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L-Edge Chromatic Number Of A Graph

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ABSTRACT

Graph colouring is one of the most popular concept in Graph Theory. The notion of "List Colouring" was introduced in 1970's by *Vizing* and *Erdös*, *Rubin* and *Taylor*. In our previous paper [2], we have defined L-edge colouring of a graph. L-edge colouring of a graph is an assignment f:E(G) $\rightarrow 2^X - \varphi$ such that no two adjacent edges receive the same label where X is the ground set. A graph G is said to be L-edge colourable if there exists an L-edge colouring of G. The minimum number of primary colours 'n' for which there is an L-edge colouring of G is called the L-edge chromatic number of G and is denoted by $\chi'_L(G)$.Some times we may use sets of specific cardinality as edge colours.In this case the L-edge chromatic number is denoted as $\chi'_{L(i,j,..)}(G)$ In this paper we determine $\chi'_{L(1,2)}(G)$ and $\chi'_{L(1,2,3)}(G)$ for some standard graphs and they are compared with $\chi'(G)$, the usual edge chromatic number of G.

Introduction

Colour is a power in our daily life. We cannot imagine a world without colour. Colour recreates our eyes and minds. Due to advanced technology we are able to use multicolours. Competing brands often use clearly different colours. Graph colorings are useful to solve various problems ranging from scheduling to the channel assignment problem. Economically L-edge colouring may be better than usual edge colouring. In this paper we have found L-edge chromatic number for some standard graphs.

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Definition1.1:

L-edge colouring of a graph is an assignment L:E(G) $\rightarrow 2^{X}-\varphi$ such that no two adjacent edges receive the same label where X is the ground set consisting of primary colours.

Definition1.2:

A graph G is said to be L-edge colourable if there exists an L-edge colouring of G.

Definition1.3:

The minimum number of primary colours 'n' for which there is an L-edge colouring of G is called the L-edge chromatic number of G and is denoted by $\chi'_{L}(G)$. Some times we may use sets of specific cardinality as edge colours.In this case the L-edge chromatic number is denoted as $\chi'_{L(i,j,...)}(G)$.If singleton sets and two element sets alone are used for edge colouring, the corresponding chromatic number is denoted by $\chi'_{L(1,2)}(G)$.

Note: $\chi'_{L(1)}(G) = \chi'(G)$.

Theorem1.4: For cycles C_n, $n \geq 3$, $\chi'_{L(1,2)}(C_n) = 2 = \Delta(C_n)$

Proof:

Let X = {1,2}. Let $\{v_1, v_2, \ldots. v_n\}$ be the vertex set of C_n. For a proper $\chi'_{L(1,2)}$-edge colouring we need at least two primary colours, since $\Delta(C_n) = 2$.

Case 1: $n = 2k, k \geq 2$.

In this case the cycle contains even number of edges.

Let $C(v_i\ v_{i+1}) = \begin{cases} \{1\} \text{ if } i \text{ is odd} \\ \{2\} \text{ if } i \text{ is even} \end{cases}$, for i = 1,2,…,2k-1

and $C(v_{2k}\ v_1) = \{2\}$

Case 2 $n = 2k+1, k \geq 1$

In this case the cycle contains odd number of edges.

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Let $C(v_i\ v_{i+1}) = \begin{cases} \{1\}\ if\ i\ is\ odd \\ \{2\}\ if\ i\ is\ even \end{cases}$, for I = 1,2,…,2k-1

and $C(v_{2k+1}\ v_1)$ = {1,2}

In both the cases no two adjacent edges receive the same colour. The number of primary colours used = 2 = $\Delta(C_n)$

$\therefore$ For cycles C_n, $\chi'_{L(1,2)}(C_n)$ = 2= $\Delta(C_n)$

Theorem1.5: For path P_n, n$\geq$2, $\chi'_{L(1,2)}(P_n)$ =2 = $\Delta(P_n)$

Proof:

Consider a path P_n, n$\geq$2. Let {v_1, v_2, …. v_n} be the vertex set and {v_1v_2, v_2v_3, …., $v_{n-1}v_n$} be the edge set of P_n. For a proper $\chi'_{L(1,2)}$ - edge colouring, we need atleast 2 primary colours and we can colour the edges of P_n as follows:

$$C(v_i\ v_{i+1}) = \begin{cases} \{1\}\ if\ i\ is\ odd \\ \{2\}\ if\ i\ is\ even \end{cases}$$

The number of primary colours used = 2

$\therefore$ For path P_n, $\chi'_{L(1,2)}(P_n)$ = 2 = $\Delta(P_n)$

Theorem1.6:

For a binary tree T, $\chi'_{L(1,2)}(T)$ = 2= $\Delta(T)-1$

Proof:

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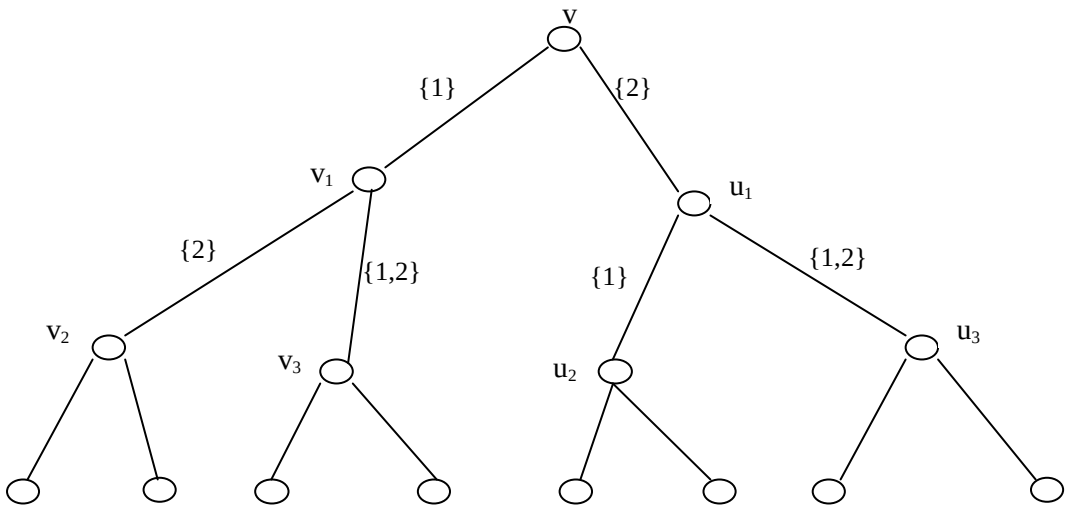

Let v be the root of the tree. Let vv_1 and vu_1, be the edges incident at v. Assign the colours {1} and {2} to the edges vv_1 and vu_1 in the first stage. In the second stage, since each vertex is of degree 3, the other two edges incident at v_1 would receive the labels {2} and {1,2}. Similarly the two edges u_1u_2, u_1u_3 incident at u_1 would receive the labels {1} and {1,2}. Suppose we have assigned labels for edges in $(n-1)^{th}$ stage. At the n^{th} stage, each vertex is of degree 3. Three edges, one in the $(n-1)^{th}$ stage and two in the n^{th} stage will be incident at each vertex. The edge in $(n-1)^{th}$ stage would have been assigned one label from $\{\{1\},\{2\},\{1,2\}\}$. The remaining two labels can be assigned to other edges.

Note:

$\chi'_{L(1,2,3)}(G) \geq 3$ for any graph G. Since $\chi'_{L(1,2)}(G) = 2$ for cycles, paths and binary trees, it is not necessary to go for a $\chi'_{L(1,2,3)}$-Edge colouring for them.

Theorem1.7:

For Peterson graph G = P(n,1), $n \geq 4$, $\chi'_{L(1,2)}(G) = 2 = \Delta(G) - 1$

Proof:

Let V = $\{v_1, v_2 \ldots v_n, \quad u_1, u_2, \ldots u_n\}$ be the vertex set and

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E=$\{v_1v_2, v_2v_3, \ldots v_{n-1}v_n, v_nv_1\} \cup \{u_1v_1, u_2v_2, \ldots, u_nv_n\} \cup$

$\{u_1u_2, u_2u_3, \ldots, u_{n-1}u_n, u_nu_1\}$ be the edge set of P(n, 1). $\Delta(G)=3$ For a proper

$\chi'_{L(1,2)}$ - edge colouring of P(n,1), we need at least two primary colours. Let X = {1,2}.

Let $C(v_i\ v_{i+1}) = \begin{cases} \{1\}\ if\ i\ is\ odd \\ \{2\}\ if\ i\ is\ even \end{cases}$, for i =1,2,…,n-1

$$C(v_n\ v_1) = \begin{cases} \{2\}\ if\ n\ is\ even \\ \{1,2\}\ if\ n\ is\ odd \end{cases}$$

and $C(u_i\ u_j) = C(v_i\ v_j)$ for all i, j.

