IJISET - International Journal of Innovative Science, Engineering & Technology, Vol. 4 Issue 5, May 2017
**ISSN (Online) 2348 – 7968 | Impact Factor (2016) – 5.264**
www.ijiset.com

# Order Statistics from Power Lomax Distribution

**Ibrahim B. Abdul-Moniem**

Department of Statistics, Higher Institute of Management sciences in Sohag,
Sohag, Egypt

**Abstract**
In this paper, we present recurrence relations between the single and the product moments for order statistics from Power Lomax distribution. Characterizations for the Power Lomax distribution are studied. Some results of means and variances are tabulated.
***Keywords:*** *moments of order statistics, Power Lomax distribution, recurrence relations, Characterizations.*

## 1. Introduction

The probability density function (*pdf*) of the Power Lomax (PL) distribution is defined by (Rady et al. [1])

$$f(x)=\alpha\beta\lambda^{-1}x^{\beta-1}\left(1+\frac{x^{\beta}}{\lambda}\right)^{-(\alpha+1)};\ x>0,(\alpha,\beta,\lambda>0). \qquad (1)$$

The corresponding reliability (survival) function of PL distribution is given by,

$$\bar{F}(x)=\left(1+\frac{x^{\beta}}{\lambda}\right)^{-\alpha};\ x>0,(\alpha,\beta,\lambda>0). \qquad (2)$$

In view of (1) and (2), we get

$$\alpha\beta\bar{F}(x)=\left(x+\lambda x^{1-\beta}\right)f(x). \qquad (3)$$

For more details on this distribution and its application one may refer to Rady et al. [1]. K-th upper record values from power Lomax distribution introduced by Abdul-Moniem [2].

Let $X_1,X_2,...,X_n$ be a random sample of size $n$ from a continuous population having *pdf* $f(x)$ and distribution function (*df*) $F(x)$. Let $X_{1:n}\le X_{2:n}\le ...\le X_{n:n}$ be the corresponding order statistics (*OS*). The *pdf* of $X_{a:n}$, the $a^{th}$ *OS* is given by David and Nagaraja [3]

$$f_{a:n}(x)=c_{a:n}\left[F(x)\right]^{a-1}\left[1-F(x)\right]^{n-a}f(x);\ -\infty<x<\infty \qquad (4)$$

where $c_{a:n}=\dfrac{n!}{(a-1)!(n-a)!}$.

The joint *pdf* of $X_{r:n},X_{s:n};1\le r<s\le n$ is given as

$$f_{r,s:n}(x,y)=c_{r,s:n}\left[F(x)\right]^{r-1}\left[F(y)-F(x)\right]^{s-r-1}\left[\bar{F}(y)\right]^{n-s} f(x)f(y);\ -\infty<x<y<\infty \qquad (5)$$

where

$x=x_{r:n}, y=x_{s:n}$ and $c_{r,s:n}=\dfrac{n!}{(r-1)!(s-r-1)!(n-s)!}$.

Moments of *OS* for some specific distributions are investigated by several authors in the literature. Athar and Nayabuddin [4] derived exact moments of order statistics from exponentiated log-logistic distribution. Hendi et al. [5] developed recurrence relations for the single and product moments of *OS* from doubly truncated Gompertz distribution. Mohie El-Din et al. [6], [7] presented recurrence relations for the single and product moments of *OS* from the doubly truncated parabolic and skewed distribution and linear-exponential distribution. Ali and Khan [8] established the ratio and inverse moments of *OS* from Weibull and exponential distribution. Khan and Ali [9] derived the ratio and inverse moments of *OS* from Burr distribution. Balakrishnan et al. [10] reviewed several recurrence relations and identities for the single and product moments of *OS* from some specific distributions. Khan and Khan [11] obtained the moments of *OS* from Burr distribution. Balakrishnan and Malik [12] established some recurrence relations of *OS* from the linear exponential distribution. Balakrishnan and Malik [13] derived some identities involving the density functions of *OS*. These identities are useful in checking the computation of the moments of *OS*. Khan et al. [14] established general result about recurrence relations between product moments of *OS*. In this paper we have obtained simple expressions for the exact single moment of *OS* from *PL* distribution. Recurrence relations between the single and the product moments for *OS* from *PL* distribution are introduced. Characterizations for the *PL* distribution are studied. Also means and variances are tabulated.

