IJISET - International Journal of Innovative Science, Engineering & Technology, Vol. 3 Issue 1, January 2016.
www.ijiset.com
ISSN 2348 – 7968

Series Solutions of Fractional Initial-Value Problems by q-Homotopy Analysis Method

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Abstract
In this paper, the q-homotopy analysis method is applied to solve linear and nonlinear fractional initial-value problems (fIVPs). The fractional derivatives are described by Caputo's sense. Exact and/or approximate analytical solutions of the fIVPs are obtained. The results of applying this procedure to the studied cases show the high accuracy and efficiency of theapproach.

Keywords:q-Homotopy analysis method; Fractional IVPs; Caputo's fractional derivative

1. Introduction

The importance of obtaining the exact and approximate solutions of fractional nonlinear equations in physics and mathematics is still a significant problem that needs new methods to discover exact and approximate solutions. But these nonlinear fractional differential equations are difficult to get their exact solutions[41, 42, 55, 58]. So, numerical methods have been used to handle these equations [9, 10, 32, 51], and some semianalytical techniques have also largely been used to solve these equations. Such as, Adomian decomposition method [5,57], variational iteration method [24,56], differential transform method [8,17],Laplace decomposition method [31,59], homotopy perturbation method [22,23,33,44,47,48] and homotopy analysis method (HAM) [34–39].The HAM initially proposed by Liao in his Ph.D. thesis [34] is a powerful method to solve nonlinear problems. In recent years, this method has been successfully employed to solve many types of nonlinear problems in science and engineering [1-3, 7, 11-13, 21, 40, 43, 45, 46, 49, 54]. HAM contains a certain auxiliary parameter$h$, which provides us with a simple way to adjust and control the convergence region and rate of convergence of the series solution. Many workers applied HAM to solve fractional differential equations [4, 6, 19, 20, 52, 53].El-Tawil and Huseen [15] established a method namely q-homotopy analysis method (q-HAM) which is a more general method of HAM, The q-HAM contains an auxiliary parameter n as well as h such that the case of $n = 1$(q-HAM;$n = 1$) the standard homotopy analysis method (HAM) can be reached. The q-HAM has been successfully applied to numerous problems in science and engineering [15, 16, 25-30]. In this paper,q-HAM is applied to solve linear and nonlinear fractional initial-value problems (fIVPs). Some test examples shall be presented to show the efficiency and accuracy of q-HAM.

2. Basic definitions

In this section, we give some definitions and properties of the fractional calculus [50].

Definition 1. A real function $h(t), t > 0$, is said to be in the space $C_\mu, \mu \in \mathbb{R}$, if there exists a real number $p > \mu$, such that $h(t) = t^p h_1(t)$, where $h_1(t) \in C(0,\infty)$, and it is said to be in the space C_μ^n if and only if $h^{(n)} \in C_\mu^n, n \in N$.

Definition 2. The Riemann–Liouville fractional integral operator J^α of order $\alpha \geq 0$, of a function if $h \in C_\mu, \mu \geq -1$, is defined as

$$J^\alpha h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} h(\tau) d\tau \quad (\alpha > 0),$$

$$J^0 h(t) = h(t) \tag{1}$$

$\Gamma(z)$ is the well-known Gamma function. Some of the properties of the operator J^α, which we will need here, are as follows

For $h \in C_\mu, \mu \geq -1, \alpha, \beta \geq 0, \gamma \geq -1$

(1) $J^\alpha J^\beta h(t) = J^{\alpha+\beta} h(t)$,

(2) $J^\alpha J^\beta h(t) = J^\beta J^\alpha h(t)$,

(3) $J^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} t^{\alpha+\gamma}$.

Definition 3. The fractional derivative (D^α) of $h(t)$ in the Caputo's sense is defined as

$$D^\alpha h(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} h^{(n)} d\tau, \tag{2}$$

for $n-1 < \alpha \leq n,\ n \in N,\ t > 0,\ h \in C_{-1}^n$.

The following are two basic properties of the Caputo's fractional derivative [18]

(1) Let $h \in C_{-1}^n, n \in N$. Then $D^\alpha h, 0 \leq \alpha \leq n$ is well defined and $D^\alpha h \in C_{-1}$.

(2) Let $n-1 < \alpha \leq n,\ n \in N$, and $h \in C_\mu^n, \mu \geq -1$. Then

$$(J^\alpha D^\alpha) h(t) = h(t) - \sum_{k=0}^{n-1} h^{(k)}(0^+) \frac{t^k}{k!} \tag{3}$$

3. The q-Homotopy Analysis Method

In q-HAM, fractional differential equations are written in the form,

$$FD[u(t)] = 0, \tag{4}$$

where FD is a fractional differential operator, t denotes an independent variable, and $u(t)$ is an unknown function.

