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Lower bounds of the number of real Roots of random algebraic polynomials

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Abstract . let $N_n(\omega)$ be the number of real roots of the equation $\sum_{\nu=0}^{n} a_\nu \xi_\nu(\omega) x^\nu = 0$ where $\xi_\nu(\omega)$'s are identically distributed random variables with mean zero and joint density function particularly defined and a_ν's are nonzero real numbers which are finite . It is shown that for any sequence of positive constants ($\epsilon_n, n \geq 0$) satisfying $\epsilon_n \to 0$ and $\epsilon_n^2 \log n \to \infty$, we have

$$\Pr\left\{\inf_{n > n_0} N_n(\omega)/\log n < \epsilon_n\right\} < \mu\left(\epsilon_{n_0} \log n_0\right)^{-1}$$

For all n_0 sufficiently large and positive constant μ .

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1 . Introduction .

Let $N_n(\omega)$ be the number of real roots of the algebraic equation

(1.1) $$f(x, \omega) = \sum_{\nu=0}^{n} a_\nu \xi_\nu(\omega) x^\nu = 0$$

where $\xi_\nu(\omega)$'s are random variables assuming real values only . Several authors have estimated bounds for $N_n(\omega)$ when the random variables satisfy different distribution laws . **Littlewood and Offord [1]** made the first attempt in this direction. They considered the cases when the $\xi_\nu(\omega)$'s are normally distributed or uniformly distributed in (-1 , 1) or assume only the values -1 and +1 with equal probability . They obtained in each case that

$$\Pr\left(N_n(\omega) > \mu\Lambda_n\right) > 1 - A/\log n$$

for large n where Λ_n = (log n)/ (log log log n) and μ , A are positive constants. **Samal [3]** has considered the general case when $\xi_\nu(\omega)$'s have identical distribution with expectation zero , the variance and the absolute moment finite and nonzero . He has shown that $N_n(\omega) \geq \epsilon_n \log n$ outside an exceptional set whose measure tends to zero as n tends to infinity where $\epsilon_n \to 0$ but $\epsilon_n \log n \to \infty$.
Evans [4] was the first to obtain ' strong result ' for this bound . He showed that the exceptional set could be independent of n , the degree of polynomial . He has proved that in case of normally distributed independent coefficients there exists an integer n_0 such that for $n > n_0$,

$$(1.2) \qquad \Pr\left(N_n(\omega) > \mu\Lambda'_n\right) > 1 - A\left(\log\log n_0\right)/\log n_0$$

where A is a positive constant and Λ'_n = (log n)/ (log log n) . Subsequently , **Samal and Pratihari ([5] & [6])** have improved the ' strong result ' for the lower bound . Considering random equations with independent coefficients they have shown that for $n > n_0$,

$$(1.3) \qquad \Pr\left\{\inf_{n > n_0} N_n(\omega)/\log n < \epsilon_n\right\} < \mu/\left(\epsilon_{n_0} \log n_0\right)$$

where $\epsilon_n \to 0$ but $\epsilon_n^2 \log n \to \infty$. The results of **Evans [4]** is a particular case of (1.3) for it is obtained from it by replacing $\epsilon_n = (\log\log n)^{-1}$. On the other hand their exceptional set is much smaller . Recently , **Sambadham, M. and Renganathan [7]** have obtained the same lower bound for $N_n(\omega)$ as Evans when the random coefficients $\xi_\nu(\omega)$'s are normally distributed with mean zero and joint density function defined by

$$(1.4) \qquad |M|^{\frac{1}{2}}.(2\Pi)^{-n/2}.\exp\left\{-(1/2)\bar{a}' M \bar{a}\right\}$$

where M^{-1} is the moment matrix with $\sigma_i = 1$; $\rho_{ij} = \rho$, $0 < \rho < 1$, $i \neq j$; i , j = 0 ,1 ,2 , . . . , n and $\bar{a}'$ is the transpose of the column vector $\bar{a}$. Another estimate of the lower bound for $N_n(\omega)$ is due to **Nayak and Patanayak [8].** Considering the polynomial

