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A Common Fixed Point Theorem For Two Weakly Compatible Pairs Of Self Maps On A Fuzzy Metric Space

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INTRODUCTION

1.

In 1965, Zadeh [1] introduced the concept of Fuzzy set as a new way to represent vagueness in our everyday life. However, when the uncertainty is due to fuzziness rather than randomness, as sometimes in the measurement of an ordinary length, it seems that the concept of a fuzzy metric space is more suitable. We can divide them into following two groups: The first group involves those results in which a fuzzy metric on a set X is treated as a map where X represents the totality of all fuzzy points of a set and satisfy some axioms which are analogous to the ordinary metric axioms. Thus, in such an approach numerical distances are set up between fuzzy objects. On the other hand in second group, we keep those results in which the distance between objects is fuzzy and the objects themselves may or may not be fuzzy.Kramosil and Michalek [2] have introduced the concept of fuzzy metric spaces in different ways.

In 1986, Jungck [3] introduced the notion of compatible maps for a pair of self mappings. However, the study of common fixed points of non-compatible maps is also very interesting. Jungck and Rhoades [4] initiated the study of weakly compatible maps in metric space and showed that every pair of compatible maps is weakly compatible but reverse is not true. In the literature, many results have been proved for weakly compatible maps satisfying some contractive condition in different settings such as probabilistic metric spaces [6,7,8]; fuzzy metric spaces [9,10,11].

In this paper, we prove a common fixed theorem for four mappings under weakly compatible condition in fuzzy metric space. While proving our results we utilize the idea of weakly compatible maps due to Jungck and Rhoades [4]. Incidentally we obtain the result of Manro [11] as a corollary.

2. PRELIMINARIES

Definition 2.1(B.Schweizer & A.Sklar [13]) A binary operation $*: [0,1] \times [0,1] \to [0,1]$ is continuous t-norm if * satisfies the following conditions:

(i) * is commutative and associative
(ii) * is continuous
(iii) a * 1 = a for all aϵ [0,1]
(iv) a * b $\leq$ c * d whenever a $\leq$ c and b $\leq$ d for all a, b, c, d ϵ [0,1]

Definition 2.2(George & Veeramani [2]) The tuple (X, M, *) is called a fuzzy metric space if X is a nonempty set, * is a continuous t-norm and M is a fuzzy set in $X^2 \times [0,\infty]$ satisfying the following conditions: for $x, y, z \in X;\ s, t > 0$

(i) $M(x, y, t) > 0$
(ii) $M(x, y, t) = 1$ for all t > 0 if and only if $x = y$
(iii) $M(x, y, t) = M(y, x, t)$
(iv) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$
(v) $M(x, y, \cdot) : [0, \infty] \rightarrow [0,1]$ is continuous

Definition 2.3 (Grabeic, [5]) Let (X, M, *) be a fuzzy metric space. Then

(i) A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in$ X if for all t > 0

$$\lim_{n\rightarrow\infty} M(x_n, x \quad, t) = 1$$

(ii) A sequence $\{x_n\}$ in X is said to be G-Cauchy sequence (or simply Cauchy sequence) if for all t > 0 and p > 0 $\lim_{n\rightarrow\infty} M\left(x_{n+p}, x_n, t\right) = 1$

Definition 2.4 (Grabeic, [5]) A fuzzy metric space (X, M, *) is said to be complete iff every Cauchy sequence in X is convergent.

Definition 2.5 (S.Manro et. al.,[10, 11]) A pair of self mappings (A, S) of a fuzzy metric space (X, M, *) is said to be commuting if M (ASx, SAx, t) = 1 for all x in X

Definition 2.6 [10] A pair of self mappings (A, S) of a fuzzy metric space (X, M, *) is said to be weakly commuting if M (ASx, SAx, t) ≥ M (Ax, SX, t) for all x in X and t > 0

Definition 2.7 (S.Manro, [11]) A pair of self mappings (A, S) of a fuzzy metric space (X, M, *) is said to be compatible if $\lim_{n\rightarrow\infty} M(ASx_n, SA\, x_n, t) = 1\ \forall\, t > 0, whenever\ \{x_n\}$ is a sequence in X such that

$\lim_{n\rightarrow\infty} Ax_n = \lim_{n\rightarrow\infty} Sx_n = u$ for some u in X.

Definition 2.8 (Schweizer & A.Sklar, [13]) Let (X, M, *) be a fuzzy metric space. A and S be self maps on X. A point x in X is called a coincidence point of A and S iff Ax = Sx. In this case w = Ax = Sx is called a point of coincidence of A and S.

Definition 2.9 (G.Jungck & B.E.Rhoades, [4]) A pair of self mappings (A, S) of a fuzzy metric space (X, M, *) is said to be weakly compatible if they commute at the coincidence points i.e., if Au = Su for some u in X, then ASu = SAu.