In the frame of HAM [15], we can construct the following zeroth-order deformation

$$(1-nq)L[\emptyset(t;q) - u_0(t)] = qhH(t)FD[\emptyset(t;q)], \tag{5}$$

where $n \geq 1, q \in [0, \frac{1}{n}]$ is the embedding parameter, $h \neq 0$ is an auxiliary parameter, $H(t) \neq 0$ is an auxiliary function, L is an auxiliary linear operator, $u_0(t)$ is an initial guess of $u(t)$ and $\emptyset(t;q)$ is an unknown function on the independent variables t and q.

Obviously, when $q = 0$ and $q = 0$, it holds

$$\emptyset(t;0) = u_0(t), \emptyset(t;1) = u(t), \tag{6}$$

respectively. Using the parameter q, we expand $\emptyset(t;q)$ in Taylor series as follows

$$\emptyset(t,q) = u_0(t) + \sum_{m=1}^\infty u_m(t) q^m, \tag{7}$$

where

$$u_m(t) = \frac{1}{m!}\frac{\partial^m \emptyset(t,q)}{\partial q^m}\Big|_{q=0} \quad (8)$$

Assume that the auxiliary linear operator, the initial guess, the auxiliary parameter $\hbar$and the auxiliary function $H(t)$are selected such that the series (7) is convergent at $q = \frac{1}{n}$, then due to (6) we have

$$u(t) = u_0(t) + \sum_{m=1}^{\infty} u_m(t)\left(\frac{1}{n}\right)^m \quad (9)$$

Let us define the vector

$$\vec{u_r}(t) = \{u_0(t), u_1(t), u_2(t), \dots, u_r(t)\}, \quad (10)$$

Differentiating (5) m times with respect to the embedding parameter q, then setting $q = 0$ and finally dividing them by $m!$, we have the so-called mth-order deformation equation

$$L[u_m(t) - k_m u_{m-1}(t)] = hH(t)R_m(\vec{u}_{m-1}(t)), \quad (11)$$

where

$$R_m(\vec{u}_{m-1}(t)) = \frac{1}{(m-1)!}\frac{\partial^{m-1} FD[\emptyset(t,q)]}{\partial q^{m-1}}\Big|_{q=0}, \quad (12)$$

and

$$k_m = \begin{cases} 0 & m \le 1 \\ n & otherwise \end{cases} \quad (13)$$

Finally, for the purpose of computation, we will approximate the q-HAM solution (9) by the following truncated series

$$U_m(t) = \sum_{i=0}^{m} u_i(t)\left(\frac{1}{n}\right)^i \quad (14)$$

4. Applications

In this section, we shall illustrate the applicability of q-HAM to linear and nonlinear fIVPs.

4.1.Problem 1

First, we consider the following linear fIVP

$$D^\alpha u = -u, \quad 0 < \alpha \le 2, \quad (15)$$

$$u(0) = 1, \quad u'(0) = 0. \quad (16)$$

The second initial condition is for $\alpha > 1$ only.

Note that the exact solution of (15) is as follows [14]

$$u(t) = \sum_{k=0}^{\infty} \frac{(-t^\alpha)^k}{\Gamma(\alpha k+1)}. \quad (17)$$

Hence, it is straightforward to use the set of base functions

$$\{t^{\alpha k} \mid k = 0,1,2,3, \dots\} \quad (18)$$

to represent the solution$u(t)$,i.e.,

$$u(t) = \sum_{k=0}^{\infty} b_k t^{\alpha k}, \quad (19)$$

where $b_k (k = 0,1,2, \dots)$is a coefficient to be determined later.

According to (5), the zeroth-order deformation can be given by

$$(1 - nq)L[\emptyset(t;q) - u_0(t)] = qhH(t)\left[D^\alpha\left(\emptyset(t;q)\right) + \emptyset(t;q)\right]. \quad (20)$$

Under the rule of solution expression denoted by (19) and according to the initial conditions (16), we can choose the initial guess of $u(t)$as follows

