{s}G(s,\tau)\dot{x}(\tau,x_0^2)d\tau)^i)]ds$$

Now taking

$$\begin{aligned}\left\|x(t,x_0^1)-x(t,x_0^2)\right\|\le\left\|x_0^1-x_0^2\right\|\left\|e^{At}\right\|+\int_0^t\left\|e^{A(t-s)}\right\|[\left\|B(s)\right\|\left\|x(s,x_0^1)-x(s,x_0^2)\right\|\\+\left\|f(s,x(s,x_0^1),\sum_{i=1}^{\infty}(\int_{-\infty}^{s}G(s,\tau)\dot{x}(\tau,x_0^1)d\tau)^i)\right.\\\left.-f(s,x(s,x_0^2),\sum_{i=1}^{\infty}(\int_{-\infty}^{s}G(s,\tau)\dot{x}(\tau,x_0^2)d\tau)^i)\right\|]ds\end{aligned}$$

By using the inequality (1.4), we get

$$\begin{aligned}\left\|x(t,x_0^1)-x(t,x_0^2)\right\|\le\left\|x_0^1-x_0^2\right\|\left\|e^{At}\right\|+\int_0^t\left\|e^{A(t-s)}\right\|[\left\|B(s)\right\|\left\|x(s,x_0^1)-x(s,x_0^2)\right\|\\+L_1\left\|x(s,x_0^1)-x(s,x_0^2)\right\|+L_2(\sum_{i=1}^{\infty}iK^iM_2^{i-1})\left\|\dot{x}(s,x_0^1)-\dot{x}(s,x_0^2)\right\|]ds\end{aligned}$$

$$\begin{aligned}\left\|x(t,x_0^1)-x(t,x_0^2)\right\|\le\left\|x_0^1-x_0^2\right\|Q+\int_0^t Q[H\left\|x(s,x_0^1)-x(s,x_0^2)\right\|\\+L_1\left\|x(s,x_0^1)-x(s,x_0^2)\right\|+L_2S_2\left\|\dot{x}(s,x_0^1)-\dot{x}(s,x_0^2)\right\|]ds\end{aligned}\qquad(3.1)$$

By integrating the right hand side of inequality (3.1), we get:

$$\|x(t,x_0^1)-x(t,x_0^2)\| \le \|x_0^1-x_0^2\|Q+TQ[H\,\|x(t,x_0^1)-x(t,x_0^2)\|$$
$$+L_1\|x(t,x_0^1)-x(t,x_0^2)\|+L_2S_2\|\dot{x}(t,x_0^1)-\dot{x}(t,x_0^2)\|]$$

Hence, we have:

$$\|x(t,x_0^1)-x(t,x_0^2)\| \le \|x_0^1-x_0^2\|Q+TQ(H+L_1)\|x(t,x_0^1)-x(t,x_0^2)\|$$
$$+TQL_2S_2\|\dot{x}(t,x_0^1)-\dot{x}(t,x_0^2)\|$$

$$\|x(t,x_0^1)-x(t,x_0^2)\| \le \|x_0^1-x_0^2\|Q+TE_1\|x(t,x_0^1)-x(t,x_0^2)\|+TE_2\|\dot{x}(t,x_0^1)-\dot{x}(t,x_0^2)\|$$

So

$$\|x(t,x_0^1)-x(t,x_0^2)\| \le (1-TE_1)^{-1}\|x_0^1-x_0^2\|Q+(1-TE_1)^{-1}TE_2\|\dot{x}(t,x_0^1)-\dot{x}(t,x_0^2)\|$$

$$\|x(t,x_0^1)-x(t,x_0^2)\| \le F_1\|x_0^1-x_0^2\|Q+F_1TE_2\|\dot{x}(t,x_0^1)-\dot{x}(t,x_0^2)\| \qquad (3.2)$$

where

$$F_1=(1-TE_1)^{-1}$$

Also, we find:

$$\|\dot{x}(t,x_0^1)-\dot{x}(t,x_0^2)\| \le \|x_0^1-x_0^2\|\|A\|\|e^{At}\|+\|e^{A(t-s)}\|[\|B(t)\|\|x(t,x_0^1)-x(t,x_0^2)\|$$
$$+\|f\ (t,x(t,x_0^1),\sum_{i=1}^{\infty}(\int_{-\infty}^{t}G(t,s)\dot{x}(s,x_0^1)ds)^i)$$
$$-f\ (t,x(t,x_0^2),\sum_{i=1}^{\infty}(\int_{-\infty}^{t}G(t,s)\dot{x}(s,x_0^2)ds)^i)\|]$$

