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Geometric coupling of vector multiplets with *D*=4, *N*=1 supergravity

Paolo Di Sia[1,2]

[1] Department of Philosophy, Education and Psychology, University of Padova,
Verona, I-37129, Italy

[2] ISEM, Institute for Scientific Methodology,
Palermo, I-90146, Italy

Abstract
In this article we consider the coupling of the vector multiplets to supergravity in four dimensions with one supersymmetric charge. Supergravity theories are the effective theories of superstring theories. We follow the so-called "geometric approach", i.e. we use the concepts of supersymmetry, superspace, rheonomic principle and consider all fields as superforms in superspace.
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1. Introduction

In high-energy physics, theories with supersymmetry and gauge symmetries inside them are often analyzed. It is therefore of great interest to find a generalization of gauge theories including this important symmetry. A good involved candidate is the vector multiplet, which is a set of fields that can be represented in the superspace by a vector superfield.

The scalar multiplets contain quarks, leptons and the Higgs particles together with their superpartner, i.e. squarks, sleptons and Higgsinos. The gauge bosons, on the contrary, belong to vector multiplets (1, 1/2). In analogy with the ordinary Yang-Mills theory, the role of vector multiplets is to "give locality" to some global symmetries groups of the matter Lagrangian. The Lagrangian of pure supergravity has normally local supersymmetry, but admits at most one group of bosonic global symmetries [1].

These symmetries are in bijective correspondence with the isometries of the Kähler metric $g_{ij^*}(z,\bar{z})$ satisfying the additional requirement to keep invariant the Kähler potential $G(z,\bar{z})$. If $K^i{}_{(\alpha)}(z)$ is a basis of holomorphic Killing vectors for the metrics $g_{ij^*}(z,\bar{z})$, the holomorphic condition means:

$$\partial_{j^*} K^i{}_{(\alpha)}(z)=0 \Rightarrow \partial_j K^{i^*}{}_{(\alpha)}(\bar{z})=0\,; \qquad (1)$$

$$K^{i^*}{}_{(\alpha)}=(K^i{}_{(\alpha)})^* . \qquad (2)$$

The vectors $K^i{}_{(\alpha)}$ are the generators of infinitesimal holomorphic coordinate transformations:

$$\delta z^i=\varepsilon^\alpha K^i{}_{(\alpha)}(z) \qquad (3)$$

which keep invariant the metrics $g_{ij^*}(z,\bar{z})$. The vector fields:

$$\vec{K}_{(\alpha)}=K^i{}_{(\alpha)}\vec{\partial}_i \qquad (4)$$

associated to such Killing vectors close a Lie algebra:

$$\left[\vec{K}_{(\alpha)},\vec{K}_{(\beta)}\right]=h_{\alpha\beta}{}^{\gamma}\vec{K}_{(\gamma)} \qquad (5)$$

and the vectors may be normalized so that the structure constants are fully antisymmetric:

$$h_{\alpha\beta}{}^{\gamma}=h_{\alpha\beta\gamma}=h_{[\alpha\beta\gamma]}\,. \qquad (6)$$

As the metrics $g_{ij^*}(z,\bar{z})$ is the derivative of other fundamental objects, so Killing vectors in a Kähler manifold are the derivatives of a convenient prepotential:

$$\vec{K}^i{}_{(\alpha)}=i\,g^{ij^*}\partial_{j^*}\mathscr{P}_{(\alpha)}\,; \quad \mathscr{P}^*{}_{(\alpha)}=\mathscr{P}_{(\alpha)}\,. \qquad (7)$$

It is therefore possible to define a Killing vector finding a real function $\mathscr{P}_{(\alpha)}$ such that $i\,g^{ij^*}\partial_{j^*}\mathscr{P}_{(\alpha)}$ is holomorphic.

The form of the isometry transformation on fermions is:

$$\delta\chi^i=\varepsilon^\alpha\partial_j K^i{}_{(\alpha)}(z)\chi^j-\frac{i}{2}\varepsilon^\alpha f_\alpha(z)\chi^i\,; \qquad (8)$$

$$\delta\psi_\bullet=\frac{i}{2}\varepsilon^\alpha\,\mathrm{Im}\,f_\alpha(z)\psi_\bullet\,. \qquad (9)$$

We consider thus holomorphic vectors $K^i{}_{(\alpha)}(z)$ which satisfy the most restrictive condition of keeping invariant the Kähler potential:

$$\partial_i G\,K^i{}_{(\alpha)}+\partial_{i^*}G\,K^{i^*}{}_{(\alpha)}=0\,. \qquad (10)$$

In the applications to particle physics, we can have the situation in which the Killing vector is a linear function in z:

$$K^i{}_{(\alpha)}=(T_\alpha)^i{}_j z^j \Rightarrow \delta z^i=\varepsilon^\alpha(T_\alpha)^i{}_j z^j\,. \qquad (11)$$