$$u_0(t) = 1,$$

and we choose the auxiliary linear operator
$L[\emptyset(t;q)] = D^\alpha \emptyset(t;q)$,
with the property
$L[C] = 0$,
where C is an integral constant. We also choose the auxiliary function to be $H(t) = 1$.
Hence, the mth-order deformation can be given by
$$D^\alpha[u_m(t) - k_m u_{m-1}(t)] = h[D^\alpha(u_{m-1}(t)) + u_{m-1}(t)], \quad (21)$$
Applying the operator J^α, the inverse operator of D^α, on both sides of (21), we obtain
$$u_m(t) = k_m u_{m-1}(t) + hJ^\alpha[D^\alpha(u_{m-1}(t)) + u_{m-1}(t)], \quad m \geq 1$$
Consequently, the first few terms of the q-HAM series solution are as follows
$$u_1(t) = h\frac{t^\alpha}{\Gamma(\alpha+1)},\ u_2(t) = h(n+h)\frac{t^\alpha}{\Gamma(\alpha+1)} + h^2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)},$$
$$u_3(t) = h(n+h)^2\frac{t^\alpha}{\Gamma(\alpha+1)} + 2h^2(n+h)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + h^3\frac{t^{3\alpha}}{\Gamma(3\alpha+1)},$$
$$u_4(t) = h(n+h)^3\frac{t^\alpha}{\Gamma(\alpha+1)} + 3h^2(n+h)^2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + 3h^3(n+h)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + h^4\frac{t^{4\alpha}}{\Gamma(4\alpha+1)},$$
$$u_5(t) = h(n+h)^4\frac{t^\alpha}{\Gamma(\alpha+1)} + 4h^2(n+h)^3\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + 6h^3(n+h)^2\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + 4h^4(n+h)\frac{t^{4\alpha}}{\Gamma(4\alpha+1)} + h^5\frac{t^{5\alpha}}{\Gamma(5\alpha+1)},$$
and so on. Hence, the q-HAM series solution is
$$u(t) = u_0 + \frac{u_1}{n} + \frac{u_2}{n^2} + \frac{u_3}{n^3} + \cdots$$
$$= 1 + \frac{h}{n}\left[1 + \frac{n+h}{n} + \left(\frac{n+h}{n}\right)^2 + \cdots\right]\frac{t^\alpha}{\Gamma(\alpha+1)} + \left(\frac{h}{n}\right)^2\left[1 + 2\frac{n+h}{n} + 3\left(\frac{n+h}{n}\right)^2 + \cdots\right]\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}$$
$$+ \left(\frac{h}{n}\right)^3\left[1 + 3\frac{n+h}{n} + 6\left(\frac{n+h}{n}\right)^2 + \cdots\right]\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots \quad (22)$$
If we take $h = -n$, then we obtain the exact solution (17). It should be noted that if we put $n = 1, h = -1$ in (22) then, we obtain the same results by the HAM solution [20].

4.2.Problem 2

Let us consider the following nonlinear fractional initial-value problem
$$D^\alpha u = u^2 + 1, \quad p - 1 < \alpha \leq p, \quad p \in N, \quad 0 < t < 1, \quad (23)$$
$$u^{(k)}(0) = 0, \quad k = 0, \dots p-1. \quad (24)$$
The exact solution of this initial-value problem for $\alpha = 1$, the ODE case, is $u = \tan t$ [20].
We will use the set of base functions
$$\{t^{(2k+1)\alpha} \mid k = 0,1,2,3,\dots\} \quad (25)$$
to represent the solution$u(t)$,i.e.
$$u(t) = \sum_{k=0}^{\infty} b_k t^{(2k+1)\alpha}, \quad (26)$$
where $b_k (k = 0,1,2,\dots)$is a coefficient to be determined later. This provides us with the Rule of solution expression. Accordingly, and by the initial conditions (24), we can choosethe initial guess of $u(t)$as follows
$$u_0(t) = \frac{t^\alpha}{\Gamma(\alpha+1)}$$
and we choose the auxiliary linear operator

$$L[\emptyset(t;q)] = D^{\alpha}\emptyset(t;q)$$

Finally, for simplicity, we choose $H(t) = 1$as the auxiliary function. Hence, we construct the zeroth-order deformation equation

$$(1-nq)L[\emptyset(t;q) - u_0(t)] = qh\, FD[\emptyset(t;q)], \qquad (27)$$

where

$$FD[\emptyset(t;q)] = D^{\alpha}\big(\emptyset(t;q)\big) - \emptyset^2(t;q) - 1 \qquad (28)$$

Differentiating (27) m times with respect to q, then setting $q = 0$and finally dividing them by $m!$, we have the mth-order deformation equation

$$L[u_m(t) - k_m u_{m-1}(t)] = hR_m(\vec{u}_{m-1}(t)), \qquad (29)$$

where

$$R_m(\vec{u}_{m-1}(t)) = \frac{1}{(m-1)!}\frac{\partial^{m-1} FD[\emptyset(t,q)]}{\partial q^{m-1}}\Big|_{q=0}, (30)$$

and k_m is defined by (13). From (28) and (30), we have

$$R_m(\vec{u}_{m-1}) = D^{\alpha}u_{m-1} - \sum_{i=0}^{m-1} u_i u_{m-1-i} - (1 - \frac{1}{n}k_m) \qquad (31)$$

Applying the fractional integral operatorJ^{α}, on both sides of (29), we obtain

$$u_m(t) = k_m u_{m-1}(t) + hJ^{\alpha}R_m(\vec{u}_{m-1}), \quad m \geq 1$$

The first few terms of the q-HAM series solution are as follows

$$u_1(t) = -h\frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)}t^{3\alpha},$$

$$u_2(t) = -h(n+h)\frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)}t^{3\alpha} + 2h^2\frac{\Gamma(2\alpha+1)\Gamma(4\alpha+1)}{\Gamma^3(\alpha+1)\Gamma(3\alpha+1)\Gamma(5\alpha+1)}t^{5\alpha},$$