Lemma 2.1 (George & Veeramani [2]) Let $\{u_n\}$ be a sequence in a fuzzy metric space (X, M, *). If there exists a constant k $\in$ (0, 1) such that M $(u_n, u_{n+1}, kt) \geq$ M (u_{n-1}, u_n, t), n = 1, 2, 3, ….. Then $\{u_n\}$ is a Cauchy sequence in X.
i.e. $\lim_{n\rightarrow\infty} M\left(u_{n+p}, u_n, t\right) = 1$

Manro proved the following theorem.

Theorem 2.1 [11] Let A, B, P and Q be self mappings of a complete fuzzy metric space (X, M, *) satisfying the following:

(2.1.1) for any x, y in X and for all t > 0 there exists k ∈ (0, 1) such that, $M(Px, Qy, kt) \geq \max\{M(Ax, By, t), 1/2(M(Px, Ax, t) + M(Qx, Bx, t))\}$

(2.1.2) P(X) ⊂ B(X) and Q(X) ⊂ A(X)

(2.1.3) one of P(X), B(X), Q(X), A(X) is a closed subset of X

Then (a) P and A have a coincidence point

(b) Q and B have a coincidence point.

If the pairs (P, A) and (Q, B) are weakly compatible then A, B, P and Q have a unique common fixed point in X.

Definition 2.10 (M.Imdad et. al., [7]) Two families of self-mappings $\{A_i\}_{i=1}^{m}$ and $\{B_j\}_{j=1}^{n}$ are said to be pair wise commuting if (2.1.4) (a) $A_i A_j = A_j A_i$, i, j ϵ {1,2,3,…m},

(b) $B_i B_j = B_j B_i$, i, j ϵ {1,2,3,…n},
(c) $A_i B_j = B_J A_i$, i ϵ {1,2,3,…m}, j ϵ {1,2,3,…n}.

Theorem 2.2 (S.Manro, [11]) Let { A_1, A_2, ……A_m}, { B_1, B_2, ……B_N}, { P_1, P_2, ……P_P} and { Q_1, Q_2, ……Q_q} be four finite families of self mappings of a complete fuzzy metric space (X, M, *) such that $A = A_1, A_2, \ldots A_m$, $B = B_1, B_2, \ldots B_n$, $P = P_1, P_2, \ldots P_p$ and $Q = Q_1, Q_2, \ldots Q_q$ satisfying the conditions (2.1.1), (2.1.2), (2.1.3) and (2.1.4) the pairs of families ({A_i}, {P_k}) and ({B_r}, {Q_t}) commute pair wise.

Then the (A, P) and (B, Q) have a point of coincidence each. Moreover, $\{A_i\}_{i=1}^{m}$, $\{P_K\}_{K=1}^{p}$ $\{B_r\}_{r=1}^{n}$ and $\{Q_t\}_{t=1}^{q}$ have a unique common fixed point.

By setting P = Q in theorem 2.1

Corollary 2.1 (S.Manro, [11]) Let A, B and P be self mappings of a complete fuzzy metric space (X, M, *) satisfying the following:

(2.1.1) for any x, y in X and for all t > 0 there exists k (0, 1) such that $M(Px, Py, kt) \geq \max\{M(Ax, By, t), 1/2(M(Px, Ax, t) + M(Px, Bx, t))\}$

(2.1.2) P(X) ⊂ B(X) and P(X) ⊂ A(X)

(2.1.3) one of P(X), B(X), A(X) is closed subset of X

Then (a) P and A have a coincidence point

(b) P and B have a coincidence point.

If the pairs (P, A) and (P, B) are weakly compatible then A, B, and P have a unique common fixed point in X.

By taking A = B = I in theorem 2.1

Corollary 2.2 (S.Manro, [11]) Let P and Q be self mapping of a fuzzy metric space (X, M, *) satisfying the following:

(2.2.4) for any x, y in X and for all t > 0 there exists k ∈ (0, 1) such that $M(Px, Qy, kt) \geq \max\{M(x, y, t),\ 1/2(M(Px, x, t) + M(Qx, x, t))\}$

(2.2.5) one of P(X), Q(X) is a closed subset of X.

If the pair (P, Q) is weakly compatible then P and Q have a unique common fixed point in X.

3. MAIN RESULTS

Now we state and prove our main result.

Theorem 3.1 Let A, B, P and Q be self mappings of a complete fuzzy metric space (X, M, *) satisfying the following: there exists k∈ (0, 1) such that for any x, y in X and for all t > 0

(3.1.1) M (Px, Qy, kt) ≥ M (Ax, By, t)

(3.1.2) P(X) ⊂ B(X) and Q(X) ⊂ A(X)

(3.1.3) one of P(X), B(X), Q(X), A(X) is a closed subset of X.