In this case Eq. (8) becomes:

$$\delta\chi^i=\varepsilon^\alpha\,(T_\alpha)^i{}_j\chi^j\,. \qquad (12)$$

In the case of linear isometries, the prepotential of Killing vectors is expressed in terms of the first derivative of the Kähler potential [2-5]:

$$\mathscr{P}_{(\alpha)} = -i\partial_i G \ (T_\alpha)^i{}_j \chi^j . \tag{13}$$

2. The vector multiplet

If we assume the existence of a m-dimensional isometry group $\mathcal{S}$, it is possible to introduce m vector multiplets:

$$(A^\alpha, \lambda^\alpha), \tag{14}$$

that belong to the adjoint representation of $\mathcal{S}$. A^α is a bosonic 1-form, λ^α is a Majorana spinor, whose chiral projections are given by:

$$\lambda^\alpha{}_\bullet = \frac{1+\gamma_5}{2}\lambda^\alpha; \quad \lambda^{\alpha\bullet} = \frac{1-\gamma_5}{2}\lambda^\alpha \ ; \tag{15 a, b}$$

$$\bar\lambda^{\alpha\bullet} = (\lambda^\alpha{}_\bullet)^+ \gamma_0 ; \tag{16}$$

$$\bar\lambda^\alpha{}_\bullet = (\lambda^{\alpha\bullet})^+ \gamma_0 . \tag{17}$$

We denote with $\hat\nabla$ the covariant derivative with respect to Lorentz transformations, the isometry group $\mathcal{S}$, Kähler transformations and holomorphic diffeomorphisms [1]. We fix:

$$\hat\nabla z^i \equiv dz^i + A^\alpha (T_\alpha)^i{}_j z^j , \tag{18}$$

$$\hat\nabla \chi^i \equiv \nabla\chi^i + A^\alpha (T_\alpha)^i{}_j z^j , \tag{19}$$

$$\hat\nabla \lambda^\alpha{}_\bullet \equiv d\lambda^\alpha{}_\bullet - \frac{1}{4}\omega^{ab}\gamma_{ab}\lambda^\alpha{}_\bullet + h^\alpha{}_{\beta\gamma} A^\beta \lambda^\gamma{}_\bullet + \frac{i}{2} Q \lambda^\alpha{}_\bullet , \tag{20}$$

$$\hat\nabla \psi_\bullet \equiv \nabla\psi_\bullet , \tag{21}$$

having assumed that the Kähler weight of $\lambda^\alpha{}_\bullet$ is the same of that of gravitino; this is consistent with Bianchi identities. In the following we will continue to write ∇ instead of $\hat\nabla$ for simplicity. The idea is to replace the new covariant derivative ∇ eveywhere in the Bianchi identities. For resolving the Bianchi identities in presence of vector supermultiplets, it is necessary to write the rheonomic parametrization of curvatures associated to A^α and λ^α. Indicating with F^α the curvature of A^α:

$$F^\alpha \overset{def}{=} dA^\alpha + h^\alpha{}_{\beta\gamma} A^\beta \wedge A^\gamma , \tag{22}$$

we write the following rheonomic parametrization:

$$F^\alpha \overset{param}{=} F^\alpha{}_{ab} V^a \wedge V^b + \frac{i}{2}\bar\lambda^\alpha{}_\bullet \gamma_m \psi^\bullet \wedge V^m + \\ + \frac{i}{2}\bar\lambda^{\alpha\bullet}\gamma_m \psi_\bullet \wedge V^m , \tag{23}$$

where the coefficient of the sector $\psi \wedge V$ defines the supersimmetric partner λ^α of A^α and it has been then arbitrarily normalized. We write the rheonomic parametrization of $\lambda^\alpha{}_\bullet$ in the following way:

$$\nabla\lambda^\alpha{}_\bullet = \nabla_a \lambda^\alpha{}_\bullet V^a + \alpha F^{(+)\alpha}{}_{ab}\gamma^{ab}\psi_\bullet + iD^\alpha \psi_\bullet , \tag{24}$$

$$\nabla\lambda^{\alpha\bullet} = \nabla_a \lambda^{\alpha\bullet} V^a + \alpha F^{(-)\alpha}{}_{ab}\gamma^{ab}\psi^\bullet - iD^{\alpha *}\psi^\bullet , \tag{25}$$

where $F^{(+)\alpha}{}_{ab}$ and $F^{(-)\alpha}{}_{ab}$ are the self-dual and antiself-dual parts of $F^\alpha{}_{ab}$ respectively:

$$F^\alpha{}_{ab} = F^{(+)\alpha}{}_{ab} + F^{(-)\alpha}{}_{ab} , \tag{26}$$

$$\varepsilon_{abcd} F^{(\pm)\alpha cd} = \pm 2i F^{(\pm)\alpha}{}_{ab} . \tag{27}$$