$u_i(t)$,$(i = 3,4,5,\ldots)$ can be calculated similarly. Then the series solution expression by q-HAM can be written in the form

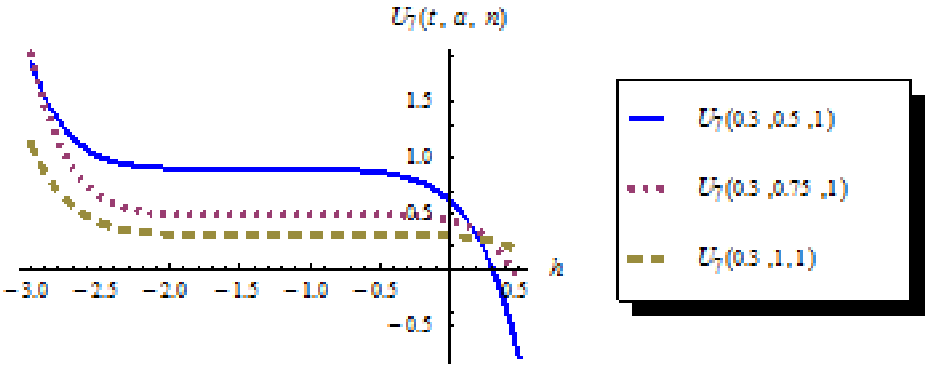
$$U_m(t;\alpha;n;h) = \sum_{i=0}^{m} u_i(t;\alpha;n;h)\left(\frac{1}{n}\right)^i \qquad (32)$$

Equation (32) is an approximate solution to the problem (23) in terms of the convergence parameters $h\ and\ n$. To find the valid region of h , the h-curves given by the 7th order q-HAM approximation at different values of α and n are drawn in figures (1-4). These figures show the interval of h at which the value of $U_7(t;\alpha;n)$ is constant at certain values of t,α and n .We choose the horizontal line parallel to $x - axis\ (h)$ as a valid region which provides us with a simple way to adjust and control the convergence region of the series solution (32). From these figures, the valid intersection region of h for the values of t,α and $n$in the curves becomes larger as n increase. The absolute errors of the 7th order solutions q-HAM approximate using different values of $n$with $\alpha = 1$are calculated by the formula

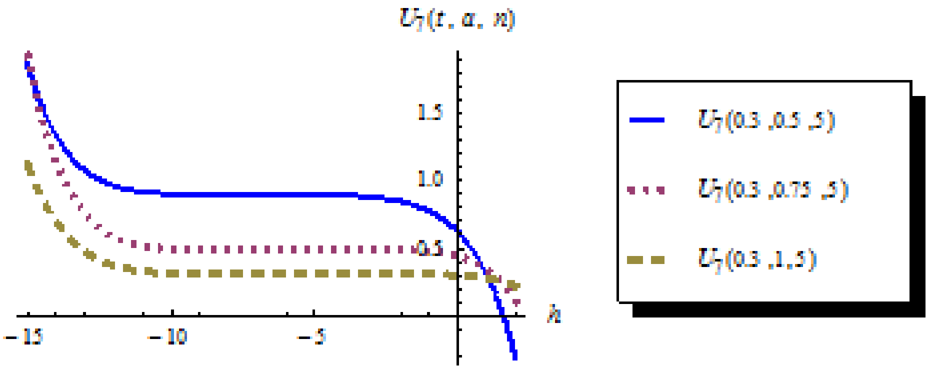
$$\text{Absolute Error} = |u_{exact} - u_{apprax}| \qquad (33)$$

Figure (5) shows that the series solutions obtained by q-HAM ($\alpha = 1$) at $n > 1$converge faster than $n = 1$(HAM).Table 1-3 shows the7th order q- HAM approximate solutionU_7 of (23) for different values of n andα), which indicates that the speed of convergence for q-HAM with $n > 1$ is faster in comparison with $n = 1$ (HAM).

