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Multi index Mittag - Leffler Function and Fractional Q-Derivative

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Abstract:

In the present paper a theorem on term by term fractional q- differentiation has been derived which an extension of the theorem is given by Yadav and Purohit. As special cases fractional q-derivative of multiindex Mittag-Leffler function and other special functions have been discussed.

Keywords and Phrases: Fractional derivative, Fractional q-derivative, Fractional Calculus, multiindex Mittag-Leffler function.

AMS Subject Classification: Primary 33A30, Secondary 33A25, 26A33, 83C99.

1. Introduction and Preliminaries:

During the last three decades Fractional Calculus has been applied to almost every field of Mathematics like Special Functions etc., Science, Engineering and Technology. Many applications of Fractional Calculus can be found in Turbulence and Fluid Dynamics, Stochastic Dynamical System, Plasma Physics and Controlled Thermonuclear Fusion, Non-linear Control Theory, Image Processing, Non-linear Biological Systems and Astrophysics.

The q-analogue of differential operator [3] is given by

$$D_{x,q} f(x) = \frac{f(xq) - f(x)}{x(q-1)} \qquad \text{...(1)}$$

If we take $q \to 1.$, we will get $\lim_{q\to 1} D_{x,q} f(x) = \frac{d}{dx} f(x) = f'(x)$

this is an inverse of the q-integral operator defined as

$$\int_x^\infty f(t)d(t;q) = x(1-q)\sum_{k=1}^\infty q^{-k} f(xq^{-k}) \qquad \text{...(2)}$$

where $0 < |q| < 1.$

The fractional q-differential operator of order α is defined as follows

$$D_{x,q}^{\alpha} f(x) = \frac{1}{\Gamma_q(-\alpha)} \int_0^x (x-yq)_{-\alpha-1} f(y) d(y;q) \qquad ...(3)$$

where $\operatorname{Re}(\alpha) < 0$.

If $f(x) = x^{\mu-1}$, then from equation (3), we get

$$D_{x,q}^{\alpha} x^{\mu-1} = \frac{\Gamma_q(\mu)}{\Gamma_q(\mu-\alpha)} x^{\mu-\alpha-1} \qquad ... (4)$$

The multiindex Mittag-Leffler function is defined by Kiryakova [9] by the power series

$$E_{(\frac{1}{\rho i}),(\mu i)}(z) = \sum_{r=0}^{\infty} \varphi_r z^r = \sum_{r=0}^{\infty} \frac{z^r}{\prod_{j=1}^{m} \Gamma(\mu_j + \frac{r}{\rho_j})} \qquad ..(5)$$

where $m > 1$ is an integer, ρ_j and μ_j are arbitrary real numbers.
The multiindex Mittag-Leffler function is an entire function and also gives its asymptotic, estimate; order and type see Kiryakova [9].

Main Results:

In this section we shall prove a theorem on term by term fractional q-differentiation of a power series. As a particular case we will obtain the fractional q-differentiation of the multiindex Mittag-Leffler function.

Theorem 1: If the series $\sum_{k=0}^{\infty} \frac{c_k x^k}{\prod_{j=1}^{m} \Gamma(\mu_j + \frac{k}{\rho_j})}$ converges absolutely for $|x| < \rho$ then

$$D_{x,q}^{\mu} \left\{ x^{\lambda-1} \sum_{k=0}^{\infty} \frac{c_n x^k}{\prod_{j=1}^{m} \Gamma(\mu_j + \frac{k}{\rho_j})!} \right\} = \sum_{k=0}^{\infty} \frac{c_k D_{x,q}^{\mu}\{x^{\lambda+k-1}\}}{\prod_{j=1}^{m} \Gamma(\mu_j + \frac{k}{\rho_j})} \qquad ...(6)$$

provided $\operatorname{Re}(\lambda) > 0$, $\operatorname{Re}(\mu) < 0, 0 < |q| < 1$.

Proof:

Taking L. H. S. of equation (6)

$$D_{x,q}^{\mu}\left\{ x^{\lambda-1}\sum_{k=0}^{\infty}\frac{c_n x^k}{\prod_{j=1}^{m}\Gamma(\mu_j+\frac{k}{\rho_j})!} \right\}$$

$$=\frac{1}{\Gamma_q(-\mu)}\int_0^x (x-yq)_{-\mu-1}\, y^{\lambda-1}\sum_{k=0}^{\infty}\frac{c_k yk}{\prod_{j=1}^{m}\Gamma(\mu_j+\frac{c}{\rho_j})}\,\mathrm{d}(y;q) \quad \{\text{by using (3)}\}$$

$$=\frac{x^{\lambda-\mu-1}}{\Gamma_q(-\mu)}\int_0^1 t^{\lambda-1}\,(1-tq)_{-\mu-1}\sum_{k=0}^{\infty}\frac{c_k t^k x^k}{\prod_{j=1}^{m}\Gamma(\mu_j+\frac{k}{\rho_j})!}\,\mathrm{d}(t;q). \qquad ...(7)$$

Now the following observations are made

(i) $\sum_{k=0}^{\infty}\frac{c_k t^k x^k}{\prod_{j=1}^{m}\Gamma(\mu_j+\frac{k}{\rho_j})}$ Converges absolutely and therefore uniformly in domain of x over the region of integration.