When n is even, let $C(v_i\ u_i) = \{1,2\}$ for i = 1,2, … n

When n is odd say n = 2k+1

let $C(v_i\ u_i) = \begin{cases} \{2\}\ for\ i=1 \\ \{1,2\}\ for\ i=2,3,\ldots,2k \\ \{1\}\ for\ i=2k+1 \end{cases}$

When n is even, colours incident at $v_i = \{C(v_iv_{i+1}), C(v_{i-1}v_i), C(u_iv_i)\}$

$= \{\{1\},\{2\},\{1,2\}\}$, if n is even.

When n is odd,

Colours incident at $v_1 = \{C(v_1v_n), C(v_1v_2), C(u_1v_1)\}$

$= \{\{2\},\{1\},\{1,2\}\}$

Colours incident at $v_n = \{C(v_nv_1), C(v_nv_{n-1}), C(u_nv_n)\}$

$= \{\{1,2\},\{2\},\{1\}\}$

Colours incident at $v_i = \{C(v_iv_{i+1}), C(v_{i-1}v_i), C(u_iv_i)\}$

$= \{\{1\},\{2\},\{1,2\}\}$

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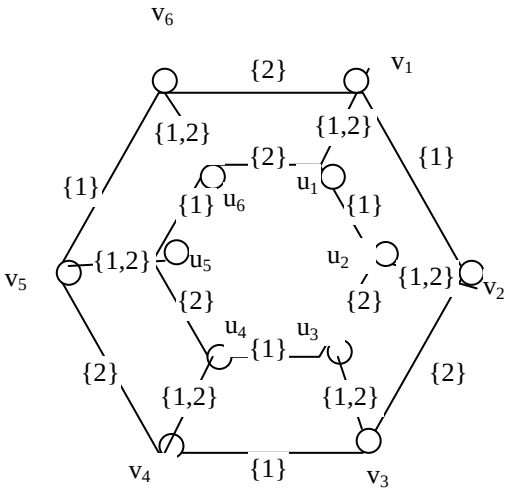
Any two adjacent edges would receive different colours and the number of primary colours used =2.

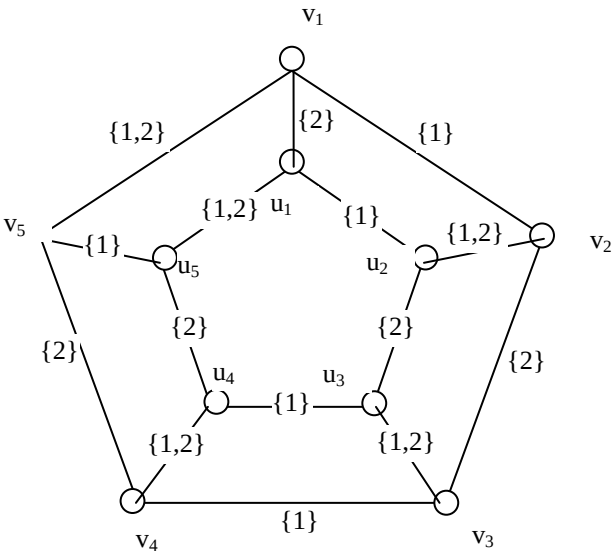
$$\therefore \quad \chi'_{L(1,2)}(G) = 2 = \Delta(G) - 1$$

Example:

$\chi'_{L(1,2)}$ -edge colouring of **P(6,1)**



$\chi'_{L(1,2)}$ -edge colouring of - **P(5,1)**



Theorem1.8:

Let G = $K_{n,m}$ be the complete bipartite graph with $n \leq m$, $m \geq 4$. Then $\chi'_{L(1,2)}(G) = k$ if $k^2 - k < 2m \leq k^2 + k$.

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Proof:

Let V = X_1UY_1 be the bipartition of $k_{n,m}$. Let $|X_1|$ = n, $|Y_1|$ = m. Let X_1 = $\{u_1,u_2,...,u_n\}$ and Y_1 = $\{v_1,v_2,...,v_m\}$. Since the graph is complete bipartite, every vertex in X_1 is adjacent to every vertex in Y_1. If there are (k-1) primary colours, we have at most $\binom{k-1}{1}+\binom{k-1}{2}$ colours for $\chi'_{L(1,2)}$ -edge colouring.

$d(u_i)$ = m for i = 1,2, …, n

∴ To colour the edges u_iv_j, j = 1,2, …, m, m primary colours are necessary.

When m > $\binom{k-1}{1}+\binom{k-1}{2}$, it is not possible for a $\chi'_{L(1,2)}$ - edge colouring with (k-1) – primary colours.

