Strongly Extending and Fully Principally Extending S-acts

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Abstract :

In this paper the notion of strongly extending and fully principally extending acts are introduced and studied which represents stronger properly than extending acts. More precisely we studied properties and characterizations of acts in which all subacts are ⋂-large in a stable retract of an S-act M_s and all subacts are P-extending respectively . Examples are given to illustrate these concepts . Relate the concept of strongly extending acts with fully stable, duo and RS property. Then lemma (4.6) explains that the concepts of fully stable, duo and RS-acts are equivalent under strongly extending act and semisimple conditions. Also, conditions are investigated under which subacts are inheriting strongly extending property. Finally, the relationship among the class of fully principally extending acts with extending acts and principally extending are introduced .

Keywords: *Strongly extending acts , Extending acts , Fully stable subacts , RS-acts , Fully principally extending acts , Principally extending acts.*

1-Introduction

In this paper , unless otherwise stated , we assume that every act is unitary right act with zero element Θ which denoted by M_s . A right S-act M_s with zero is a non-empty set with a function f : M× S→M , f(m,s) ⟼ ms such that the following properties hold : (1) m·1= m (2) m(st)= (ms)t , for all m ∈ M and s , t ∈ S, 1 is the identity element of S . A subact N of an S-act M_s , is a non-empty subset of M such that xs∈ N for all x ∈ N and s ∈ S . Note that, we will use terminology and notations from [10] freely .

An S-act M_s is called decomposable if there exist two subacts A , B of M_s such that M_s = A⋃B and A⋂B = Θ . In this case , A⋃B is called decomposition of M_s . Otherwise M_s is called indecomposable ([5],p.65) . Every cyclic act is indecomposable .

A non-zero S-act M_s over a monoid S is called reversible (⋂-reversible) if every non-zero subact of M_s is large (⋂-large) , it is clear that every nonzero reversible act is ⋂-reversible act , but the converse is not true in general and they are coincide when $\psi_M = i$ [1] .

A subact N of a right S-act M_s is called fully invariant if f(N) ⊆ N for every endomorphism f of M_s and M_s is called duo if every subact of M_s is fully invariant [7] . As stronger than that of duo acts M.S.Abbas and R.B.Hiba introduced the concept of fully stable acts . A subact N of an S-act M_s is called stable if f(N) ⊆ N for each S-homomorphism f:N→M_s . An S-act M_s is called fully stable if each subact of M_s is stable [3] . It was proved that every fully stable subact is fully invariant subact .

An S-act M_s is multiplication if each subact of M_s is of the form MI , for some right ideal I of S . This is equivalent to saying that every principal subact is of this form [7] .

Injective and quasi injective acts play an important role in acts theory . In 1967 , P.Berthiaume introduced the concepts of injective [2] . An S-act M_s is called injective (H-injective) if for any given an Smonomorphism $\alpha : N \to H_s$ where N is a subact of H_s and every S-homomorphism $\beta: N \to M_s$, can be extended to an S-homomorphism $\sigma: H_s \to M_s$. Quasi injective S-acts have been studied by Lopez and Luedeman [6] such that an S-act M_s is called quasi injective if and only if it is M-injective .

A subact N of S-act M_s is called closed if it has no proper ∩-large extension in M_s that is the only solution of $N \hookrightarrow^{\cap l} L \;_{\neq}\!\hookrightarrow M_s$ is N = L [9] .

In [10] , the author introduced the concepts of extending and principally extending acts as follows : an S-act M_s is called extending act (simply CS-act) if every subact of M_s is ∩-large in a retract of M_s . Also, an S-act M_s is called principally extending act (simply P-extending) if every cyclic subact of M_s is ∩-large in a retract of M_s . In other words , every cyclic-closed subact of M_s is a retract of M_s .

In this paper , we adopt another generalizations of injective act which is strongly extending and fully principally extending acts .

One of the interesting results on the study of strongly extending acts , is that many of the important properties that hold for (quasi) injective acts still hold for strongly extending for example every subact of an S-act is ∩-large in a retract of M_s , this simple property still hold for strongly extending acts .

Like extending acts , we explain that a direct sum of strongly extending acts need not strongly . However , we obtain sufficient conditions for a direct sum of strongly acts to be strongly extending . Moreover , the inherited property for strongly extending acts is studied .

Fully stable acts used as a link between extending and strongly extending acts . Also, we assert that extending and strongly extending acts are linked by RS-act . Corollary(4.4):let M_s be RS-act . M_s is strongly extending if and only if M_s is extending .

In this work ,we explore conditions semisimple and strongly extending to versus duo and fully stable acts .

An S-act M_s is called semisimple if and only if every subact of M_s is a retract or it is union of simple subacts [10] .