$$(1.5) \qquad f(x, \omega) = \sum_{\nu=0}^{n} a_\nu \xi_\nu(\omega) x^\nu$$

where a_ν's are nonzero real numbers such that $(k_n / t_n) = O(\log n)$; $k_n = \max_\nu |a_\nu|$, $t_n = \min_\nu |a_\nu|$ and $\xi_\nu(\omega)$'s are identically distributed dependent random variables with mean zero and the joint density function given by (1.4), they have shown that there exists a positive integer n_0 such that for $n > n_0$,

$$\boxed{N_n(\omega) > \mu \log n / \log\{(k_n/t_n)\log n\}}$$

the exceptional set being at most $\mu'\{\log((k_{n0}/t_{n0})\log n_0)/\log n_0\}^{1/2}$.

In this paper our object is to improve the ' strong result ' for the lower bound of $N_n(\omega)$ in case of the dependent coefficients . In fact , considering the random polynomial (1. 5) with dependent coefficients under the condition (1.4) we establish that for any sequence of positive constants $(\epsilon_n, n \geq 0)$ satisfying $\epsilon_n \to 0$ and $\epsilon_n^2 \log n \to \infty$, we have

$$\Pr\left\{\begin{matrix}\inf \\ n > n_0\end{matrix} N_n(\omega)/\log n < \epsilon_n\right\} < \mu\left(\epsilon_{n_0} \log n_0\right)^{-1}$$

for all n_0 sufficiently large and a positive constant μ.

Throughout the paper n is considered to be sufficiently large in order that the inequalities are satisfied and μ 's denote positive absolute constants assuming not necessarily the same value at every place of occurrence , [x] denote the greatest integer not exceeding x . We wish to prove the following theorem .

Theorem . *Let $N_n(\omega)$ be the number of real roots of the random algebraic equation* $f(x, \omega) = \sum_{\nu=0}^{n} a_\nu \xi_\nu(\omega) x^\nu = 0$ *in which $\xi_\nu(\omega)$'s are identically distributed random variables with mean zero and joint density function given by (1.4) and a_ν's are nonzero real numbers such that* $(k_n / t_n) = O(\log n)$; *where* $k_n = \max_\nu |a_\nu|$, $t_n = \min_\nu |a_\nu|$. *Then , for any sequence of positive constants $(\epsilon_n, n \geq 0)$ satisfying* $\epsilon_n \to 0$ and $\epsilon_n^2 \log n \to \infty$, *we have*

$$\Pr\left\{\begin{matrix}\inf \\ n > n_0\end{matrix} N_n(\omega)/\log n < \epsilon_n\right\} < \mu\left(\epsilon_{n_0} \log n_0\right)^{-1}$$

for all n_0 sufficiently large and positive constant μ.

2 . Proof of the theorem . Let A and B be constants such that

(2.1) $0 < B < 1$ and $A > 1$

Let

(2.2) $$\beta_n = (t_n / k_n)\exp\{c' / (\in_n{}^2 \log n)\}$$

where c’ is a constant to be chosen later , and

(2.3) $$M_n = [\beta_n{}^2 (k_n / t_n)^2 (Ae/B)] + 1,$$ where $e = \exp(1)$

so

(2.4) $$\boxed{\mu (k_n / t_n)^2 \beta_n{}^2 \le M_n \le \mu' (k_n / t_n)^2 \beta_n{}^2}$$

We define

(2.5) $\varphi(x) = x^x$.