## 2. Single moments of *OS* from *PL* distribution

The single moments of *OS* for *PL* distribution can be obtained from (4) (when *b* is positive integer) as follows:

$$\mu_{a:n}^{(b)}=\int_0^\infty x^b f_{a:n}(x)dx=c_{a:n}\int_0^\infty x^b\left[1-\bar{F}(x)\right]^{a-1}\left[\bar{F}(x)\right]^{n-a}f(x)dx \qquad (6)$$

**Lemma 2.1.** For *PL* distribution, the $b^{th}$ moment of the $a^{th}$ *OS* for $b=1,2,...$ denoted by $\mu_{a:n}^{(b)}$ is

$$\mu_{a:n}^{(b)}=c_{a:n}\sum_{i=0}^{a-1}\frac{\binom{a-1}{i}b(-1)^i\lambda^{\frac{b}{\beta}}}{\beta(n-a+i+1)}B\left(\frac{b}{\beta},\alpha(n-a+i+1)-\frac{b}{\beta}\right) \qquad (7)$$

**Proof.** From (6), expanding $\left[1-\bar{F}(x)\right]^{a-1}$ binomially, we get

$$\mu_{a:n}^{(b)}=c_{a:n}\sum_{i=0}^{a-1}\binom{a-1}{i}(-1)^{i}\int_{0}^{\infty}x^{b}\left[\bar{F}(x)\right]^{n-a+i}f(x)dx$$

Integrating by parts, we have

$$\mu_{a:n}^{(b)}=c_{a:n}\sum_{i=0}^{a-1}\frac{\binom{a-1}{i}b(-1)^{i}}{n-a+i+1}\int_{0}^{\infty}x^{b-1}\left[\bar{F}(x)\right]^{n-a+i+1}dx$$

Using (2),

$$\mu_{a:n}^{(b)}=c_{a:n}\sum_{i=0}^{a-1}\frac{\binom{a-1}{i}b(-1)^{i}}{n-a+i+1}\int_{0}^{\infty}x^{b-1}\left(1+\frac{x^{\beta}}{\lambda}\right)^{-\alpha(n-a+i+1)}dx$$

Set $y=\left(1+\frac{x^{\beta}}{\lambda}\right)^{-1}$ and after simplification yields (7). □

Now, recurrence relation for single moments is obtained.

**Lemma 2.2.** For PL distribution, the following recurrence relation is satisfied.

$$\mu_{a:n}^{(b)}=\frac{\alpha\beta(n-a+1)\mu_{a-1:n}^{(b)}+b\lambda\mu_{a:n}^{(b-\beta)}}{\alpha\beta(n-a+1)-b} \tag{8}$$

**Proof.** From (6), we have

$$\mu_{a:n}^{(b)}=c_{a:n}\int_{0}^{\infty}x^{b}\left[1-\bar{F}(x)\right]^{a-1}\left[\bar{F}(x)\right]^{n-a}f(x)dx$$

Integrating by parts, we get

$$\mu_{a:n}^{(b)}=\frac{c_{a:n}}{n-a+1}\int_{0}^{\infty}\left[\bar{F}(x)\right]^{n-a+1}\left\{(a-1)x^{b}\left[1-\bar{F}(x)\right]^{a-2}f(x)+bx^{b-1}\left[1-\bar{F}(x)\right]^{a-1}\right\}dx$$

Using (3),

$$\mu_{a:n}^{(b)}=\frac{c_{a:n}}{n-a+1}\left\{(a-1)\int_{0}^{\infty}x^{b}\left[1-\bar{F}(x)\right]^{a-2}f(x)\left[\bar{F}(x)\right]^{n-a+1}dx+\frac{b}{\alpha\beta}\int_{0}^{\infty}\left(x^{b}+\lambda x^{b-\beta}\right)\left[1-\bar{F}(x)\right]^{a-1}f(x)\left[\bar{F}(x)\right]^{n-a}dx\right\}$$

$$=\mu_{a-1:n}^{(b)}+\frac{b}{\alpha\beta(n-a+1)}\left(\mu_{a:n}^{(b)}+\lambda\mu_{a:n}^{(b-\beta)}\right)$$