For the following we will name "sector (m,n)" of an equation between forms that sector which contains m vierbeins V e n gravitinos ψ. From the Bianchi identity:

$$(\nabla F^\alpha)_{(1,2)} = 0 , \tag{28}$$

we get:

$$2iF^\alpha{}_{ab}\bar\psi^\bullet \wedge \gamma^a \psi_\bullet \wedge V^b + \frac{i}{2}\bar\psi^\bullet \wedge \gamma_m (\alpha F^{(+)\alpha}{}_{ab}\gamma^{ab} + \\ + iD^\alpha)\psi_\bullet \wedge V^m + \frac{i}{2}\bar\psi_\bullet \wedge \gamma_m (\alpha F^{(-)\alpha}{}_{ab}\gamma^{ab} - iD^{\alpha *})\psi^\bullet \wedge V^m = 0 \tag{29}$$

From this we get:

$$(D^\alpha)^* = D^\alpha , \tag{30}$$

which is a real function, called "auxiliary field", and:

$$2i(F^{(+)\alpha}{}_{ab} + F^{(-)\alpha}{}_{ab})\bar\psi^\bullet \wedge \gamma^a \psi_\bullet \wedge V^b + 2i\alpha F^{(+)\alpha}{}_{ab}\bar\psi^\bullet \wedge \\ \wedge \delta^{ma}\gamma^b \psi_\bullet \wedge V_m + 2iF^{(-)\alpha}{}_{ab}\bar\psi_\bullet \wedge \delta^{ma}\gamma^b \psi^\bullet \wedge V_m = 0 , \tag{31}$$

It follows:

$$\alpha = 1 . \tag{32}$$

With regard to the curvatures of z^i and χ^i, in accordance to what stated before, we replace d with ∇. The definitions of "gauged" curvatures of z^i and χ^i become:

$$R(z^i) \overset{def}{=} dz^i + A^\alpha (T_\alpha)^i{}_j z^j = \nabla z^i \overset{param}{=} Z^i{}_a V^a + \bar\chi^i \psi_\bullet ; \tag{33}$$

$$\nabla\chi^i \overset{def}{=} \mathscr{D}\chi^i - \frac{i}{2}Q\chi^i + \Gamma^i{}_{jk} dz^j \chi^k + A^\alpha (T_\alpha)^i{}_j \chi^j \overset{param}{=} \\ \overset{param}{=} (\nabla\chi^i)_a V^a + iZ^i{}_a \gamma^a \psi^\bullet + \mathscr{H}^i \psi_\bullet . \tag{34}$$

The Bianchi identity of z^i has the same solution as before.

3. "Off-shell" parametrization of gravitino

At this point we solve the Bianchi identities of supergravity in the presence not only of the Wess-Zumino multiplet (scalar multiplets), but also of vector multiplets. With the replacement of the new covariant derivative "∇" instead of "d", the parametrization of the gravitino curvature is [6-8]:

$$\rho_\bullet = \rho_{\bullet ab} V^a \wedge V^b + iA_a \psi_\bullet \wedge V^a + iA'_a \gamma^a{}_b \psi_\bullet \wedge V^b + \\ + S\gamma_a \psi^\bullet \wedge V^a . \tag{35}$$

The corresponding Bianchi identity is given by:

$$\mathscr{D}\rho_\bullet + \frac{1}{4}R^{ab} \wedge \gamma_{ab}\psi_\bullet + \frac{1}{2}g_{ij^*}\nabla z^i \wedge \nabla \bar z^{j^*} \wedge \psi_\bullet = 0 , \tag{36}$$

with the "off-shell" parametrization of R^{ab}:

$$R^{ab}=R^{ab}_{cd}V^c\wedge V^d+\bar\theta^{ab}{}_{c\bullet}\psi_\bullet\wedge V^c+\bar\theta^{ab\bullet}_{c}\psi^\bullet\wedge V^c-$$
$$-iS^*\bar\psi_\bullet\wedge\gamma^{ab}\psi_\bullet+iS\bar\psi^\bullet\wedge\gamma^{ab}\psi^\bullet-2iA'_c\bar\psi^\bullet\wedge\gamma_d\psi_\bullet\varepsilon^{abcd}\,. \quad (37)$$

Since the parametrization of ∇z^i is the same of that in absence of vector multiplets, the explicit form of the Bianchi identity of gravitino in the sector $\psi\wedge\psi\wedge\psi$ has that form; but it is different because in the case of only Wess-Zumino multiplet it exists only a current $\bar\chi^i\gamma_a\chi^{j^*}$ identifiable with A_a e A'_a fields, but now we can build also the current:

$$Q_{\alpha\beta}\bar\lambda^\alpha{}_\bullet\gamma_a\lambda^{\beta\bullet}\,. \quad (38)$$