Then

(a) P and A have a coincidence point
(b) Q and B have a coincidence point

If the pairs (P, A) and (Q, B) are weakly compatible then A, B, P and Q have a unique common fixed point in X

Proof: Let $x_0 \in X.\ Then\ there\ exists\ x_1\ \ni P(x_0) =\ B(x_1).$

Write $y_1 = Px_0 = Bx_1, now\ there\ exists\ x_2 \in X\ \ni\ Q(x_1) = A(x_2).\ Write\ \ y_2 = Qx_1 = Ax_2$ Thus inductively we define sequences {x_n} and {y_n} such that $y_{2n+1} = Px_{2n} = Bx_{2n+1}, y_{2n+2} = Qx_{2n+1} = Ax_{2n+2}$. By (3.1.1)

$$M(Px_{2n}, Qx_{2n+1}, kt) \geq M(Ax_{2n}, Bx_{2n+1}, t),$$

$$hence\ M(y_{2n+1}, y_{2n+2}, kt) \geq M(y_{2n}, y_{2n+1}, t)$$

Similarly

$$M(y_{2n+2}, y_{2n+3}, kt) \geq M(y_{2n+1}, y_{2n+2}, t)$$

Therefore

$$M(y_n, y_{n+1}, kt) \geq M(y_{n-1}, y_n, t)$$

Hence by Lemma 2.1, {y_n} is Cauchy in X. Suppose $y_n \to z$ as $n \to \infty$.

Assume A (X) is closed. Then z ∈ A(X). Then there exists w ∈ X ∋ $z = Aw. From$ (3.1.1)

$$M(Pw, Qx_{2n+1}, kt) \geq M(Aw, Bx_{2n+1}, t)$$

$$M(Pw, y_{2n+2}, kt) \geq M(z, y_{2n+1}, t)$$

$$M(Pw, z, kt) \geq M(z, z, t)$$

$$M(Pw, z, kt) = 1 \; \forall \; t > 0$$

Therefore Pw = z= Aw. But (P, A) is weakly compatible so that APw = PAw. Thus Az=Pz

Since P(X) ⊂ $B(X)$ *implies* $z = Pw \in P(X) \subset$ B(X) implies z ∈ B(X)

Let Bv = z. Substitute x = w, y = v in (3.1.1)

$$M(Pw, Qv, kt) \geq M(Aw, Bv, t)$$

$$M(Pw, Qv, kt) \geq M(z, z, t)$$

$$M(z, Qv, kt) \geq 1. \, then \; Qv = z = Bv.$$

Again (Q, B) is weakly compatible. Hence QBv = BQv implies Qz = Bz

Now we will show Pz = z. Substitute x = z, y = v in (3.1.1)

$$M(Pz, Qv, kt) \geq M(Az, Bv, t)$$

$$M(Pz, Qv, kt) \geq M(Az, z, t)$$

$$M(Pz, Qv, kt) \geq M(Pz, z, t) \; \forall t > 0. \therefore Pz = z$$

Similarly we prove Qz = z. Substitute x = w, y = z in (3.1.1)

$$M(Pw, Qz, kt) \geq M(Aw, Bz, t)$$

$$M(z, Qz, kt) \geq M(z, Bz, t)$$

$$M(z, Qz, kt) \geq M(z, Qz, t) \forall t > 0. \; Hence \; Qz = z = Bz.$$

Therefore z is common fixed point of P, A, Q and B.

Uniqueness:

Let x, y be two fixed points. Then

$$M(Px, Qy, kt) \geq M(Ax, By, t)$$

$$M(x, y, kt) \geq M(x, y, t) \; \forall \; t > 0$$

Therefore x = y. Thus P, Q, A and B have unique common fixed point.

Corollary 3.1: If (2.1.1) is replaced by (3.1.1) in theorem 2.1, then A, B, P, Q have unique common fixed point.

Proof: Since (2.1) implies (3.1) the result follows.

Thus the result of Manro [11] is a simple corollary of our result.

By taking P = Q and A = T in Theorem 3.1, we get

Corollary 3.2: Let A, B and P be self mappings of a complete fuzzy metric space (X, M, *) satisfying the following:

(3.2.1) there exists $k \in (0, 1)$ such that for any x, y in X and for all $t > 0$ $M(Px, Qy, kt) \geq M(Ax, By, t)$

(3.2.2) $P(X) \subset B(X)$ and $P(X) \subset A(X)$

(3.2.3) one of P(X), B(X), A(X) is a closed subset of X.

Then (a) P and A have a coincidence point

(b) P and B have a coincidence point.

If the pairs (P, A) and (P, B) are weakly compatible then A, B, and P have a unique common fixed point in X.