Let $C\left(u_1 v_{1+\frac{j(j-1)}{2}}\right)=\{j\}$, j = 1, 2, …, k

$C\left(u_1 v_{2+\frac{j(j-1)}{2}}\right)=\{1, j\}$, j = 2, 3, … k

$C\left(u_1 v_{3+\frac{j(j-1)}{2}}\right)=\{2, j\}$, j = 3, 4, … k

⋮

$C\left(u_1 v_{\frac{j(j+1)}{2}}\right)=\{j-1, j\}$, j = k

For $2 \le i \le n$, $C(u_iv_j)$ = $\begin{Bmatrix} C(u_1v_{m-i+j+1}), j=1,2,...,i-1 \\ C(u_1v_{j-i+1}), j=i,i+1,...,m \end{Bmatrix}$

Let A_1 = $\left\{1+\frac{j(j-1)}{2} \middle/ 1 \le j \le k\right\}$,

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$$A_2 = \left\{ 2 + \frac{j(j-1)}{2} \,/\, 2 \le j \le k \ \text{if} \ 2 + \frac{k(k-1)}{2} \le m \right\}$$
$$2 \le j \le k-1 \ \text{otherwise}$$

$$A_3 = \left\{ 3 + \frac{j(j-1)}{2} \,/\, 3 \le j \le k \ \text{if} \ 3 + \frac{k(k-1)}{2} \le m \right\}$$
$$3 \le j \le k-1 \ \text{otherwise}$$

$$\vdots \qquad\qquad \vdots$$

$$A_k = \left\{ k + \frac{j(j-1)}{2}, j = k \ \text{if} \ k + \frac{j(j-1)}{2} \le m \right\}$$

Note that $\bigcup_{l=1}^{k} A_l = \{1, 2, 3, \ldots, m\}$ and $A_s \bigcap A_t = \varphi$, $s \neq t$.

Claim: $C(u_i v_{l_1}) \neq C(u_i v_{l_2})$ for $l_1 \neq l_2$

Case 1:

Suppose $l_1, l_2 \in A_s$, $s \geq 2$.

Then $l_1 = s + \frac{j_1(j_1-1)}{2}$, $l_2 = s + \frac{j_2(j_2-1)}{2}$, for $j_1 \neq j_2$

$$C(u_i v_{l_1}) = \{s-1, j_1\}$$

$$C(u_i v_{l_2}) = \{s-1, j_2\}$$

$\therefore$ $C(u_i v_{l_1}) \neq C(u_i v_{l_2})$ Since, they differ in the second place.

Case 2:

Suppose $l_1 \in A_s$, $l_2 \in A_t$, $s,t \geq 2$, $s \neq t$

Then, $l_1 = s + \frac{j_1(j_1-1)}{2}$, $l_2 = t + \frac{j_2(j_2-1)}{2}$

(j_1 may be equal to j_2 in this case).

$$C(u_i v_{l_1}) = \{s-1, j_1\}$$

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$$C(u_i v_{l_2}) = \{t-1, j_2\}$$

$\therefore$ $C(u_i v_{l_1}) \neq C(u_i v_{l_2})$ Since, they differ in the first place.

Similar proof can be given for $l_1 \in A_1, l_2 \in A_s$ for $s \geq 2$, l_1 and $l_2 \in A_1$

Claim : $C(u_i v_j) \neq C(u_l v_j)$ for $i \neq l$

With out loss of generality let $i < l$

Case 1: $j \leq i-1$, then $j \leq l-1$

Then $C(u_i v_j) = C(u_1 v_{m-i+j+1})$ and $C(u_l v_j) = C(u_1 v_{m-l+j+1})$

$$C(u_i v_j) = C(u_l v_j)$$

$\Rightarrow C(u_1 v_{m-i+j+1}) = C(u_1 v_{m-l+j+1})$

$\Rightarrow m-i+j+1 = m-l+j+1$

$\Rightarrow i = l$

$\Rightarrow\Leftarrow$ since $i<l$

$$\therefore C(u_i v_j) \neq C(u_l v_j)$$

Case2: $i \leq j \leq l-1$

In this case, $C(u_i v_j) = C(u_1 v_{j-i+1})$ and $C(u_l v_j) = C(u_1 v_{m-l+1+j})$

$$C(u_i v_j) = C(u_l v_j)$$

$\Rightarrow C(u_1 v_{j-i+1}) = C(u_1 v_{m-l+1+j})$

$\Rightarrow j-i+1 = m-l+1+j$

$\Rightarrow m-l = -i$

$\Rightarrow m = l-i$

But l,i lie between 1 and n. So $l-i<n \Rightarrow m < n$ which is a contradiction since by our assumption $n \leq m$.