By using also the inequality (1.4), we get :

$$\|\dot{x}(t,x_0^1)-\dot{x}(t,x_0^2)\| \le \|x_0^1-x_0^2\|\|A\|\|e^{At}\|+\|e^{A(t-s)}\|[\|B(t)\|\|x(t,x_0^1)-x(t,x_0^2)\|$$
$$+L_1\|x(t,x_0^1)-x(t,x_0^2)\|+L_2(\sum_{i=1}^{\infty}iK^iM_2^{i-1})\|\dot{x}(t,x_0^1)-\dot{x}(t,x_0^2)\|]$$

$$\|\dot{x}(t,x_0^1)-\dot{x}(t,x_0^2)\| \le \|x_0^1-x_0^2\|NQ+Q[H\,\|x(t,x_0^1)-x(t,x_0^2)\|$$
$$+L_1\|x(t,x_0^1)-x(t,x_0^2)\|+L_2S_2\|\dot{x}(t,x_0^1)-\dot{x}(t,x_0^2)\|]$$

Hence

$$\|\dot{x}(t,x_0^1)-\dot{x}(t,x_0^2)\| \le \|x_0^1-x_0^2\| NQ + Q(H+L_1)\|x(t,x_0^1)-x(t,x_0^2)\|$$
$$+QL_2S_2\|\dot{x}(t,x_0^1)-\dot{x}(t,x_0^2)\|$$

$$\|\dot{x}(t,x_0^1)-\dot{x}(t,x_0^2)\| \le \|x_0^1-x_0^2\| NQ + E_1\|x(t,x_0^1)-x(t,x_0^2)\| + E_2\|\dot{x}(t,x_0^1)-\dot{x}(t,x_0^2)\|$$

and

$$\|\dot{x}(t,x_0^1)-\dot{x}(t,x_0^2)\| \le (1-E_2)^{-1}\|x_0^1-x_0^2\| NQ + (1-E_2)^{-1}E_1\|x(t,x_0^1)-x(t,x_0^2)\|$$

$$\|\dot{x}(t,x_0^1)-\dot{x}(t,x_0^2)\| \le F_2\|x_0^1-x_0^2\| NQ + F_2E_1\|x(t,x_0^1)-x(t,x_0^2)\| \;. \qquad (3.3)$$

where

$$F_2 = (1-E_2)^{-1}$$

Now, by substituting inequality (3.3) in (3.2), we get:

$$\|x(t,x_0^1)-x(t,x_0^2)\| \le F_1\|x_0^1-x_0^2\| Q + F_1F_2TE_2\|x_0^1-x_0^2\| QN$$
$$+F_1F_2TE_1E_2\|x(t,x_0^1)-x(t,x_0^2)\|$$

So

$$\|x(t,x_0^1)-x(t,x_0^2)\| \le (1-F_1F_2TE_1E_2)^{-1}F_1\|x_0^1-x_0^2\| Q$$
$$+(1-F_1F_2TE_1E_2)^{-1}F_1F_2TE_2\|x_0^1-x_0^2\| QN$$

$$\|x(t,x_0^1)-x(t,x_0^2)\| \le F_1F_3\|x_0^1-x_0^2\| Q + F_1F_2F_3TE_2\|x_0^1-x_0^2\| QN$$

where

$$F_3 = (1-F_1F_2TE_1E_2)^{-1}$$

$$\|x(t,x_0^1)-x(t,x_0^2)\| \le (F_1F_3Q + F_1F_2F_3TE_2QN)\|x_0^1-x_0^2\|$$

for all $t \in R^1$, $x_0^1, x_0^2 \in D_f$.

4. **Existence and Uniqueness solution of the equation** (1.1).

In this section, we study the existence and uniqueness solution of equation (1.1) by using Banach fixed point theorem .

Theorem 4. Let the function $f(t,x,y)$ in equation (1.1) be defined and continuous in the domain (1.2), and satisfies all assumptions and conditions of theorem 1, then equation (1.1) has a unique continuous solution on the domain (1.2).

Proof. Let $(S,\|.\|)$ be a Banach space which is given in Lemma 1[1] ,we defined a mapping T on S=[0,T] as follows :

$$Tz(t,x_0)=x_0e^{At}+\int_0^t e^{A(t-s)}[B(s)z(s,x_0)+f(s,z(s,x_0),\sum_{i=1}^{\infty}(\int_{-\infty}^{s}G(s,\tau)\dot{z}(\tau,x_0)d\tau)^i)]ds \tag{4.1}$$

Since $z(t,x_0)$ is continuous on the domain (1.2), then $\int_{-\infty}^{t}G(t,s)\dot{z}(s,x_0)ds$ is also continuous on the same domain, and hence the function

$$f(t,z(t,x_0),\sum_{i=1}^{\infty}(\int_{-\infty}^{t}G(t,s)\dot{z}(s,x_0)ds)^i)$$

is continuous on the domain (1.2) .