It is therefore not more possible to write $A_a = 0$; we set:

$$A_a=Q_{\alpha\beta}\bar\lambda^\alpha{}_\bullet\gamma_a\lambda^{\beta\bullet}\,, \quad (39)$$

and therefore:

$$A'_a=\frac{1}{2}Q_{\alpha\beta}\bar\lambda^\alpha{}_\bullet\gamma_a\lambda^{\beta\bullet}+\frac{1}{8}g_{ij^*}\bar\chi^i\gamma_a\chi^{j^*}\,. \quad (40)$$

Let's consider the sector (1,2) of Eq. (36):

$$2i\rho_{\bullet ab}\bar\psi^\bullet\wedge\gamma^a\psi_\bullet\wedge V^b-i\psi_\bullet\wedge(\bar\psi_\bullet\nabla_{(0,1)\bullet}A_b+\bar\psi^\bullet\nabla^\bullet{}_{(0,1)}A_b)\wedge V^b-$$
$$-i\gamma^a{}_b\psi_\bullet\wedge(\bar\psi_\bullet\nabla_{(0,1)\bullet}A'_a+\bar\psi^\bullet\nabla^\bullet{}_{(0,1)}A'_a)\wedge V^b-\gamma_b\psi^\bullet\wedge \quad (41)$$
$$\wedge(\nabla_m S\bar\psi_\bullet\chi^m+\nabla_{m^*}S\bar\psi^\bullet\chi^{m^*})\wedge V^b+\frac{1}{4}\gamma_{ac}\psi_\bullet\wedge(\bar\psi_\bullet\theta^{ac}_{\bullet b}+$$
$$+\bar\psi^\bullet\theta^\bullet{}^{ac}_{b})\wedge V^b+\frac{1}{2}g_{ij^*}\psi_\bullet\wedge(\bar\psi_\bullet\chi^i\bar Z^{j^*}{}_b-\bar\psi^\bullet\chi^{j^*}Z^i{}_b)\wedge V^b=0$$

The cancellation of the current $\bar\psi^\bullet\wedge\gamma^{lm}\psi^\bullet$ brings to:

$$S=iee^{G/2}\,. \quad (42)$$

In the $\bar\psi_\bullet\wedge\gamma^{lm}\psi_\bullet$ sector we have:

$$\frac{i}{8}\bar\psi_\bullet\wedge\gamma^{lm}\psi_\bullet\gamma_{lm}(\nabla_{(0,1)\bullet}A_b)+\frac{i}{8}\gamma^a{}_b\gamma_{lm}(\nabla_{(0,1)\bullet}A'_a)$$
$$\bar\psi_\bullet\wedge\gamma^{lm}\psi_\bullet-\frac{1}{32}\gamma_{ac}\gamma_{lm}\theta^{ac}_{\bullet b}\bar\psi_\bullet\wedge\gamma^{lm}\psi_\bullet-$$
$$-\frac{1}{16}g_{ij^*}\gamma_{lm}\chi^i\bar Z^{j^*}{}_b\bar\psi_\bullet\wedge\gamma^{lm}\psi_\bullet=0\,. \quad (43)$$

Multiplying both sides of Eq. (43) by γ^{lm} and considering the gamma-matrices algebra, we get:

$$-\frac{3}{2}\nabla_{(0,1)\bullet}A_b+\frac{i}{2}\gamma^a{}_b\nabla_{(0,1)\bullet}A'_a-$$
$$-\frac{1}{8}\gamma_{ac}\theta^{ac}_{\bullet b}+\frac{3}{4}g_{ij^*}\chi^i\bar Z^{j^*}{}_b=0\,, \quad (44)$$

with A_a, A'_a and $\theta^{ac}_{\bullet b}$ given by Eqs (39), (40) respectively:

$$\theta^{ac/b}_\bullet=-2i\gamma^{[a}\rho^{c]b}{}_\bullet+i\gamma^b\rho^{ac\bullet}\,, \quad (45)$$

and $Q_{\alpha\beta}$ fucntion of fields z^i, $\bar z^{i^*}$. In the sector of one-index current of Eq. (36) we have lastly:

$$2i\rho_{\bullet ab}-\frac{i}{2}\gamma_a(\nabla^\bullet{}_{(0,1)}A_b)-\frac{i}{2}\gamma^l{}_b\gamma_a(\nabla^\bullet{}_{(0,1)}A'_l)-$$
$$-\frac{ie}{4}\gamma_b\gamma_a\chi^m(\partial_m G)\,e^{G/2}+\frac{1}{8}\gamma_{lm}\gamma_a\theta^{\bullet lm}{}_b-$$
$$-\frac{1}{4}g_{ij^*}Z^i{}_b\gamma_a\chi^{j^*}=0\,. \quad (46)$$