2- **Strongly Extending S-act :**

Definition(2.1): An S-act M_s is called strongly extending if every subact of M_s is ∩-large in a stable retract of M_s . A monoid S is called right (left) strongly extending if S is a strongly extending right (left) act .

Remarks and Examples(2.2) :

1- Every strongly extending act is extending , but the converse is not true in general for example : The Z-act $M = Z_2 \dot{\cup} Z_2$ is semisimple act , so it is extending but it is not strongly extending since if $N = Z_2 \dot{\cup} \{\Theta\}$ is a subact of M , then N is closed subact in M but it is not stable subact of M_s (For if

$f: N \to M_s$ defined by $f(2x,0) = (0,2x)$, then $(0,2x) \notin N = Z_2 \dot{\cup} \{\Theta\}$, so $f(N) \not\subseteq N$ and N is not stable subact of M_s) .

2- Every ∩-reversible act is strongly extending , (since every subact of ∩-reversible act M_s is ∩-large in M_s , so M_s is a stable and retract of M_s) but the converse is not true in general for example Z_6 as Z-act with multiplication is strongly extending act but not ∩-reversible .

3- Let M_s be indecomposable S-act . An S-act M_s is ∩-reversible if and only if M_s is strongly extending .

Proof : ⟹) Let M_s be ∩-reversible S-act . By (2) M_s is strongly extending .

⟸) Let N be subact of M_s . By strongly extending property of M_s , N is ∩-large in a stable retract of M_s .

Since M_s is indecomposable , so Θ and M_s are the only retract of M_s . Hence N is ⋂-large in M_s . Hence M_s is ⋂-reversible S-act . ■

4- Let an S-act M_s is indecomposable . An S-act M_s is strongly if and only if M_s is extending .

Proof : Let M_s be strongly extending ,so by (1) M_s is extending act . Conversely, let M_s be an extending act . Since every indecomposable and extending act is ⋂-reversible (for if N is a subact of M_s . By extending property of M_s , N is ⋂-large in a retract of M_s . Since M_s is indecomposable , so Θ and M_s are the only retract of M_s . Hence N is ⋂-large in M_s . Hence M_s is ⋂-reversible S-act) , so by (3) M_s is strongly extending act . ■

It is known from [10] , an S-act M_s is extending if and only if every closed subact of M_s is a retract of M_s. An analogous, we have the following characterization of strongly extending acts :

Proposition(2.3): An S-act M_s is strongly extending if and only if every closed subact of M_s is a stable retract of M_s .

Proof : ⟹) Assume that M_s is strongly extending S-act . Let N be a closed subact of M_s . By strongly extending property of M_s , there exists a stable retract B of M_s such that N is ⋂-large in B . But N is closed subact of M_s , so N=B . This implies that N is a stable retract of M_s .

⟸) Let A be a subact of M_s . Thus , by Zorn's lemma , there exists a closed subact B of M_s such that A is ⋂-large in B . Since B is closed , so by hypothesis B is a stable retract of M_s and then A is ⋂-large in stable retract of M_s . Thus M_s is strongly extending act . ■

The following theorem gives us another characterization of strongly extending acts :

Theorem(2.4): The following statements are equivalent for an S-act M_s :

1- M_s is strongly extending S-act ;

2- Every closed subact of M_s is a stable retract ;

3- If N is a retract of $E(M_s)$, then $N \cap M_s$ is a stable retract of M_s .

Proof: : (1⟶2) By proposition(2.3) .

(2⟶3) Let $E(M_s) = A \dot{\cup} B$, where B is a subact of $E(M_s)$. Suppose that $A \cap M_s$ is ⋂-large in K , where K is a subact of M_s and then of $E(M_s)$ and let $k \in K$. Then $k \in E(M_s)$, which implies that $k \in A$ or $k \in B$ (this means that k = a or k =b) . Now , consider $k \notin A$ and $k = b \neq \Theta$. Since M_s is ⋂-large in E(Ms) , so there exists $\Theta \neq s \in S$ such that $ks = bs \in M_s$. But , $\Theta \neq b \in B$ and , so $bs \in B$. Thus , we have $bs \in M_s \cap B$. On the other hand , we have $A \cap M_s$ is ⋂-large in K and B is ⋂-large in B , so $A \cap M_s \cap B$ is ⋂-large in $K \cap B$. But $M_s \cap A \cap B = \Theta$, so $K \cap B = \Theta$ and then $bs = \Theta$ which is a contradiction . Therefore $A \cap M_s$ is closed of M_s and hence by (2) it is a stable retract of M_s .