Let k be an integer determined by

(2.6) $$\phi(8k+7) M_n{}^{(8k+7)} \le n \le \phi(8k+11) M_n{}^{(8k+11)}$$

The first inequality gives

$$(8k+7) . \{ \log(8k+7) + \log M_n \} \le \log n$$

and ultimately $k \le \mu''(\log n)/(\log \beta_n)$. The second inequality gives

$$\log n < (8k+11) . \{ \log(8k+11) + \log M_n \}$$
$$< (8k+11)^2 + \log(8k+11) + \log M_n$$
$$< \mu k^2 \log M_n$$

so that

$$k > \mu' \{(\log n)/(\log \beta_n)\}^{1/2} .$$

Thus

$$\frac{\mu_1}{\sqrt{c'}} \in_n \log n \le k \le \frac{\mu_2}{c'} . (\in_n \log n)^2 ,$$

so that k approaches infinity less rapidly than n . We consider

$$f(x_{m,}\omega) = U_m(\omega) + R_m(\omega)$$

at the points

(2.7) $$x_m = \left\{ 1 - \frac{1}{\phi(4m+1)M\,\frac{4m}{n}} \right\}^{1/2}$$

for m = [k / 2] + 1 , [k / 2] + 2 , … , k where

$$U_m(\omega) = \sum_{\nu=\nu 1+1}^{\nu 2} a_\nu \xi_\nu(\omega) x_m{}^{\nu}$$

and $$R_m(\omega) = \left\{ \sum_{\nu=0}^{\nu 1} + \sum_{\nu=\nu 2+1}^{n} \right\} a_\nu \xi_\nu(\omega) x_m{}^{\nu}$$

where $\nu 1 = \phi(4m-1)M\,\frac{4m-1}{n}$, $\nu_2 = \phi(4m+3)M\,\frac{4m+3}{n}$

2.1 . We shall use the fact that each $\xi_\nu(\omega)$ has marginal frequency function $(1/\sqrt{2\Pi}) \exp(-\omega^2/2)$.

We shall need the following lemmas .

Lemma 1 . For $\alpha_1 > 0$, $\sigma_m > \alpha_1 t_n \, \phi(4m+1)M\,\frac{4m}{n}$ where

$$\sigma_m{}^2 = (1-\rho)\sum_{\nu=\nu 1+1}^{\nu 2} a_\nu{}^2 x_m{}^{2\nu} + \rho\left(\sum_{\nu=\nu 1+1}^{\nu 2} a_\nu x_m{}^{\nu}\right)^2 ; \quad 0 < \rho < 1.$$

Proof.

$$\sum_{\nu=\nu 1+1}^{\nu 2} a_\nu x_m{}^{\nu} > t_n \sum_{\nu = \phi(4m-1)M\frac{(4m-1)}{n}+1}^{\phi(4m+3)M\frac{(4m+3)}{n}} x_m{}^{\nu}$$

$$> t_n \left\{ \phi(4m+1)M\,\frac{4m}{n} - \phi(4m-1)M\,\frac{4m-1}{n} \right\}^{1/2} . \qquad x_m{}^{\phi(4m+1)M\frac{4m}{n}}$$

$> t_n \, \phi(4m+1) M \frac{4m}{n} \, (B / (A\sqrt{e}))$; e = exp (1) ,

where m is large . Again

$$\sum_{\nu=\nu1+1}^{\nu2} a_\nu^{\ 2} x_m^{\ 2\nu} > t^2 \phi(4m+1) M \frac{4m}{n} . \left(\frac{B}{Ae} \right)$$

so that

$$\sigma_m^{\ 2} > (1-\rho) t_n^{\ 2} \phi(4m+1) M \frac{4m}{n} . \left(\frac{B}{Ae} \right) + \rho t_n^{\ 2} \phi^2 (4m+1) M \frac{8m}{n} \left(\frac{B^2}{A^2 e} \right);$$

$$> \alpha_1^{\ 2} t_n^{\ 2} \phi^2 (4m+1) M \frac{8m}{n}$$; α_1 being a positive constant .

Hence the proof of lemma 1 is complete .

Lemma 2 .

$$\Pr\left\{ \omega : | \sum_{\nu=0}^{\nu1} a_\nu \xi_\nu(\omega) x_m^{\ \nu} | > \lambda_n \tilde{\sigma}_m \right\} < \sqrt{\frac{2}{\pi}} \frac{e^{-\lambda_n^{\ 2}/2}}{\lambda_n} = 0$$

where $\lambda_n = m^2 \beta_n$ and $\tilde{\sigma}_m^{\ 2} = (1-\rho) \sum_{\nu=0}^{\nu1} a_\nu^{\ 2} x_m^{\ 2\nu} + \rho \left(\sum_{\nu=0}^{\nu1} a_\nu x_m^{\ \nu} \right)^2$; $0 < \rho < 1$.