Eq. (46), by multiplication with γ^a, leads to the gravitino equation of motion, which now includes also the coupling to the vector multiplets. It is:

$$2i\gamma^a\rho_{\bullet ab}-2i(\nabla^\bullet{}_{(0,1)}A_b)+\frac{ie}{2}\gamma_b\chi^m(\partial_m G)\,e^{G/2}-$$
$$-g_{ij^*}Z^i{}_b\chi^{j^*}=0\,, \quad (47)$$

having used the gamma-matrices algebra. The spinor derivative of A_b results:

$$(\nabla^\bullet{}_{(0,1)}A_b)=\nabla^\bullet{}_{(0,1)}(Q_{\alpha\beta}\bar\lambda^\alpha{}_\bullet\gamma_b\lambda^{\beta\bullet})=$$
$$=(\partial_i Q_{\alpha\beta}\chi^i+\partial_{i^*}Q_{\alpha\beta}\chi^{i^*})\bar\lambda^\alpha{}_\bullet\gamma_b\lambda^{\beta\bullet}+ \quad (48)$$
$$+Q_{\alpha\beta}(-4\overset{(-)}{F}{}^\beta{}_{bn}\gamma^n+iD^\beta\gamma_b)\lambda^\alpha{}_\bullet\,,$$

considering the parametrization of $\lambda^\alpha{}_\bullet$, $\lambda^{\beta\bullet}$ and:

$$(\nabla_{(0,1)})_\bullet\lambda^{\alpha\bullet}=0\,, \quad (49)$$

$$(\nabla_{(0,1)})^\bullet\lambda^\alpha{}_\bullet=0\,. \quad (50)$$

The motion equation results:

$$2i\gamma^a\rho_{\bullet ab}-2i((\partial_i Q_{\alpha\beta}\chi^i+\partial_{i^*}Q_{\alpha\beta}\chi^{i^*})\bar\lambda^\alpha{}_\bullet\gamma_b\lambda^{\beta\bullet}+$$
$$+Q_{\alpha\beta}(-4\overset{(-)}{F}{}^\beta{}_{bn}\gamma^n+iD^\beta\gamma_b)\lambda^\alpha{}_\bullet)+\frac{ie}{2}\gamma_b\chi^m(\partial_m G)\,e^{G/2}-$$
$$-g_{ij^*}Z^i{}_b\chi^{j^*}=0\,. \quad (51)$$

4. Bianchi identities of z^i and χ^i

We analize now the new Bianchi identities of the Wess-Zumino multiplet [6-8]. In relation to χ^i we have:

$$\nabla\chi^i\overset{def}{=}\mathscr{D}\chi^i-\frac{i}{2}Q\chi^i+\Gamma^i{}_{jk}dz^j\chi^k+A^\alpha(T_\alpha)^i{}_j\chi^j\overset{param}{=}$$
$$\overset{param}{=}(\nabla\chi^i)_aV^a+iP^i{}_a\gamma^a\psi^\bullet+\mathscr{H}^i\psi_\bullet+iL^i{}_{ab}\gamma^{ab}\psi_\bullet\,. \quad (52)$$

The Bianchi identity of $R(z^i)$ results:

$$\nabla^2 z^i=F^\alpha(T_\alpha)^i{}_jz^j=\nabla Z^i{}_a\wedge V^a+iZ^i{}_a\bar\psi_\bullet\wedge\gamma^a\psi^\bullet-$$
$$-\bar\psi_\bullet\wedge\nabla\chi^i+\bar\chi^i\rho_\bullet\,. \quad (53)$$

The (0,2) sector brings to:

$$P^i{}_a=Z^i{}_a\,;\quad\mathscr{H}^i=\text{free};\quad L^i{}_{ab}=0\,. \quad (54)$$

From (0,1) sector it is possible to find the spinor derivative of $Z^i{}_a$:

$$(\nabla_{(0,1)}Z^i{}_a)^\bullet=S\gamma_a\chi^i-\frac{i}{2}\gamma_a\lambda^\alpha{}_\bullet(T_\alpha)^i{}_jz^j\,; \quad (55)$$

$$(\nabla_{(0,1)}Z^i{}_a)_\bullet=\nabla_a\chi^i-iA_a\chi^i+iA'_b\gamma^b{}_a\chi^i-$$
$$-\frac{i}{2}\gamma_a\lambda^{\alpha\bullet}(T_\alpha)^i{}_jz^j\,. \quad (56)$$

For $\nabla_{(0,1)}\bar Z^{i^*}{}_a$ similarly we get:

$$(\nabla_{(0,1)}\ \bar{Z}^{i^*}{}_a)_\bullet = -S^*\gamma_a\chi^{i^*} - \frac{i}{2}\gamma_a\lambda^{\alpha\bullet}\ (T_\alpha)^{i^*}{}_{j^*}z^{j^*}\ ; \quad (57)$$

$$(\nabla_{(0,1)}\ \bar{Z}^{i^*}{}_a)^\bullet = \nabla_a\chi^{i^*} - iA_a\chi^{i^*} + iA'_b\gamma^b{}_a\chi^{i^*} - -\frac{i}{2}\gamma_a\lambda^\alpha{}_\bullet\ (T_\alpha)^{i^*}{}_{j^*}z^{j^*}\ . \quad (58)$$