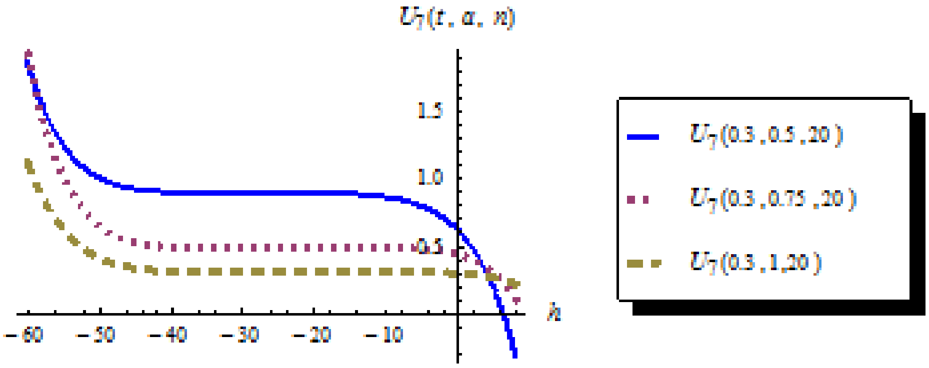
IJISET - International Journal of Innovative Science, Engineering & Technology, Vol. 3 Issue 1, January 2016.
www.ijiset.com
ISSN 2348 – 7968



Figure(1) : h - curve for the 7th order q- HAM approximate solution of problem (23) at $t = 0.3, n = 1$ with different values of α.



Figure(2) : h - curve for the 7th order q- HAM approximate solutionof problem (23) at $t = 0.3, n = 5$ with different values of α.



Figure(3) : h - curve for the 7th order q- HAM approximate solution of problem (23) at $t = 0.3, n = 20$ with different values of α.

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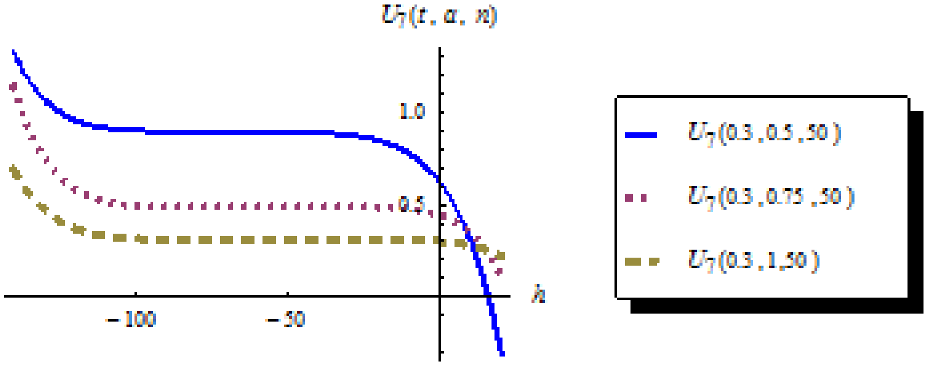
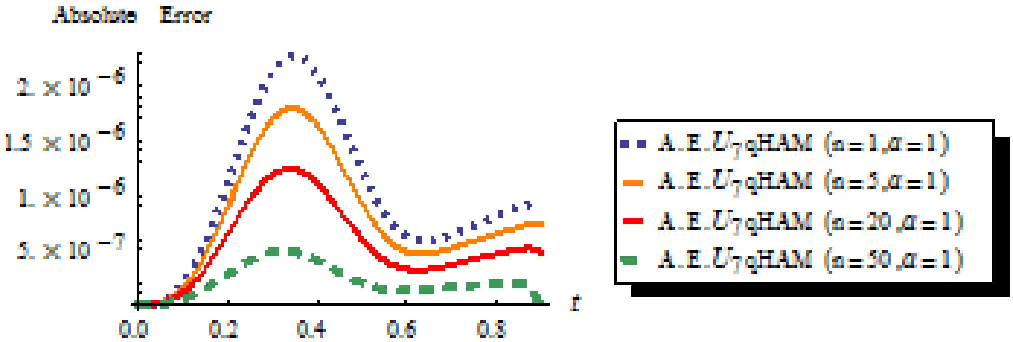


Figure(4) : h - curve for the 7th order q- HAM approximate solution of problem (23) at $t = 0.3, n = 50$ with different values of α.



Figure(5): The absolute errorsfor the 7th order q- HAM approximate solution ($n = 1, 5, 20$ *and* 50) for problem (23) at $\alpha = 1$ using ($h = -1.35$, $h = -6.7$, $h = -26.5$, $h = -64.5$) respectively.

Table 1: Approximate solution of (23) for some values of n using 7th order q- HAM approximate solution U_7, when $\alpha = 0.25$.

t	$n = 1$; $h = -1.35$	$n = 5$; $h = -6.7$	$n = 20$; $h = -26.5$	$n = 50$; $h = -64.5$	Exact
0.1	1.397969	1.395297	1.391261	1.381698	0.100334
0.2	5.645199	5.576077	5.473959	5.242819	0.202710
0.3	20.128929	19.714423	19.107395	17.757485	0.309336
0.4	55.677935	54.252447	52.175091	47.601064	0.422793
0.5	126.743551	123.088968	117.780308	106.167432	0.546302
0.6	251.214070	243.399946	232.075399	207.418539	0.684136
0.7	450.266233	435.495206	414.126201	367.765772	0.842288
0.8	748.244430	722.702152	685.802311	605.974561	1.029638
0.9	1172.559796	1131.268786	1071.685777	943.085390	1.260158
1.0	1753.604027	1690.280964	1598.993326	1402.346755	1.557407