$$\therefore C(u_i v_j) \neq C(u_l v_j)$$

Case 3: $j \geq l$

Since $i < l, j \geq i$

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$\therefore$ $C(u_iv_j) = C(u_1v_{j-i+1})$ and $C(u_iv_j) = C(u_1v_{j-l+1})$

$$C(u_iv_j) = C(u_lv_j)$$

$\Rightarrow j - i + 1 = j - l + 1 \quad \Rightarrow i = l$

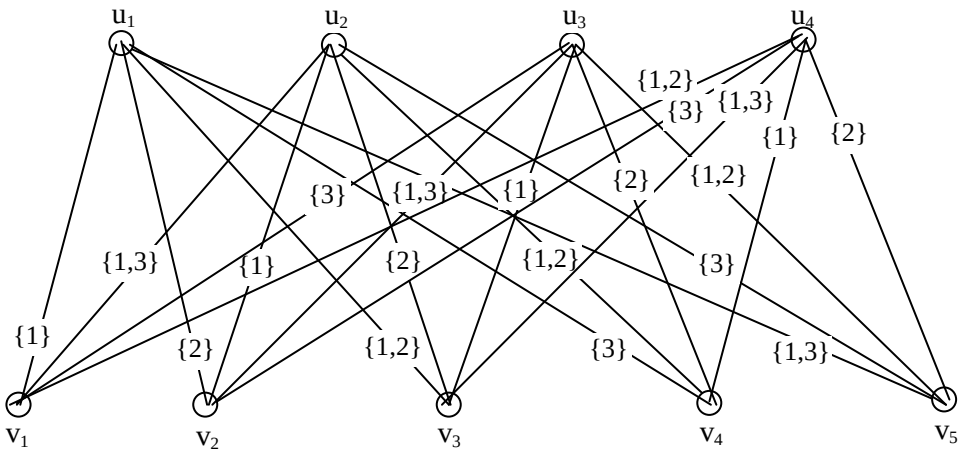
$\Rightarrow\Leftarrow$ since $i \neq l$

$\therefore$ $C(u_iv_j) \neq C(u_lv_j)$

Therefore for $k_{n,m}$, $\chi'_{L(1,2)} = k$ if $\binom{k-1}{1} + \binom{k-1}{2} < 2m \leq \binom{k}{1} + \binom{k}{2}$.

That is, if $k^2 - k < 2m < k^2 + k$.

Example: $G = k_{4,5}$



Theorem1.9:

For wheel W_n, $n \geq 4$, $\chi'_{L(1,2)}(W_n) = k$ if $k^2 - k < 2n \leq K^2 + k$.

Proof:

Let $v = \{u, v_1, v_2, \ldots, v_{n-1}, v_n\}$ be the vertex set where u is the centre vertex and $E = \{uv_1, uv_2, \ldots, uv_n\} \cup \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n\}$ be the edge set. Now colour the edges $\{uv_1, uv_2, \ldots, uv_n\}$ as follows:

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If there are (k-1) primary colours we have at most $\binom{k-1}{1}+\binom{k-1}{2}$ colours to colour the edges. d(u) = n.

$\therefore$ to colour the edges uv_j, j = 1,2, … n, n colours are necessary.

When n $>\binom{k-1}{1}+\binom{k-1}{2}$, it is not possible for a $\chi'_{L(1,2)}$ -edge colouring with (k – 1)-primary colours.

Now colour the edges $uv_1, uv_2, \ldots, uv_n$ as follows:

$C\left(uv_{\frac{i(i-1)}{2}+1}\right)=\{i\},$ i = 1, 2, …, k

$C\left(uv_{\frac{i(i-1)}{2}+2}\right)=\{1,i\},$ i = 2, 3, … k

$C\left(uv_{\frac{i(i-1)}{2}+3}\right)=\{2,i\},$ i = 3, 4, … k

$C\left(uv_{\frac{i(i-1)}{2}+4}\right)=\{3,i\},$ i = 4, 5, … k

⋮ ⋮

$C\left(uv_{\frac{i(i+1)}{2}}\right)=\{i-1,i\},$ i = k

Now we colour the edges $\{v_1v_2, v_2v_3, \ldots v_{n-1}v_n\}$ as follows:

$C(v_i v_{i+1})=C(uv_{i+2})$ for i = 1, 2, …, n-2

$C(v_{n-1} v_n)=C(uv_1)$

$C(v_n v_1)=C(uv_2)$

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Let $A_1 = \left\{1+\frac{i(i-1)}{2} \middle/ 1 \le i \le k\right\}$,

$$A_2 = \left\{\begin{matrix} 2+\frac{i(i-1)}{2} / 2 \le i \le k \ if \ 2+\frac{k(k-1)}{2} \le n \\ 2 \le i \le k-1 \ \text{otherwise} \end{matrix}\right\}$$

$$A_3 = \left\{\begin{matrix} 3+\frac{i(i-1)}{2} / 3 \le i \le k \ if \ 3+\frac{k(k-1)}{2} \le n \\ 3 \le i \le k-1 \ \text{otherwise} \end{matrix}\right\}$$

$\vdots \qquad\qquad \vdots$

$$A_k = \left\{k+\frac{i(i-1)}{2}, \middle/ i = k, k+\frac{i(i-1)}{2} \le n\right\}$$

$\bigcup_{i=1}^{k} A_i = \{1,2,3,...,n\}$

and $A_i \cap A_m = \varphi$, $i \neq m$

Claim: $C(uv_{l_1}) \neq C(uv_{l_2})$

Case 1:

Suppose $l_1, l_2 \in A_s$, $s \geq 2$

Then $l_1 = s + \frac{i_1(i_1-1)}{2}$ and $l_2 = s + \frac{i_2(i_2-1)}{2}$ for some i_1, i_2

$$C(uv_{l_1}) = \{s-1, i_1\}$$

$$C(uv_{l_2}) = \{s-1, i_2\}$$

$\therefore \quad C(uv_{l_1}) \neq C(uv_{l_2})$ Since, they differ in the second place.

Case 2:

Suppose $l_1 \in A_s$, $l_2 \in A_t$, $s \geq 2$

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Then, $l_1 = s + \frac{i(i-1)}{2}$,

$l_2 = t + \frac{i(i-1)}{2}$

$C(uv_{l_1}) = \{s-1, i\}$

$C(uv_{l_2}) = \{t-1, i\}$

$\therefore \quad C(uv_{l_1}) \neq C(uv_{l_2})$ Since, they differ in the first place.

Case 3:

If $l_1, l_2 \in A_1$, then $C(uv_{l_1}) = \{l_1\}$ and $C(uv_{l_2}) = \{l_2\}$

$\therefore \quad C(uv_{l_1}) \neq C(uv_{l_2})$

Also for $2 \leq i \leq n-2$,

colours incident at $v_i = \{C(v_{i-1}v_i), C(v_iv_{i+1}), C(uv_i)\}$

$= \{C(uv_{i+1}), C(uv_{i+2}), C(uv_i)\}$

colours incident at $v_n = \{C(v_{n-1}v_n), C(v_nv_1), C(uv_n)\}$

$= \{\{1\}, \{2\}, \{C(uv_n)\}\}$, $C(uv_n) \neq \{1\}, \{2\}$

colours incident at $v_1 = \{C(v_nv_1), C(v_1v_2), C(uv_1)\} = \{\{2\}, \{1,2\}, \{1\}\}$

colours incident at $v_{n-1} = \{C(v_{n-2}v_{n-1}), C(v_{n-1}v_n), C(uv_{n-1})\}$

$= \{C(u\ v_n), \{1\}, C(uv_{n-1})\}$

In this colouring process no two adjacent edges receive the same colour.

Hence, for wheel W_n, $n \geq 4$, $= k\ \chi'_{L(1,2)}(W_n)$ if $k^2 - k < 2n \leq K^2 + k$.