Since $B(t)$ and $e^{A(t-s)}$ are continuous on t, then we have:

$$\int_0^t e^{A(t-s)}[B(s)z(s,x_0)+f(s,z(s,x_0),\sum_{i=1}^{\infty}(\int_{-\infty}^{s}G(s,\tau)\dot{z}(\tau,x_0)d\tau)^i)]ds$$

is continuous on the domain (1.2). So $Tz(t,x_0)\in S$ and hence $Tz(t,x_0):S\to S$.

Next, we claim that $Tz(t,x_0)$ is a contraction mapping on S, let z_1,z_2 be any two functions in S .Taking

$$\|Tz_1(t,x_0)-Tz_2(t,x_0)\|=\max_{t\in[0,T]}\{|Tz_1(t,x_0)-Tz_2(t,x_0)|\}$$

$$\|Tz_1(t,x_0)-Tz_2(t,x_0)\|=\max_{t\in[0,T]}\{|\int_0^t e^{A(t-s)}[B(s)(z_1(s,x_0)-z_2(s,x_0))$$

$$+(f(s,z_1(s,x_0)),\sum_{i=1}^{\infty}(\int_{-\infty}^{s}G(s,\tau)\dot{z}_1(\tau,x_0)d\tau)^i)$$

$$-f\,(s, z_2(s,x_0), \sum_{i=1}^{\infty}(\int_{-\infty}^{s} G(s,\tau)\dot{z}_2(\tau,x_0)d\tau)^i))]ds\,\Big|\Big\}$$

By using the inequality (1.4), we get:

$$\|Tz_1(t,x_0) - Tz_2(t,x_0)\| \le TQ[H \max_{t\in[0,\,T]}|z_1(t,x_0) - z_2(t,x_0)|$$

$$+L_1 \max_{t\in[0,\,T]}|z_1(t,x_0) - z_2(t,x_0)| + L_2 S_2 \max_{t\in[0,\,T]}|\dot{z}_1(t,x_0) - \dot{z}_2(t,x_0)|] \quad (4.2)$$

But

$$\|\dot{z}_1(t,x_0) - \dot{z}_2(t,x_0)\| \le TQ(H+L_1)\|z_1(t,x_0) - z_2(t,x_0)\| + TQL_2S_2\|\dot{z}_1(t,x_0) - \dot{z}_2(t,x_0)\|$$

So

$$\|\dot{z}_1(t,x_0) - \dot{z}_2(t,x_0)\| \le \frac{TQ(H+L_1)}{(1-TQL_2S_2)}\|z_1(t,x_0) - z_2(t,x_0)\| \,. \quad (4.3)$$

Substituting the inequality (4.3) in (4.2), we get:

$$\|Tz_1(t,x_0) - Tz_2(t,x_0)\| \le TQ[H \max_{t\in[0,\,T]}|z_1(t,x_0) - z_2(t,x_0)| + L_1 \max_{t\in[0,\,T]}|z_1(t,x_0) - z_2(t,x_0)|$$

$$+L_2S_2\frac{TQ(H+L_1)}{1-TQL_2S_2}\max_{t\in[0,\,T]}|z_1(t,x_0) - z_2(t,x_0)|]$$

$$\|Tz_1(t,x_0) - Tz_2(t,x_0)\| \le TQ[H + L_1 + L_2S_2\frac{TQ(H+L_1)}{1-TQL_2S_2}]\max_{t\in[0,\,T]}|z_1(t,x_0) - z_2(t,x_0)|$$

$$\|Tz_1(t,x_0) - Tz_2(t,x_0)\| \le TQ[\frac{(H+L_1) - TQL_2S_2(H+L_1) + L_2S_2TQ(H+L_1)}{1-TQL_2S_2}]$$

$$\max_{t\in[0,\,T]}|z_1(t,x_0) - z_2(t,x_0)|$$

$$\|Tz_1(t,x_0) - Tz_2(t,x_0)\| \le \frac{TQ(H+L_1)}{1-TQL_2S_2}\max_{t\in[0,\,T]}|z_1(t,x_0) - z_2(t,x_0)|$$

Thus $\|Tz_1(t,x_0) - Tz_2(t,x_0)\| \le \lambda\|z_1(t,x_0) - z_2(t,x_0)\|$

where

$$\lambda = \frac{TQ(H+L_1)}{1-TQL_2S_2}$$

So T is a contraction mapping if $0 < \lambda < 1$. Thus, by Banach fixed point theorem , there exists a fixed point $z(t)$ in S such that $Tz(t) = z(t)$ and therefore,

$$z(t,x_0) = x_0 e^{At} + \int_0^t e^{A(t-s)}[B(s)z(s,x_0) + f(s,z(s,x_0), \sum_{i=1}^{\infty}(\int_{-\infty}^{s} G(s,\tau)\dot{z}(\tau,x_0)d\tau)^i)]ds$$

is a unique solution of equation (1.1) on the domain (1.2).

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