(3⟶1) Let A be a subact of M_s and B be a relative complement of A . Then , by lemma(2.6) in [10] $A \dot{\cup} B$ is ⋂-large subact of M_s . As M_s is ⋂-large in $E(M_s)$, so $A \dot{\cup} B$ is ⋂-large in $E(M_s)$ by lemma(3.1) in [4] and so $E(A) \dot{\cup} E(B) = E(A \dot{\cup} B) = E(M_s)$. Since E(A) is a retract of $E(M_s)$, then by (3) $E(A) \cap M_s$ is a stable retract of M_s . But A is ⋂-large in E(A) and M_s is ⋂-large in M_s , then $A = A \cap M_s$ is ⋂-large in $E(A) \cap M_s$. Thus , M_s is strongly extending act . ■

Remark(2.5): Let A , M_1 , M_2 be an S-acts . If A is ⋂-large in M_1 , then $A \dot{\cup} M_2$ is ⋂-large in $M_1 \dot{\cup} M_2$.

Proof: Let $x \in M_1 \dot{\cup} M_2$, then $x \in M_1$ or $x \in M_2$. Now , we have two cases :

Case(1): If $x \in M_1$ and since A is ⋂-large in M_1 , then there exists $s \in S$ such that $xs \in A$.

Case(2): If $x \in M_2$ and since M_2 is $\cap$-large in M_2 , then there exists $s \in S$ such that $xs \in M_2$.

From both cases , we have $xs \in A$ or $xs \in M_2$. Thus $xs \in A\dot{\cup}M_2$ and $A\dot{\cup}M_2$ is $\cap$-large in $M_1\dot{\cup}M_2$.■

In [8] , Nicholson define a submodule when it lies under a direct summand of R-module M which motivate us to generalize this concept to an act as follows :

Definition(2.6): A subact N of an S-act M_s is said to be lie under a retract of M_s if there exists a direct decomposition $M_s= M_1\dot{\cup}M_2$ with $N \subseteq M_1$ and N is $\cap$-large in M_s .

From above definition and definition of strongly extending , we can conclude that an S-act M_s is strongly extending if every subact of M_s lie under a stable retract of M_s .

Theorem(2.7): An S-act M_s is strongly extending if and only if for each subact N of M_s , there is a direct decomposition $M_s = M_1\dot{\cup}M_2$ such that $N \subseteq M_1$ where M_1 is a stable subact of M_s and $N\dot{\cup}M_2$ is $\cap$-large in M_s.

Proof: $\Longrightarrow$) Assume that M_s is strongly extending act . Let N be a subact of M_s . By strongly extending property , we have N is $\cap$-large in a stable retract say M_1 of M_s . This means $M_s = M_1\dot{\cup}M_2$, where M_2 is a subact of M_s . Also , since N is $\cap$-large in M_1 and M_2 is $\cap$-large in M_2 , so by remark(2.5) , $N\dot{\cup}M_2$ is $\cap$-large in $M_1\dot{\cup}M_2 = M_s$. Hence $N\dot{\cup}M_2$ is $\cap$-large subact of M_s .

$\Longleftarrow$) Let A be subact of M_s . By hypothesis, there exists a direct decomposition $M_s = M_1\dot{\cup}M_2$ such $A \subseteq M_1$ where M_1 is a stable subact of M_s and $A\dot{\cup}M_2$ is $\cap$-large in M_s . We claim that A is $\cap$-large in M_1 . Let B be a non-zero subact of M_1 and hence of M_s . Thus , $(A\dot{\cup}M_2) \cap B \neq \Theta$. This implies that there exists $x(\neq \Theta)\in (A\dot{\cup}M_2)\cap B$ and when $x \in M_2\cap B$, then $x = \Theta$ (since $B\subseteq M_1$) . Hence , $x \in A\cap B$ and then $A\cap B \neq \Theta$. Therefore A is $\cap$-large in M_1 and M_s is strongly extending S-act . ■

It is well known that every stable subact of any act is fully invariant and the converse is not true in general .In the following lemma , we obtain a condition under which the converse is true .

Lemma(2.8): Every fully invariant retract of S-act are stable .

Proof : Let N be a fully invariant retract of an S-act M_s Assume that $f : N \longrightarrow M_s$ be any S-homomorphism . As N is a retract of M_s . Thus , there is a projection map $\pi : M_s\longrightarrow N$. So $(f\circ\pi) : M_s\longrightarrow M_s$. Since N is fully invariant , hence $(f\circ\pi)(N) \subseteq N$ and thus $f(N) = f(\pi(N)) = (f\circ\pi)(N) \subseteq N$. Therefore , N is a stable retract of M_s.■

Corollary(2.9): Every duo semisimple act is fully stable . ■

Remarks(2.10):

1- From lemma(2.8) , if a subact N of an S-act is either fully invariant or retract but not both , then N need not be stable subact . For example in the Z-act with multiplication Z , the subact 2Z is fully invariant , but not stable and it is not retract of Z . On the other hand , consider Q as Z-act , the subact Z is not fully invariant and Z is not retract of Q . Also Z is not stable .