By taking A = B = I in Theorem 3.1, we get

Corollary 3.3: Let P and Q be self mapping of a fuzzy metric space (X, M, *) satisfying the following:

(3.2.4) there exists $k \in (0, 1)$ such that for any x, y in X and for all $t > 0$ $M(Px, Qy, kt) \geq M(Ax, By, t)$

(3.2.5) one of P(X), Q(X) is a closed subset of X.

If the pair (P, Q) is weakly compatible then P and Q have a unique common fixed point in X.

Corollary 3.4: If A and P commute and B and Q commute then (A, P) and (B, Q) are weakly compatible hence the other conditions of theorem 3.1 on hold then A, B, P, Q have unique common fixed point

Theorem 3.2: Let $\{A_1, A_2, \ldots\ldots A_m\}$, $\{B_1, B_2, \ldots\ldots B_N\}$, $\{P_1, P_2, \ldots\ldots P_P\}$ and $\{Q_1, Q_2, \ldots\ldots Q_q\}$ be four finite families of self mappings of a complete fuzzy metric space (X, M, *) such that $A = A_1, A_2, \ldots\ldots A_m$, $B = B_1, B_2, \ldots\ldots B_n$, $P = P_1, P_2, \ldots\ldots P_p$ and $Q = Q_1, Q_2, \ldots\ldots Q_q$ satisfying the conditions (3.1.1), (3.1.2), (3.1.3) and (2.1.4) the pairs of families $(\{A_i\}, \{P_k\})$ and $(\{B_r\}, \{Q_t\})$ commute pair wise.

Then the (A, P) and (B, Q) have a point of coincidence each. Moreover, $\{A_i\}_{i=1}^{m}$, $\{P_K\}_{K=1}^{p}$ $\{B_r\}_{r=1}^{n}$ and $\{Q_t\}_{t=1}^{q}$ have a unique common fixed point.

Proof: By using (2.1.4), first we show that AP = PA

$$AP = (A_1A_2 \ldots . A_m)(P_1P_2 \ldots \ldots P_p)$$

$= (A_1A_2 \ldots . A_{m-1})(A_mP_1P_2 \ldots . . P_p)$

$= (A_1A_2 \ldots . A_{m-1})(P_1P_2 \ldots . . P_pA_m)$

$= (A_1A_2 \ldots . A_{m-2})(A_{m-1}P_1P_2 \ldots . . P_pA_m)$

$= (A_1A_2 \ldots . A_{m-2})(P_1P_2 \ldots . . P_pA_{m-1}A_m)$

…………………………………………………………..

$= A_1(P_1P_2 \ldots . . P_pA_2 \ldots . . A_{m-1}A_m)$

$= (P_1P_2 \ldots . . P_p)(A_1A_2 \ldots . A_m) = PA.$

Similarly one can prove that BQ = QB. And hence, the pairs (A, P) and (B, Q) are weakly compatible. Now using Theorem 3.1 we conclude that A, B, P and Q have a unique common fixed point in X, say z. Now we need to prove that z remains the fixed point of all the component mappings.

For this consider $A(A_iz) = (A_iA)z$

$= A_i(Az) = A_iz$ $(\because Az = z)$.

$\therefore A_iz$ is a fixed point of A.

Similarly we prove that P_kz is a fixed point of P. B_rz is a fixed point of B. Q_tz is a fixed point of Q.

Also $A(P_kz) = P_k(Az) = P_kz$ also a fixed point of A. similarly Q_t z is also a fixed point of B.

Now put x = P_kz, y = Q_tz in (3.1.1), we get

$$M(Px, Qy, kt) \geq M(Ax, By, t)$$

$$M(PP_kz, QQ_tz, kt) \geq M(AP_kz, BQ_tz, t)$$

$$M(P_kz, Q_tz, kt) \geq M(P_kz, Q_tz, t) \ \forall\ t > 0$$

Therefore $P_kz = Q_tz$. implies P_kz is a fixed point for A, P, Q and B, that is P_kz= z since z is a unique common fixed point for A, P, Q and B.

Similarly Q_tz = z, A_kz = z and B_rz = z. Hence z is a common fixed point for$\{A_i\}_{i=1}^{m}$, $\{P_K\}_{K=1}^{p}$ $\{B_r\}_{r=1}^{n}$ and $\{Q_t\}_{t=1}^{q}$

Uniqueness: Suppose z^1 is a common fixed point for$\{A_i\}_{i=1}^{m}$, $\{P_K\}_{K=1}^{p}$ $\{B_r\}_{r=1}^{n}$ and $\{Q_t\}_{t=1}^{q}$. Then z^1 is a fixed point for A, P, Q and B. thus z^1 is a common fixed point for A, P, Q and B.

Therefore z^1 = z since A, P, Q and B have unique common fixed point.

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