This is implies that

$$\mu_{a:n}^{(b)}=\frac{\alpha\beta(n-a+1)\mu_{a-1:n}^{(b)}+b\lambda\mu_{a:n}^{(b-\beta)}}{\alpha\beta(n-a+1)-b}. \quad □$$

**Remark 2.1** When $b=1$,

$$\mu_{a:n}=\frac{\alpha\beta(n-a+1)\mu_{a-1:n}+\lambda\mu_{a:n}^{(1-\beta)}}{\alpha\beta(n-a+1)-1}, \tag{9}$$

and

$$\mu_{a:n}=c_{a:n}\sum_{i=0}^{a-1}\frac{\binom{a-1}{i}(-1)^{i}\lambda^{\frac{1}{\beta}}}{\beta(n-a+i+1)}B\left(\frac{1}{\beta},\alpha(n-a+i+1)-\frac{1}{\beta}\right). \tag{10}$$

**Remark 2.2** When b =1 and $\beta=1$, we have the recurrence relation of single moments *OS* from Lomax distribution.

## 3. Product moments of *OS* from *PL* distribution

The product moments of *OS* for *PL* distribution can be obtained from (5) (when *j*,*k* are positive integer) as follows:

$$\mu_{r,s:n}^{(j,k)}=c_{r,s:n}\int_{0}^{\infty}\int_{x}^{\infty}x^{j}y^{k}\left[F(x)\right]^{r-1}\left[F(y)-F(x)\right]^{s-r-1}\left[\bar{F}(y)\right]^{n-s}f(x)f(y)dydx \tag{11}$$

**Lemma 3.1.** let $X$ be a random variable has *pdf* (1). Then for positive integers *j*,*k*, the following recurrence relation is satisfied.

$$\mu_{r,s:n}^{(j,k)}-\mu_{r,s-1:n}^{(j,k)}=\frac{\mu_{r,s-1:n}^{(j,k)}+\lambda\mu_{r,s-1:n}^{(j,k-\beta)}}{\alpha\beta(s-r-1)} \tag{12}$$

**Proof.** We have from (Khan et al. [14]) that

$$\mu_{r,s:n}^{(j,k)}-\mu_{r,s-1:n}^{(j,k)}=c_{r,s-1:n}^{*}\int_{0}^{\infty}\int_{x}^{\infty}x^{j}y^{k-1}\left[F(x)\right]^{r-1}\left[F(y)-F(x)\right]^{s-r-1}\left[\bar{F}(y)\right]^{n-s+1}f(x)dydx$$

where $c_{r,s:n}^{*}=\frac{n!}{(r-1)!(s-r-1)!(n-s+1)!}$.

Using (3), after simplification we get required result. □

**Remark 3.1** When $\beta=1$, we have the recurrence relation of product moments *OS* from Lomax distribution.

## 4. Characterization

In this section, characterization the *PL* distribution using the recurrence relation of single *OS* is obtained.

**Lemma 4.1.** Let $X$ be a non-negative random variable having an absolutely continuous distribution function F (x) with F (0) = 0 and 0 < F (x) < 1 for all $x>0$, then

$$\mu_{a:n}=\frac{\alpha\beta(n-a+1)\mu_{a-1:n}+\lambda\mu_{a:n}^{(1-\beta)}}{\alpha\beta(n-a+1)-1}, \tag{13}$$

if and only if $\bar{F}(x)=\frac{1}{\alpha\beta}\left(x+\lambda x^{1-\beta}\right)f(x)$.