Let's consider now the Bianchi identity of χ^i:

$$\nabla^2\chi^i = \nabla(\nabla_a\chi^i V^a + iZ^i{}_a\gamma^a\psi^\bullet + \mathcal{H}^i\ \psi_\bullet)\ . \quad (59)$$

We note that the auxiliary field $\mathcal{H}^i$ appearing in the parametrization (59) generally does not coincide with the found value without coupling to vector multiplets. Therefore we write:

$$\mathcal{H}^i = \mathcal{H}^{\wedge\, i} + \Delta\mathcal{H}^i, \quad (60)$$

where $\mathcal{H}^{\wedge\, i}$ is the expression in absence of gauge fields, whereas $\Delta\mathcal{H}^i$ has in general the form:

$$\Delta\mathcal{H}^i = N^i{}_{\alpha\beta}(z,\bar{z})\bar{\lambda}^{\alpha\bullet}\lambda^{\beta\bullet} + M^i{}_{\alpha\beta}(z,\bar{z})\bar{\lambda}^\alpha{}_\bullet\lambda^\beta{}_\bullet, \quad (61)$$

which appears to be the rheonomic form compatible with the rigid scale "- 1" and with the Kähler scale "- 1".

Computing the (0,2) sector of Eq. (59) and considering the $\bar{\psi}_\bullet\wedge\gamma_{ab}\psi_\bullet$ sector, we get:

$$\frac{1}{8}\nabla_l\ \mathcal{H}^{\wedge\, i} - \frac{i}{4}\delta^i{}_l S^* = 0\ . \quad (62)$$

We have also:

$$(\nabla_{(0,1)}\ \Delta\mathcal{H}^i)_\bullet = 0\ ; \quad (63)$$

$$\Rightarrow \nabla_{(0,1)\bullet}(\ N^i{}_{\alpha\beta}(z,\bar{z})\bar{\lambda}^{\alpha\bullet}\lambda^{\beta\bullet} + M^i{}_{\alpha\beta}(z,\bar{z})\bar{\lambda}^\alpha{}_\bullet\lambda^\beta{}_\bullet) = 0 \quad (64)$$

From Eq. (64), considering that Eq. (49) holds, it results:

$$M^i{}_{\alpha\beta} = 0\ ; \quad (65)$$

$$\nabla_l\ N^i{}_{\alpha\beta} = 0\ . \quad (66)$$

Multiplying both sides of Eq. (66) with g_{ij^*} we get:

$\nabla_l(\ g_{ij^*}N^i{}_{\alpha\beta}) = 0 = \nabla_l\ N_{j^*\alpha\beta} = \partial_l N_{j^*\alpha\beta} +$ terms in which it appears the Riemann affine connection with indexes that are never "all starry" or "all not starry". This implies that all these additional terms are zero; It is therefore:

$$\partial_l N_{j^*\alpha\beta} = 0\ , \quad (67)$$

i.e. $N_{j^*\alpha\beta}$ is an anti-holomorphic function.

In the sector $\bar{\psi}^\bullet\wedge\gamma^{ab}\psi^\bullet$, considering the gamma-matrices algebra, we have identically zero, and the same holds for the term $\gamma^a\psi^\bullet\wedge\bar{\psi}^\bullet\gamma_a\ \lambda^\alpha{}_\bullet\ (T_\alpha)^{i^*}{}_j z^j$. Also the term $\gamma_{ab}\chi^i S\ \bar{\psi}^\bullet\wedge\gamma^{ab}\psi^\bullet$ is zero because $\gamma_{ab}\chi^i$ and $\bar{\psi}^\bullet\wedge\gamma^{ab}\psi^\bullet$ have opposite self-duality. The study of the sector $\bar{\psi}^\bullet\wedge\gamma^a\psi_\bullet$ brings to an implicit motion equation, where we don't know the functions $N^i{}_{\alpha\beta}$ and D^β.

5. Bianchi identities of F^α

We analyze now the Bianchi identities of the vector multiplet of significant interest [6-8]. Considering the Fierz identities, the (0,3) sector gives identically 0 = 0; similarly we can say for the (1,2) sector. The (2,1) sector gives the spinor derivative of $F^\alpha{}_{ab}$. Contracting $(\nabla_{(0,1)}\overset{(+)}{F^\alpha{}_{ab}})$ with γ^{ab}, we get:

$$(\nabla_{(0,1)}\overset{(+)}{F^\alpha{}_{ab}}\gamma^{ab})_\bullet = \frac{i}{2}\gamma^{ab}\gamma_a\nabla_b\lambda^{\alpha\bullet} - 6iS^*\lambda^\alpha{}_\bullet - -\frac{3}{2}A^a\gamma_a\lambda^{\alpha\bullet} - \frac{3}{2}A'^a\gamma_a\lambda^{\alpha\bullet}\ . \quad (68)$$