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Table 2: Approximate solution of (23) for some values of n using 7th order q- HAM approximate solution U_7, when $\alpha = 0.5$.

t	$n = 1$; $h = -1.35$	$n = 5$; $h = -6.7$	$n = 20$; $h = -26.5$	$n = 50$; $h = -64.5$	Exact
0.1	0.391977	0.391977	0.391977	0.391976	0.100334
0.2	0.624113	0.624113	0.624113	0.624112	0.202710
0.3	0.897445	0.897432	0.897409	0.897347	0.309336
0.4	1.317889	1.317476	1.316834	1.315219	0.422793
0.5	2.159492	2.155250	2.148813	2.133439	0.546302
0.6	4.151219	4.126289	4.089008	4.002537	0.684136
0.7	9.024494	8.920348	8.766064	8.414927	0.842288
0.8	20.447487	20.101949	19.593418	18.451261	1.029638
0.9	45.484309	44.513396	43.091403	39.928687	1.260158
1.0	96.730507	94.324345	90.813385	83.062966	1.557407

Table 3: Approximate solution of (23) for some values of n using 7th order q- HAM approximate solution U_7, when $\alpha = 0.75$.

t	$n = 1$; $h = -1.35$	$n = 5$; $h = -6.7$	$n = 20$; $h = -26.5$	$n = 50$; $h = -64.5$	Exact
0.1	0.197049	0.197049	0.197049	0.197049	0.100334
0.2	0.343180	0.343179	0.343179	0.343178	0.202710
0.3	0.488351	0.488351	0.488351	0.488350	0.309336
0.4	0.645709	0.645709	0.645708	0.645708	0.422793
0.5	0.828281	0.828281	0.828281	0.828280	0.546302
0.6	1.055572	1.055571	1.055569	1.055563	0.684136
0.7	1.364033	1.364002	1.363951	1.363806	0.842288
0.8	1.832852	1.832491	1.831919	1.830442	1.029638
0.9	2.649372	2.646851	2.642970	2.633427	1.260158
1.0	4.264848	4.252176	4.232986	4.187345	1.557407

4.3. Problem 3

Consider the nonlinear fIVP

$$D^{\alpha}u = \frac{9}{4}\sqrt{u} + u, \quad 1 < \alpha \leq 2, \quad p \in N, \quad t \geq 0, \tag{34}$$

$$u(0) = 1, \;\; u'(0) = 2. \tag{35}$$

The exact solution of the initial-value problem (34) and (35) for $\alpha = 2$, i.e. the ODE case, is [20]

$$u(t) = \frac{9}{4}\left[\frac{3e^{0.5t}}{2} + \frac{e^{-0.5t}}{6} - 1\right]^2. \tag{36}$$

Expanding the nonlinear term, $\sqrt{u}$, in (34) by using the Taylor series, we obtain

ISSN 2348 – 7968

$$\sqrt{u} \approx 1 + \frac{1}{2}(u-1) - \frac{1}{8}(u-1)^2 + \frac{1}{16}(u-1)^3. \quad (37)$$

Then, the fIVP (34) can be approximated by

$$D^\alpha u = \frac{45}{64} + \frac{199}{64}u - \frac{45}{64}u^2 + \frac{9}{64}u^3 \quad (38)$$

We will use the set of base functions

$$\{t^{(k\alpha+l)} \mid l = 0, \dots, 2k+1, k = 0,1,2,3, \dots\}, \quad (39)$$

to represent the solution $u(t)$, i.e.,

$$u(t) = \sum_{k=0}^{\infty} \sum_{l=0}^{2k+1} b_{k,l} t^{(k\alpha+l)}, \quad (40)$$

where $b_{k,l}$ are coefficients to be determined. This provides us with the rule of solution expression. In view of q-HAM technique, we construct the zeroth-order deformation equation as follows

$$(1-nq)L[\emptyset(t;q) - u_0(t)] = qh\,H(t)\,FD[\emptyset(t;q)], \quad (41)$$

where

$$FD[\emptyset(t;q)] = D^\alpha\big(\emptyset(t;q)\big) - \frac{45}{64} - \frac{199}{64}\emptyset(t;q) + \frac{45}{64}\emptyset^2(t;q) - \frac{9}{64}\emptyset^3(t;q). \quad (42)$$