Proof . Let $F(x)$ be the distribution function of $\sum_{\nu=0}^{\nu1} a_\nu \xi_\nu(\omega) x_m^{\ \nu}$

Then

$$\Pr\left\{ \omega : | \sum_{\nu=0}^{\nu1} a_\nu \xi_\nu(\omega) x_m^{\ \nu} | > \lambda_n \tilde{\sigma}_m \right\}$$

$$= 1 - \left\{ F(\lambda_n \tilde{\sigma}_m) - F(-\lambda_n \tilde{\sigma}_m) \right\}$$

$$= \sqrt{\frac{2}{\pi}} \int_{\lambda_n}^{\infty} e^{-t^2/2} dt$$

$$\le \sqrt{\frac{2}{\pi}} \frac{e^{-\lambda_n^{\ 2}/2}}{\lambda_n} .$$

hence the proof of Lemma 2 is complete .

Lemma-3.

$$\Pr\left\{\omega : \left| \sum_{\nu=\nu 2+1}^{n} a_\nu \xi_\nu(\omega) x_m^{\nu} \right| > \lambda_n \tilde{\tilde{\sigma}}_m \right\} < \sqrt{\frac{2}{\pi}} \frac{e^{-\lambda_n^2/2}}{\lambda_n}$$

Where $\tilde{\tilde{\sigma}}_m^{\,2} = (1-\rho)\sum_{\nu=\nu 2+1}^{n} a_\nu^{\,2} x_m^{\,2\nu} + \rho\left(\sum_{\nu=\nu 2+1}^{n} a_\nu x_m^{\,\nu}\right)^2$;

$0 < \rho < 1$.

The proof is similar to that of Lemma 2 .

Lemma 4 . For a fixed m ,

$$\Pr\{\omega : |R_m(\omega)| < \sigma_m\} > 1 - 2\sqrt{\frac{2}{\pi}} \frac{e^{-\lambda_n^2/2}}{\lambda_n}.$$

***Proof*.** For a given m we have

$$|R_m(\omega)| < \lambda_n\left(\tilde{\sigma}_m + \tilde{\tilde{\sigma}}_m\right) .$$

Again we have

$$\sum_{\nu=0}^{\nu 1} a_\nu^{\,2} x_m^{\,2\nu} < \frac{2k_n^{\,2}\phi(4m+1)M \begin{matrix} 4m-1 \\ n \end{matrix}}{(4m+1)^2}$$

and

$$\sum_{\nu=0}^{\nu 1} a_\nu^{\,2} x_m^{\,\nu} < \frac{2k_n\phi(4m+1)M \begin{matrix} 4m-1 \\ n \end{matrix}}{(4m+1)^2}$$

Hence , for positive constants α_2 and α_3 ,

$$\tilde{\sigma}_m^{\,2} \le \frac{\alpha_2^{\,2} k_n^{\,2}}{(4m+1)^2}\left\{\phi(4m+1)M \begin{matrix} (4m-1) \\ n \end{matrix}\right\}^2$$

and similarly

$$\tilde{\tilde{\sigma}}_m^{\,2} \le \frac{\alpha_3^{\,2} k_n^{\,2}}{(4m+1)^2}\left\{\phi(4m+1)M \begin{matrix} (4m-1) \\ n \end{matrix}\right\}^2 .$$

Therefore

$$|R_m(\omega)| < \frac{\lambda_n(\alpha_2+\alpha_3)k_n\phi(4m+1)M\dfrac{(4m-1)}{n}}{(4m+1)^2}$$

$$< \frac{\lambda_n}{16m^2}\left(\frac{\alpha_2+\alpha_3}{\alpha_1}\right).\frac{k_n}{t_n}.\frac{\sigma_m}{M_n} \qquad \text{(by Lemma 1)}$$

using the statement (2.3) . Thus $|R_m(\omega)| < \sigma_m$ except for a set of measure at most $2\sqrt{2/\pi}\exp(-\lambda_n{}^2/2)/\lambda_n$ for m = [k / 2] + 1 , [k / 2] + 2 , . . . , k . Hence , Lemma 4 is proved .