We note also that:

$$(\nabla_{(0,1)}F^\alpha{}_{ab})_\bullet \equiv (\nabla_{(0,1)}\overset{(+)}{F^\alpha{}_{ab}})_\bullet\ , \quad (69)$$

$$(\nabla_{(0,1)}F^\alpha{}_{ab})^\bullet \equiv (\nabla_{(0,1)}\overset{(-)}{F^\alpha{}_{ab}})^\bullet\ , \quad (70)$$

because:

$$(\nabla_{(0,1)}\overset{(-)}{F^\alpha{}_{ab}})_\bullet = 0\ , \quad (71)$$

$$(\nabla_{(0,1)}\overset{(+)}{F^\alpha{}_{ab}})^\bullet = 0\ . \quad (72)$$

6. Bianchi identity of gaugino

The Bianchi identity of gaugino is given by:

$$\nabla^2\lambda^\alpha{}_\bullet = \nabla(\nabla_a\lambda^\alpha{}_\bullet V^a + \overset{(+)}{F^\alpha{}_{ab}}\gamma^{ab}\psi_\bullet + iD^\alpha\psi_\bullet)\ . \quad (73)$$

Considering the (0,2) sector of Eq. (73) and of this the sector $\bar{\psi}_\bullet\wedge\gamma_{lm}\psi_\bullet$, we get:

$$\frac{i}{4}\gamma^{lm}\lambda^\alpha{}_\bullet S^*\ \bar{\psi}_\bullet\wedge\gamma_{lm}\psi_\bullet = \frac{1}{8}\gamma^{ab}\gamma^{lm}\ (\nabla_{(0,1)\bullet}\overset{(+)}{F^\alpha{}_{ab}})\ \bar{\psi}_\bullet\wedge\gamma_{lm}\psi_\bullet + \frac{i}{8}\gamma^{lm}\ (\nabla_{(0,1)\bullet}D^\alpha)\ \bar{\psi}_\bullet\wedge\gamma_{lm}\psi_\bullet\ . \quad (74)$$

Multiplying both members of Eq. (74) by γ_{lm} and using the gamma-matrices algebra, we get:

$$-3i\lambda^\alpha{}_\bullet S^* = \frac{1}{2}\gamma^{ab}\ (\nabla_{(0,1)\bullet}\overset{(+)}{F^\alpha{}_{ab}}) - \frac{3}{2}i\ (\nabla_{(0,1)\bullet}D^\alpha)\ . \quad (75)$$

Let's do now the following ansatz, which is the only one compatible with rheonomy, Kähler weight "0" and scale weight "- 1":

$$D^\alpha = D^{\wedge\alpha} + L^\alpha{}_{\beta i}\bar{\chi}^i\lambda^\beta{}_\bullet + \bar{L}^\alpha{}_{\beta i^*}\bar{\chi}^{i^*}\lambda^{\beta\bullet}\ . \quad (76)$$

Computing the quantity $\overline{(\nabla_{(0,1)}D^\alpha)_\bullet\ \psi_\bullet}$ and using it, togheter with Eq. (76) inside Eq. (75), we get the motion equation of $\lambda^{\alpha\bullet}$. We can get the motion equation of $\lambda^\alpha{}_\bullet$ from the sector $\bar{\psi}_\bullet\wedge\gamma^a\psi^\bullet$ of the (0,2) sector of Eq. (73).

Computing the spinor derivative of $\nabla\!\!\!\!/\, \lambda^{\alpha\bullet}$, i.e. $(\nabla_{(0,1)\bullet}(\nabla\!\!\!\!/\,\lambda^{\alpha\bullet}))$, in two different way:

a) using the motion equation of $\lambda^{\alpha\bullet}$,

b) directly from the Bianchi identity of gaugino (it is the complex conjugate of Eq. (73)) in the (1,1) sector, and equating the two results, we get informations on the introduced arbitrary functions $L^{\alpha}{}_{\beta i}$, $Q_{\alpha\beta}$ e $N^{i}{}_{\alpha\beta}$.