**Proof.**

The necessary part follows immediately from Lemma 2.2. On the other hand if the recurrence relation in equation (13) is satisfied, then on using equation (6), we have

$$c_{a:n}\int_{0}^{\infty}x\left[1-\bar{F}(x)\right]^{a-1}\left[\bar{F}(x)\right]^{n-a}f(x)dx=\frac{1}{\alpha\beta(n-a+1)-1}\left\{\alpha\beta(n-a+1)c_{a-1:n}\int_{0}^{\infty}x\left[1-\bar{F}(x)\right]^{a-2}\left[\bar{F}(x)\right]^{n-a+1}f(x)dx+\lambda c_{a:n}\int_{0}^{\infty}x^{1-\beta}\left[1-\bar{F}(x)\right]^{a-1}\left[\bar{F}(x)\right]^{n-a}f(x)dx\right\}$$

Integrating the first integral in right hand side by parts, we get

$$c_{a:n}\int_0^\infty x\left[1-\bar F(x)\right]^{a-1}\left[\bar F(x)\right]^{n-a} f(x)dx=\frac{1}{\alpha\beta(n-a+1)-1}$$
$$\Big\{-\alpha\beta c_{a:n}\int_0^\infty\left[1-\bar F(x)\right]^{a-1}\left[\bar F(x)\right]^{n-a}\left[\bar F(x)-(n-a+1)xf(x)\right]dx$$
$$+\lambda c_{a:n}\int_0^\infty x^{1-\beta}\left[1-\bar F(x)\right]^{a-1}\left[\bar F(x)\right]^{n-a} f(x)dx\Big\}$$

This is implies that

$$c_{a:n}\int_0^\infty\left[1-\bar F(x)\right]^{a-1}\left[\bar F(x)\right]^{n-a}\left[\alpha\beta\bar F(x)-\left(x+\lambda x^{1-\beta}\right)f(x)\right]dx=0.$$

Now applying a generalization of the Muntz-Szasz theorem (Hwang and Lin [15]), we get

$$\bar F(x)=\frac{1}{\alpha\beta}\left(x+\lambda x^{1-\beta}\right)f(x).\ \square$$

## 5. Some numerical results

In this section, some results of moments and variance of OS from PL distribution are calculated and tabulated.

**Table 1:** Mean of order statistics

| $n$ | $a$ | $\alpha=3,\ \beta=1.5,\ \lambda=2$ | $\alpha=3.5,\ \beta=2,\ \lambda=2.5$ | $\alpha=4,\ \beta=2.5,\ \lambda=3$ | $\alpha=4.5,\ \beta=3,\ \lambda=3.5$ |
|---|---|---|---|---|---|
| 1 | 1 | 0.853 | 0.843 | 0.853 | 0.866 |
| 2 | 1 | 0.479 | 0.560 | 0.622 | 0.669 |
| | 2 | 1.227 | 1.126 | 1.084 | 1.063 |
| 3 | 1 | 0.353 | 0.449 | 0.522 | 0.579 |
| | 2 | 0.731 | 0.783 | 0.821 | 0.848 |
| | 3 | 1.475 | 1.298 | 1.216 | 1.170 |
| 4 | 1 | 0.287 | 0.385 | 0.462 | 0.524 |
| | 2 | 0.553 | 0.640 | 0.701 | 0.745 |
| | 3 | 0.909 | 0.927 | 0.941 | 0.951 |
| | 4 | 1.663 | 1.421 | 1.308 | 1.243 |
| 5 | 1 | 0.245 | 0.342 | 0.421 | 0.485 |
| | 2 | 0.455 | 0.555 | 0.626 | 0.679 |
| | 3 | 0.699 | 0.767 | 0.812 | 0.843 |
| | 4 | 1.050 | 1.033 | 1.027 | 1.027 |
| | 5 | 1.817 | 1.518 | 1.379 | 1.298 |

Note that: the results in Table 1 are consistent with property of order statistics $\sum_{i=1}^{n}\mu_{i:n}=n\mu_{1:1}$ given by David and Nagaraja [3].