Lemma 5 . *Let* $\eta_1, \eta_2, \eta_3, \ldots$ *be a sequence of independent random variables with variance* $V(\eta_i) < 1$ *for all i . Then , for each* $\epsilon > 0$,

$$\Pr\left\{\begin{matrix}\sup \\ k \geq k_0\end{matrix}\left|\frac{1}{K}\sum_{i=0}^{K}\{\eta_i - E(\eta_i)\}\right| \geq \in\right\} \leq \frac{D}{\in^2 K_0}$$

where D is a positive constant.

This form of ‘ the strong law of large numbers ’ is a consequence of **Hajek-Renye inequality [2] .**

2 . 2 . We define S_m^+ , S_m^- as sets of ω in which , respectively

$U_m(\omega) > \sigma_m$, $U_m(\omega) < -\sigma_m$.

Hence

E_m Ù F_m = ($S_{2m}^+ \cap S_{2m+1}^-$) Ù ($S_{2m}^- \cap S_{2m+1}^+$) , say ,

As in **Samal and Pratihari ([5] & [6])** . Obviously , the two sets within the braces on the right hand side are disjoint . Therefore

(2 .8) Pr (E_m Ù F_m) = ($S_{2m}^+ \cap S_{2m+1}^-$) Ù ($S_{2m}^- \cap S_{2m+1}^+$) .

We shall use the fact that $\xi_\nu(\omega)$ has marginal frequency functions $\left(1/\sqrt{2\pi}\right)\exp\left(-t^2/2\right)$. The distribution function is $\left(1/\sqrt{2\pi}\right)\int_{-\infty}^{x}\exp\left(-t^2/2\right)dt.$

The distribution function of $U_m(\omega) = \sum_{\nu=\nu1+1}^{\nu2} a_\nu \xi_\nu(\omega) x_m{}^\nu$ is

$$\left(\frac{1}{\sqrt{2\pi}\sigma_m}\right)\int_{-\infty}^{x}\exp\left\{-\left(t^2/2\sigma_m{}^2\right)\right\}dt.$$

where

$$\sigma_m{}^2 = (1-\rho)\sum_{\nu=\nu1+1}^{\nu2} a_\nu{}^2 x_m{}^{2\nu} + \rho\left(\sum_{\nu=\nu1+1}^{\nu2} a_\nu x_m{}^\nu\right)^2;$$

Therefore , the distribution function of U_m / σ_m is

$$F_m(x) = \left(1/\sqrt{2\pi}\right)\int_{-\infty}^{x} \exp\left(-t^2/2\right)dt.$$

Thus

$$F_{2m}\left(\frac{U_{2m}}{\sigma_{2m}} < -1\right) = \left(1/\sqrt{2\pi}\right)\int_{-\infty}^{-1} \exp\left(-t^2/2\right)dt. = F_{2m}(-1)$$

and

$$F_{2m}\left(\frac{U_{2m}}{\sigma_{2m}} > 1\right) = 1 - F_{2m}\left(\frac{U_{2m}}{\sigma_{2m}} \le 1\right) = 1 - F_{2m}(1)$$

Now $\Pr\left(S_{2m}{}^+ \cap S_{2m+1}{}^-\right) = \Pr\left\{\frac{U_{2m}}{\sigma_{2m}} > 1 and \frac{U_{2m+1}}{\sigma_{2m+1}} \le -1\right\}$

$$= \frac{1}{2\pi}\int_{x=1}^{\infty}\int_{y=-\infty}^{-1} \exp\left\{-\left(x^2+y^2\right)/2\right\}dxdy = \delta' \text{ , (say) ,}$$

where δ' is independent of m.
Similarly Pr ($S_{2m}{}^- \cap S_{2m+1}{}^+$) = δ'
Hence from (2.8) we get that

Pr (E_m Ů F_m) = 2 δ' = δ , (say)

Obviously δ > 0.