Considering that $S = i\, e \exp(G/2)$,

$\mathscr{H}^i = 2e(g^{ij^*}\partial_{j^*}G)\exp(G/2)$,

and putting:

$$D^{\wedge\alpha} = (M^{-1})^{\alpha\beta}\,\mathscr{P}_{\beta}, \quad (77)$$

where $\mathscr{P}_{\beta}$ is the prepotential of the Killing vectors (Eq. (7)), in the case of *m* multiplets, we get:

$$L^{\alpha}{}_{\beta i} = -\frac{i}{2}(M^{-1})^{\alpha}{}_{\gamma}\,\partial_i M^{\gamma}{}_{\beta}\,. \quad (78)$$

The *M* function is a harmonic function, i.e. the real part of an analytic function:

$$M = \frac{1}{2}(f(z) + \bar{f}(\bar{z}))\,. \quad (79)$$

It is:

$$\partial_{k^*} M = \frac{1}{2}\partial_{k^*}\bar{f}(\bar{z})\,, \quad (80)$$

therefore:

$$N_{k^*} = \frac{1}{4}\partial_{k^*}\bar{f}(\bar{z})\,. \quad (81)$$

The found results is generalizable in the case of *m* vector multiplets as:

$$N_{k^*\alpha\beta} = \frac{1}{4}\partial_{k^*}\bar{f}_{\alpha\beta}(\bar{z})\,. \quad (82)$$

With $Q_{\alpha\beta}$ all the unknown functions that appear in the initial ansatz on the parameterization of curvatures are determined in terms of the analytic function $f_{\alpha\beta}(z)$. To determine this latter function, we analyze a sector in which explicitly appear the auxiliary fields of supergravity [1]. We obtain:

$$Q_{\alpha\beta} = M_{\alpha\beta} = \mathrm{Re}\, f_{\alpha\beta}\,. \quad (83)$$

7. Conclusions

The obtained results from the analysis of the Bianchi identity of the gaugino equation can be synthesized in the following way:

- from the equation:

$$(\nabla_{(0,1)}(\nabla\!\!\!\!/\,\lambda^{\alpha\bullet}))_{\bullet\,\substack{Equations\\ of\ motion}} = (\nabla_{(0,1)}(\nabla\!\!\!\!/\,\lambda^{\alpha\bullet}))_{\bullet\,\substack{Bianchi\\ identities}}$$

we get, in the $1\psi_{\bullet}$ sector:

a) *Bosonic terms*:

$$L^{\alpha}{}_{\beta i} = -\frac{i}{2}(M^{-1})^{\alpha}{}_{\gamma}\,\partial_i M^{\gamma}{}_{\beta}\,. \quad (84)$$

b) *Bilinear terms in* $\chi\chi 1\psi_{\bullet}$: we get an identically satisfied equation with the previous positions.

c) *Bilinear terms in* $\chi\lambda 1\psi_{\bullet}\bar{Z}^{i^*}{}_{a}$:

$$\partial_l\,\partial_{i^*} M^{\alpha}{}_{\beta} = 0\,. \quad (85)$$

d) *Terms in* $\lambda\lambda\, boson\, 1\psi_{\bullet}$:

$$N_{k^*\alpha\beta} = \frac{1}{4}\partial_{k^*}\bar{f}_{\alpha\beta}(\bar{z})\,, \quad (86)$$

where:

$$M_{\alpha\beta} = \mathrm{Re}\, f_{\alpha\beta}\,. \quad (87)$$

e) $\chi\lambda\lambda\lambda$ *sector*:

$$Q_{\alpha\beta} = \mathrm{Re}\, f_{\alpha\beta}\,. \quad (88)$$

The equations of motion of χ^i and $\lambda^{\alpha\bullet}$, written in implicit form, can be made explicit, having determined the form of the auxiliary fields $\mathscr{H}^i$, D^{α}, S, A_a, A'_a.

To derive the equations of motion of all fields of the theory it is easier to use the variational principle, writing the complete Lagrangian which includes the interaction of the system “supergravity + Wess-Zumino multiplets” with the “vector multiplets”.

The complete theory of $N = 1$ supergravity, as effective theory of the heterotic superstring compactified in 4 dimensions, ultimately contains two arbitrary functions: the real function $G(z,\bar{z})$, i.e. the Kähler potential of the Kähler manifold of Wess-Zumino multiplets and the analytic function $f_{\alpha\beta}(z)$. Information on them can be obtained from the analysis of the fundamental superstring theory, of which supergravity is the effective theory.

The presented results, obtained through a geometric formulation of the coupling “supergravity + matter”, are in complete agreement with the results obtained by means of the superconformal tensor calculus in the components approach [1,9-13].

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Paolo Di Sia is currently Adjunct Professor by the University of Padova (Italy), Professor at ISSR in Bolzano (Italy) and Member of ISEM (Institute for Scientific Methodology) in Palermo (Italy). He obtained a 1st level Laurea in Metaphysics, a 2nd level Laurea in Theoretical Physics, a PhD in Mathematical Modelling applied to Nano-Bio-Technology. He interested in Classical Quantum Relativistic Nanophysics, Nano-Neuroscience, Planck Scale Physics, Supergravity, History and Philosophy of Science, Science Education. He wrote 207 publications at today, is reviewer of 2 mathematics books, reviewer of 12 international journals, 8 Awards obtained, included in Who's Who in the World 2015 and 2016, member of 7 international scientific societies, member of 27 International Advisory/Editorial Boards.
www.paolodisia.com