**Table 2:** Variance of order statistics

| $n$ | $a$ | $\alpha=3,\ \beta=1.5,\ \lambda=2$ | $\alpha=3.5,\ \beta=2,\ \lambda=2.5$ | $\alpha=4,\ \beta=2.5,\ \lambda=3$ | $\alpha=4.5,\ \beta=3,\ \lambda=3.5$ |
|---|---|---|---|---|---|
| 1 | 1 | 0.626 | 0.289 | 0.179 | 0.125 |
| 2 | 1 | 0.138 | 0.103 | 0.081 | 0.066 |
| | 2 | 0.835 | 0.315 | 0.169 | 0.106 |
| 3 | 1 | 0.068 | 0.062 | 0.055 | 0.048 |
| | 2 | 0.183 | 0.110 | 0.075 | 0.055 |
| | 3 | 0.977 | 0.329 | 0.164 | 0.098 |
| 4 | 1 | 0.043 | 0.044 | 0.042 | 0.038 |
| | 2 | 0.091 | 0.066 | 0.050 | 0.039 |
| | 3 | 0.212 | 0.112 | 0.071 | 0.049 |
| | 4 | 1.090 | 0.340 | 0.161 | 0.093 |
| 5 | 1 | 0.030 | 0.034 | 0.034 | 0.032 |
| | 2 | 0.057 | 0.047 | 0.038 | 0.031 |
| | 3 | 0.105 | 0.067 | 0.047 | 0.035 |
| | 4 | 0.233 | 0.114 | 0.068 | 0.045 |
| | 5 | 1.186 | 0.350 | 0.159 | 0.089 |

## 6. Conclusions

In this paper, we introduce recurrence relations between the single and the product moments of the OS from Power Lomax distribution. Characterizations for the Power Lomax distribution are studied. Some results of means and variances are tabulated.

## References

[1] E. A. Rady, W. A. Hassanein, and T. A. Elhaddad, The power Lomax distribution with an application to bladder cancer data. *SpringerPlus 5:1838*, 2016.

[2] I. B. Abdul-Moniem, K-th Upper Record Values from Power Lomax Distribution, Int. J Math. Statist. Inv.,4(10), 2016, 41-43.

[3] H.A. David, and H.N. Nagaraja, Order Statistics, John Wiley, New York, 2003.

[4] H. Athar and Nayabuddin, A note on exact moments of order statistics from exponentiated log-logistic distribution. ProbStat Forum, 7, 2014, 39-44.

[5] M.I. Hendi, S.E. Abu-Youssef and A.A. Alraddadi, Order Statistics from Doubly Truncated Gompertz Distribution and its Characterizations. The Egyptian Statistical Journal, 50(1), 2006, 21‐ 31.

[6] M.M. Mohie El-Din, M.A.W. Mahmoud, S.E. Abu-Youssef and K.S. Sultan, Order Statistics from the doubly truncated linear exponential distribution and its characterizations. Commun. Statist.-Simula., 26, 1997, 281‐ 290.

[7] M.M. Mohie El-Din, M.A.W. Mahmoud and S.E. Abu-Youssef, Moments of order statistics from parabolic and skewed distributions and characterization of Weilbull distribution. Commun. Statist. Simul. Comput., 20(2,3), 1991, 639‐ 645.

[8] M.A. Ali and A. H. Khan, Ratio and inverse moments of order statistics from Weibull and exponential distribution. J. Applied statistical science 4(1), 1996, 1-7.
[9] A. H. Khan and M.A. Ali, Ratio and inverse moments of order statistics from Burr distribution, J. Ind. Soc. Prob. Statist. 2, 1995, 97-102.
[10] N. Blakrishnan, H.J. Malik and S.E. Ahmed, Recurrence relations and identities for moments of order statistics, II: Specific continuous distributions. Commun. Statist. Theor. Meth., 17(8), 1988, 2657‐ 2694.
[11] A.H. Khan and I.A. Khan, Moments of order statistics from Burr distribution and its characterizations, Metron XLV, 1987, 21-29.
[12] N. Balakrishnan and H.J. Malik, Order statistics from the linear-exponential distribution, part I: Increasing hazard rate case. Commun. Statist. Theor. Meth., 15(1), 1986, 179‐203.
[13] N. Balakrishnan and H.J. Malik, Some general identities involving order statistics. Commun. Statist. Theor. Meth., 14(2), 1985, 333‐ 339.
[14] A.H. Khan, S. Parvez and M. Yaqub, Recurrence relations between product moments of order statistics. J. Statist. Plan. Inf., 8, 1983, 175‐ 183.
[15] J. S. Hwang and G. D. Lin, On a generalized moments problem II. Proc. Amer. Math. Soc., 91, 1984, 577-580.

**The Author** received the Ph.D degree in Statistics from Cairo University, Egypt, in 2006. His research interests are mathematical statistics.