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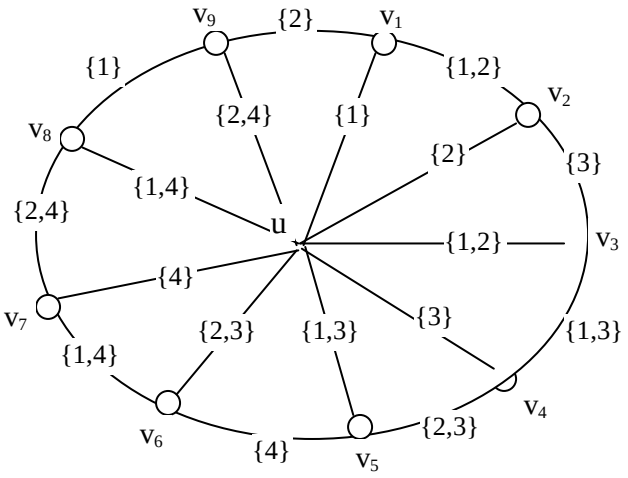
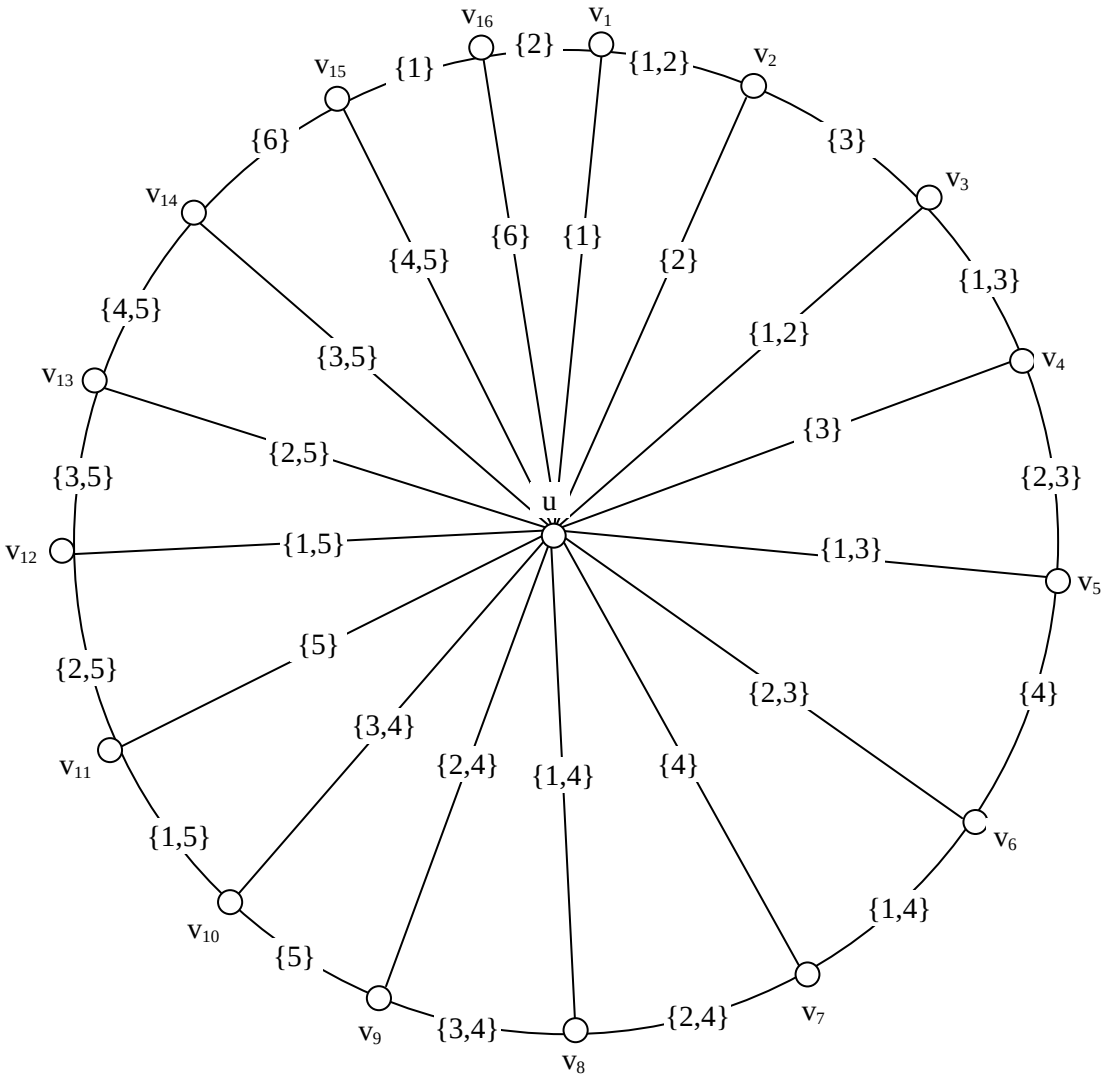
Example: $G = W_9$



Example: $\chi'_{L(1,2)}(W_{16}) = 6$. A $\chi'_{L(1,2)}$ -edge colouring of W_{16} is shown below:



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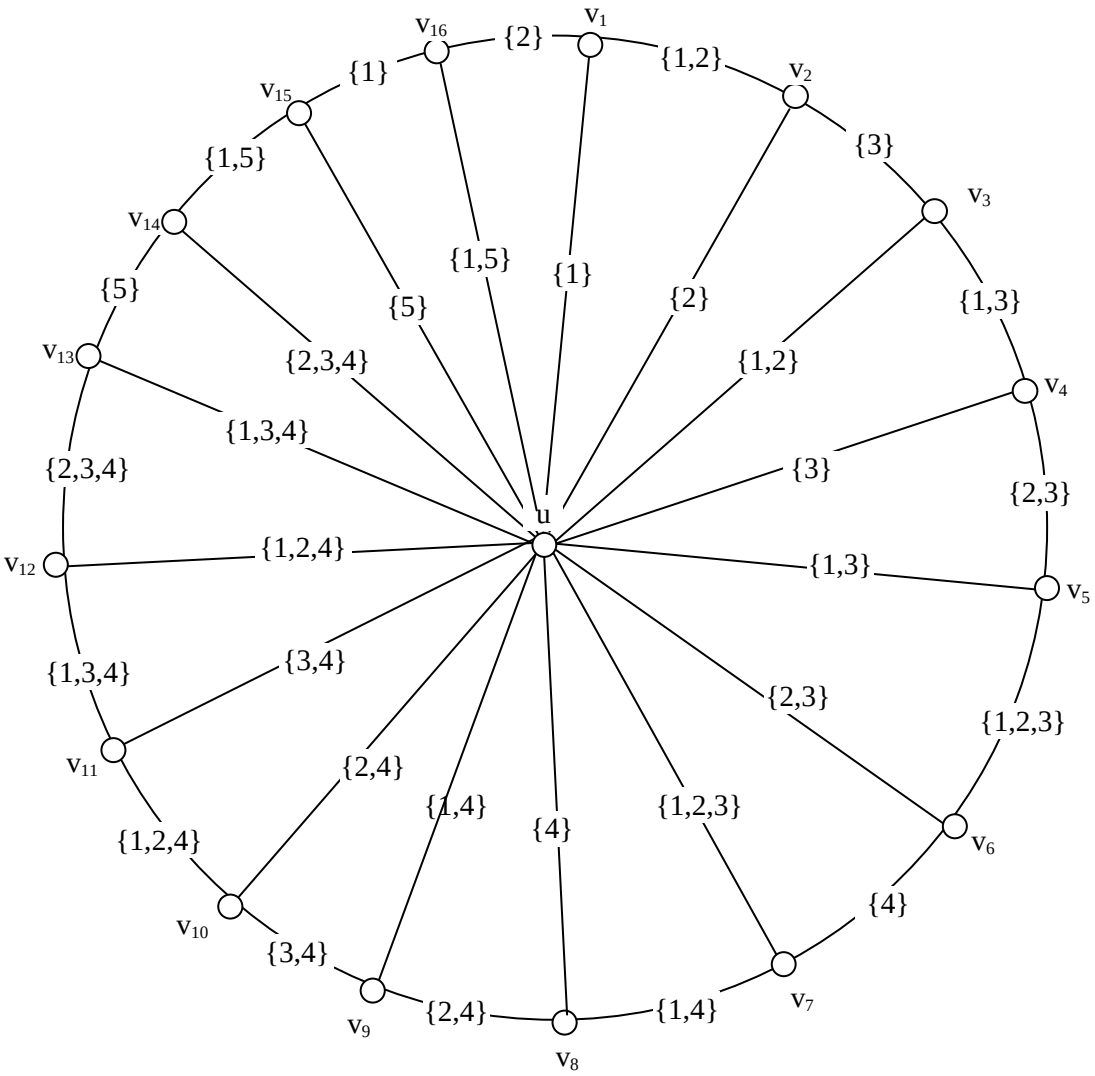
But when we go for a $\chi'_{L(1,2,3)}$-edge colouring it is enough if we use five primary colours.

Example:

A $\chi'_{L(1,2,3)}$ -edge colouring of W_{16} with five primary colours is shown below:



For the wheel W_{16} , $\chi'_{L(1,2)}(W_{16})$ = 6, whereas $\chi'_{L(1,2,3)}(W_{16})$ = 5. For such graph $\chi'_{L(1,2,3)}$-edge colouring is better than $\chi'_{L(1,2)}$-edge colouring.

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Conclusion:

Generalization of L-edge colouring for standard families may lead to several applications. In all the above results we note that $\chi'_L \leq \chi'$, *for* $L \geq 2$ and in many of them $\chi'_L < \chi'$. So our aim is to check the theorems related with edge chromatic number χ' for L-edge chromatic number χ'_L.

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