Let η_m , ρ_m , θ_m be defined as in **Samal and Pratihari [6]** , that is ,

$$\eta_m = \begin{cases} 1 & with \quad probability \quad \delta \\ 0 & with \quad probability \quad 1-\delta \end{cases}$$

Then quite obviously η_m' s are independent random variables with E (η_m) = δ and Var (η_m) = $\delta - \delta^2 < 1$.

Let $\rho_m = \begin{cases} 0 \quad if \quad |R_{2m}| < \sigma_{2m} \quad and \quad |R_{2m+1}| < \sigma_{2m+1} \\ 1 \quad otherwise \end{cases}$

And $\theta_m = \eta_m - \eta_m \rho_m$.

Then the number of roots in the interval (x_{2m0} , x_{2k+1}) where m_0 =[k /2] +1, must exceed $\sum_{m=m_0}^{k} \theta_m$.

2 . 3 . We define ω-sets A (ω) , B (ω) and C (ω) as in [6]. We have

$$E(\rho_m) \le \Pr(R_{2m} \ge \sigma_{2m}) + \Pr(R_{2m+1} \ge \sigma_{2m+1}).$$

By lemmas

$$\Pr\{| R_{2m} |\ge \sigma_{2m}\} < 2\sqrt{\frac{2}{\pi}}\,\frac{e^{-\lambda_n^2/2}}{\lambda_n}.$$

and

$$\Pr\{| R_{2m+1} |\ge \sigma_{2m+1}\} < 2\sqrt{\frac{2}{\pi}}\,\frac{e^{-\lambda_n^2/2}}{\lambda_n}.$$

Thus

$$E(\rho_m) < \frac{\mu' \exp(-\lambda_n^2/2)}{\lambda_n} = \frac{\mu' \exp(-m^2\beta_n^2/2)}{m^2\beta_n^2}$$

$< \mu / m^2$, (by definition of β_n).

Hence

$$\frac{1}{k-m_0+1}\sum_{m=m_0}^{k} E(\rho_m) \le \frac{1}{k-m_0+1}\sum_{m=m_0}^{k}(\mu/m^2) < \mu/m_0^2.$$

Therefore

$$\Pr\{C(\omega)\} < \frac{2\mu'}{\in}\sum_{k-m_0+1\ge k_0}(1/m_0^2).$$

Applying Lemma 5 , we have

$$\Pr\{ B(\omega) \} \le 4D/(\varepsilon^2 k_0) = \mu / k_0 .$$

Since A (ω) ⊆ B (ω) Ů C (ω) ,

$$\text{(2.9)} \qquad \Pr\{A(\omega)\} \le \mu/k_0 + \frac{2\mu'}{\in}\sum_{k-m_0+1\ge k_0}(1/m_0^2).$$

2 . 4 . If ω ∉ A (ω) ,

$$\sum_{m=m_0}^{k} \theta_m > \sum_{m=m_0}^{k} E(\eta_m) - (k - m_0 + 1) \in$$

for all k such that $k - m_0 + 1 \geq k_0$. Therefore

$$N_n(\omega) > \frac{1}{2}(\delta - \in)k \geq \frac{\mu(\delta - \in)}{2\sqrt{c'}} \in_n \log n \quad .$$

For $n > n_0$,

$$\Pr\{A(\omega)\} < \mu/k_0 + \frac{2\mu'}{\in} \sum_{k-m_0+1 \geq k_0} (1/m_0{}^2)$$

$$< \mu/k_0 < \left(\frac{\mu\sqrt{c'}}{\mu'}\right) \Big/ (\in_{n_0} \log n_0)$$

Taking $c' = (\delta - \varepsilon)^2 \mu_1{}^2 / 4$, the desired result follows.

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