According to the rule of solution expression (40) and the initial conditions (35), we can choose $u_0(t) = 1 + 2t$as the initial guess of $u(t), L[\emptyset(t;q)] = D^\alpha[\emptyset(t;q)]$ as the auxiliary linear operator, and for simplicity, we choose $H(t) = 1$ as the auxiliary function.Differentiating (41) m times with respect to q, then setting $q = 0$and finally dividing them by $m!$, we have the mth-order deformation equation

$$L[u_m(t) - k_m u_{m-1}(t)] = hR_m(\vec{u}_{m-1}(t)), \quad (43)$$

where

$$R_m(\vec{u}_{m-1}(t)) = \frac{1}{(m-1)!}\frac{\partial^{m-1} FD[\emptyset(t,q)]}{\partial q^{m-1}}\Big|_{q=0}, \quad (44)$$

and k_m is defined by (13). Applying the fractional integral operatorJ^α, on both sides of (43), we obtain

$$u_m(t) = k_m u_{m-1}(t) + hJ^\alpha R_m(\vec{u}_{m-1}), \quad m \geq 1$$

The first few terms of the q-HAM series solution are as follows

$$u_1(t) = -hC_1 t^\alpha - hC_2 t^{\alpha+1} + hC_3 t^{\alpha+2} - hC_4 t^{\alpha+3},$$

$$u_2(t) = -h(n+h)C_1 t^\alpha - h(n+h)C_2 t^{\alpha+1} + h(n+h)C_3 t^{\alpha+2} - h(n+h)C_4 t^{\alpha+3}$$
$$+hC_5 t^{2\alpha} + hC_6 t^{2\alpha+1} + hC_7 t^{2\alpha+2} + hC_8 t^{2\alpha+3} - hC_9 t^{2\alpha+4} + hC_{10} t^{2\alpha+5},$$

where

$$C_1 = \frac{13}{4\Gamma(\alpha+1)}, C_2 = \frac{17}{4\Gamma(\alpha+2)}, C_3 = \frac{9}{4\Gamma(\alpha+3)}, C_4 = \frac{27}{4\Gamma(\alpha+4)}, C_5 = \frac{17\Gamma(\alpha+1)}{8\Gamma(2\alpha+1)}C_1,$$
$$C_6 = \frac{\Gamma(\alpha+2)}{8\Gamma(2\alpha+2)}(17C_2 - 9C_1), C_7 = \frac{\Gamma(\alpha+3)}{16\Gamma(2\alpha+3)}(27C_1 - 18C_2 - 34C_3),$$
$$C_8 = \frac{\Gamma(\alpha+4)}{16\Gamma(2\alpha+4)}(27C_2 + 18C_3 + 34C_4), C_9 = \frac{\Gamma(\alpha+5)}{16\Gamma(2\alpha+5)}(27C_3 + 18C_4), C_{10} = \frac{27\Gamma(\alpha+6)}{16\Gamma(2\alpha+6)}C_4.$$

$u_i(t)$, $(i = 3,4,5, \dots)$ can be calculated similarly. Then the series solution expression by q-HAM can be written in the form

$$U_m(t;\alpha;n;h) = \sum_{i=0}^{m} u_i(t;\alpha;n;h)\left(\frac{1}{n}\right)^i \quad (45)$$

Tables 4-7 show 3rd order q- HAM approximate solution$U_3$of fIVP (34) and (35) with different values of n, α. It is clear that when we take $n = 50$, we obtain the best results for the

ISSN 2348 – 7968

case $\alpha = 2$which has an exact solution. Figure (6) shows the absolute error of the 3th order q-HAM approximate solution for various $n$and$\alpha = 1.75$.

Table 4: Approximate solution of (34) for some values of n using 3th order q- HAM approximate solution U_3, when $\alpha = 2$.

t	$n = 1$; $h = -1.1$	$n = 5$; $h = -4.4$	$n = 20$; $h = -16.1$	$n = 50$; $h = -36.5$	Exact
0.0	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
0.1	1.21699422	1.21694776	1.21684967	1.21663943	1.21697814
0.2	1.47105017	1.47085406	1.47042557	1.46952342	1.47099037
0.3	1.76719157	1.76671999	1.76564277	1.76343324	1.76704923
0.4	2.11114557	2.11022072	2.10803931	2.10370779	2.11074085
0.5	2.50967720	2.50797380	2.50401990	2.49646208	2.50828744
0.6	2.97118609	2.96798260	2.96124832	2.94892985	2.96661653
0.7	3.50669738	3.50032922	3.48924774	3.46997633	3.49343764
0.8	4.13141671	4.11821076	4.10028324	4.07083913	4.09732726
0.9	4.86706730	4.83943775	4.81062032	4.76616912	4.78782298
1.0	5.74528696	5.68854501	5.64227690	5.57546182	5.57552764

Table 5: Approximate solution of (34) for some values of n using 3th order q- HAM approximate solution U_3, when $\alpha = 1.25$.