(ii) $\int_0^1 \left| t^{\lambda-1}(1-tq)_{-\mu-1}\right| \mathrm{d}(t;q)$ is convergent,

provided $\mathrm{Re}(\lambda)>0, \mathrm{Re}(\mu)<0, 0<|q|<1.$

Therefore the order of integration and summation can be interchanged in (7) to obtain

$$=\frac{x^{\lambda-\mu-1}}{\Gamma_q(-\mu)}\sum_{k=0}^{\infty}\frac{c_k x^k}{\prod_{j=1}^{m}\Gamma(\mu_j+\frac{k}{\rho_j})}\int_0^1 t^{\lambda+k-1}(1-tq)_{-\mu-1}\,d(t;q)$$

$$=\sum_{k=0}^{\infty}\frac{c_k}{\prod_{j=1}^{m}\Gamma(\mu_j+\frac{k}{\rho_j})}D_{x,q}^{\mu}\{x^{\lambda+k-1}\} \qquad ..(8)$$

This proves the theorem (6).

Special Cases:

Let us consider some special cases of above result:

(i) If we take $c_k = 1$ in equation (6) it reduces to fractional q-derivative of multiindex Mittag-Leffler function [9]

$$D_{x,q}^{\mu}\left\{ x^{\lambda-1}\sum_{k=0}^{\infty}\frac{x^k}{\prod_{j=1}^{m}\Gamma(\mu_j+\frac{k}{\rho_j})} \right\}=\sum_{k=0}^{\infty}\frac{1}{\prod_{j=1}^{m}\Gamma(\mu_j+\frac{k}{\rho_j})}D_{x,q}^{\mu}\{x^{\lambda+k-1}\}$$

... (9)

Or

Equivalently,

$$D_{x,q}^{\mu}\left\{ x^{\lambda-1}E_{(\frac{1}{\rho i}),(\mu i)}(z) \right\}=\sum_{k=0}^{\infty}\frac{1}{\prod_{j=1}^{m}\Gamma(\mu_j+\frac{k}{\rho_j})}D_{x,q}^{\mu}\{x^{\lambda+k-1}\}$$

... (10)

(ii) If we take m = 2, $c_k = 1$ in (6) it reduces to

$$D_{x,q}^{\mu}\left\{ x^{\lambda-1}\sum_{n=0}^{\infty}\frac{z^k}{\Gamma(\mu_1+\frac{k}{\rho_1})\Gamma(\mu_2+\frac{k}{\rho_2})} \right\}=\sum_{k=0}^{\infty}\frac{1}{\Gamma(\mu_1+\frac{k}{\rho_1})\Gamma(\mu_2+\frac{k}{\rho_2})}D_{x,q}^{\mu}\{x^{\lambda+k-1}\}$$

… (11)

which is the fractional q-derivative of the function shown by Dzrbashjan[10] that it is an entire function of order

$\rho=\frac{\rho_1\rho_2}{\rho_1+\rho_2}$ and type $\sigma=\left(\frac{\rho_1}{\rho}\right)^{\frac{\rho}{\rho_1}}\left(\frac{\rho_2}{\rho}\right)^{\frac{\rho}{\rho_2}}$

(iii) If we put $m=1, \rho_j=\frac{1}{\alpha}, \mu_j=\beta$ in (9), we get

$$D_{x,q}^{\mu}\left\{ x^{\lambda-1}\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)} \right\}=\sum_{k=0}^{\infty}\frac{1}{\Gamma(\alpha k+\beta)}D_{x,q}^{\mu}\{x^{\lambda+k-1}\}$$ … (12)

which is the fractional q- derivative of generalized Mittag-Leffler function denoted by $E_{\alpha,\beta}(z)$ see [10].

(iv) If we take $m=1, \rho_j = \frac{1}{\alpha}, \mu_j = 1$ in (9), we get

$$D^{\mu}_{x,q}\left\{ x^{\lambda-1}\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)} \right\} = \sum_{k=0}^{\infty}\frac{1}{\Gamma(\alpha k+1)} D^{\mu}_{x,q}\{x^{\lambda+k-1}\} \quad \ldots(13)$$

which is the fractional q- derivative of Mittag-Leffler function denoted by $E_{\alpha}(z)$ see [10].

(v) If we take $\alpha = 1$ in (13), we get the fractional q-derivative of the Exponential function [8] e^x

$$D^{\mu}_{x,q}\left\{ x^{\lambda-1}\sum_{k=0}^{\infty}\frac{z^k}{k!} \right\} = \sum_{k=0}^{\infty}\frac{1}{k!} D^{\mu}_{x,q}\{x^{\lambda+k-1}\} \quad \ldots (14)$$

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