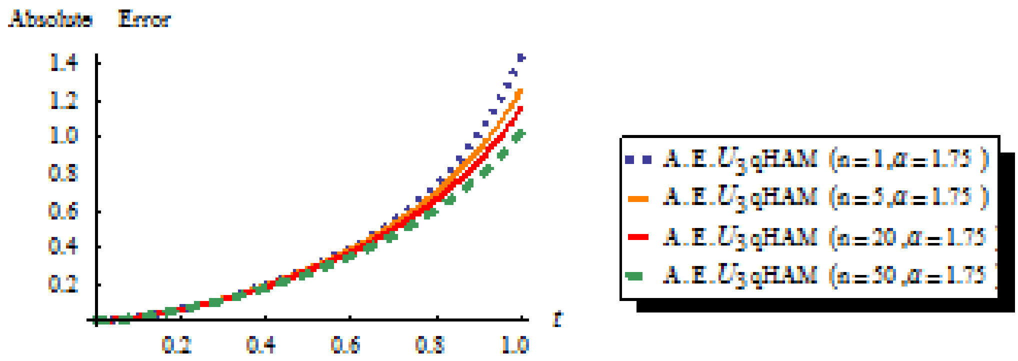
t	$n = 1$; $h = -1.1$	$n = 5$; $h = -4.4$	$n = 20$; $h = -16.1$	$n = 50$; $h = -36.5$	Exact
0.0	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
0.1	1.37721580	1.37664564	1.37526946	1.37263715	1.21697814
0.2	1.86568779	1.86339373	1.85851323	1.85008030	1.47099037
0.3	2.45603230	2.44862802	2.43698904	2.41847069	1.76704923
0.4	3.16375601	3.14280469	3.11928073	3.08453045	2.11074085
0.5	4.01895484	3.96627694	3.92261385	3.86251021	2.50828744
0.6	5.07099809	4.95137053	4.87422031	4.77493635	2.96661653
0.7	6.39611393	6.14654599	6.01459157	5.85507708	3.49343764
0.8	8.10737100	7.62195896	7.40188339	7.15035297	4.09732726
0.9	10.36742085	9.47652153	9.11750787	8.72667178	4.78782298
1.0	13.40468314	11.84679674	11.27315683	10.67385775	5.57552764

Table 6: Approximate solution of (34) for some values of n using 3th order q- HAM approximate solution U_3, when $\alpha = 1.5$.

t	$n = 1$; $h = -1.1$	$n = 5$; $h = -4.4$	$n = 20$; $h = -16.1$	$n = 50$; $h = -36.5$	Exact
0.0	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
0.1	1.28250094	1.28226799	1.28173649	1.28064631	1.21697814
0.2	1.65030916	1.64951479	1.64757613	1.64386332	1.47099037
0.3	2.09470874	2.09267840	2.08803271	2.07971234	1.76704923
0.4	2.62204315	2.61697226	2.60759259	2.59185499	2.11074085
0.5	3.24595124	3.23332577	3.21597617	3.18872377	2.50828744
0.6	3.98918254	3.95892103	3.92832736	3.88341308	2.96661653
0.7	4.88774245	4.81947610	4.76699413	4.69502084	3.49343764
0.8	5.99693910	5.85269881	5.76429106	5.65080978	4.09732726
0.9	7.39955369	7.11294045	6.96622659	6.78913332	4.78782298
1.0	9.21659783	8.67726339	8.43734689	8.16322655	5.57552764

Table 7: Approximate solution of (34) for some values of n using 3th order q- HAM approximate solution U_3, when $\alpha = 1.75$.

t	$n = 1$; $h = -1.1$	$n = 5$; $h = -4.4$	$n = 20$; $h = -16.1$	$n = 50$; $h = -36.5$	Exact
0.0	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
0.1	1.23783896	1.23773482	1.23750971	1.23703316	1.21697814
0.2	1.53427829	1.53389758	1.53301881	1.53122713	1.47099037
0.3	1.88856602	1.88768375	1.88555280	1.88140110	1.76704923
0.4	2.30565107	2.30382093	2.29953850	2.29159866	2.11074085
0.5	2.79367805	2.78981005	2.78199119	2.76824998	2.50828744
0.6	3.36478975	3.35623686	3.34268869	3.32022553	2.96661653
0.7	4.03713632	4.01796219	3.99513230	3.95959253	3.49343764
0.8	4.8380570	4.79603987	4.75811650	4.70287594	4.09732726
0.9	5.80866925	5.72044940	5.65795847	5.57284543	4.78782298
1.0	7.01022722	6.83382197	6.73151994	6.60092268	5.57552764



Figure(6): The absolute errors for the 3th order q- HAM approximate solution ($n = 1, 5, 20$ *and* 50) for problem (34) at $\alpha = 1.75$ using($h = -1.1$, $h = -4.4, h = -16.1, h = -36.5$) respectively.

5. Conclusions

In this paper, the q-homotopy analysis method (q-HAM) has been successfully employed to obtain the exact and approximate analytical solutions of both linear and nonlinear fIVPs. The result reveals that the series solutions become more accurate with the increase of n.The reliability of q-HAM and the reduction in computations give q-HAM a wider applicability.

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