2- The condition in corollary (2.9) of semisimple act is necessary , for example Z-act with multiplication is duo , but it is not semisimple and it is not fully stable .

By using lemma(2.8), we have the following characterization of strongly extending acts :

Proposition(2.11) : An S-act M_s is strongly extending if and only if every subact of M_s is $\cap$-large in a fully invariant retract of M_s . ■

Recall that an S-act M_s is called quasi injective if for every homomorphism from subact N of M_s into M_s can be extended to an S-endomorphism of M_s . It is clear that every strongly extending act is quasi injective , but the converse is not true in general and the following proposition give a condition under which the converse is true :

It is known that the concepts of multiplication acts and quasi injective acts are different . For example Z which is Z-act with multiplication is multiplication act and it is not quasi injective .

Proposition(2.12): Every multiplication quasi injective (extending) act is strongly extending .

Proof: Let M_s be a multiplication quasi injective act and N be a closed subact of M_s . Since M_s is quasi injective , so N is a retract of M_s by [10] . It is enough to show that N is fully invariant of M_s . Let $\beta : M_s \longrightarrow M_s$ be any S-endomorphism of M_s . As M_s is multiplication, thus N = MI for some ideal I of S . Now , $\beta(N) = \beta(MI) = \beta(M)I \subseteq MI = N$. Hence, by lemma(2.8) N is stable and M_s is strongly extending act.■

In [1] , explain that every cyclic act over a commutative monoid is multiplication act . Thus , we have the following corollaries from above proposition .

Corollary(2.13): Every cyclic quasi injective act over commutative monoid is strongly extending . ■

Corollary(2.14): Every commutative self-injective monoid is strongly extending . ■

Recall that an S-act M_s is C-quasi injective if each S-homomorphism from closed subact of M_s into M_s can be extended to S-endomorphism of M_s [11] .The following proposition asserts that a class of strongly extending acts contained in a class of C-quasi injective acts . The proof is routine , so it is omitted .

Proposition(2.15): Every strongly extending act is C-quasi injective . ■

The following results are partial answering of the equation : when is the strongly extending property inherited by the subacts ?

Proposition(2.16): A closed subact (and hence retract) of strongly extending act is strongly extending .

Proof: Let N be a closed subact of strongly extending S-act M_s and A be closed subact of N (and then closed subact of M_s by lemma(2.4) in [10]) . As M_s is strongly extending , so A is stable retract of M_s and since $A \subseteq N$, then A is retract of N . We claim that A is a stable subact of N . Let $f : A \longrightarrow N$ be an S-homomorphism and $i : N \longrightarrow M_s$ be the inclusion map . Now, $(i\circ f): A \longrightarrow M_s$ and since A is stable subact of M_s , then $(i\circ f)(A) \subseteq A$, so $f(A) \subseteq A$ and A is a stable retract of N . Therefore N is strongly extending . ■

The following proposition give us another answer of the above equation :

Proposition(2.17): Every subact N of a strongly extending S-act M_s with the property that the intersection of N with any stable retract of M_s is stable retract of N , is strongly extending .

Proof: Let A be subact of N(and then subact of M_s) . Since M_s is strongly extending , so there exists a stable retract B of M_s such that A is $\cap$-large in B . But $A \subseteq B\cap N \subseteq B$, thus by lemma(3.1) in [4] A is $\cap$-large in $B\cap N$ and by hypothesis $B\cap N$ is stable retract of N . Hence N is strongly extending . ■

Proposition(2.18): Let M_s be an S-act and $M_s = M_1 \dot{\cup} M_2$, where M_1 and M_2 are both strongly extending acts . Then M_s is strongly extending if and only if every closed subact A of M_s with $A\cap M_1 = \Theta$ or $A\cap M_2 = \Theta$ is a stable retract .

Proof : The necessity is clear by proposition(2.3) . Conversely, suppose that every closed subact N of M_s

with $N\cap M_1 = \Theta$ or $N\cap M_2 = \Theta$ is a stable retract of M_s . Let A be a closed subact of M_s . Then , there exists a complement B in A such that $A\cap M_2$ is $\cap$-large in B and since A is closed of M_s , so B is closed of M_s by lemma(2.4) in [10] . Since $(A\cap M_2)\cap M_1$ is $\cap$-large in $B\cap M_1$ whence M_1 is $\cap$-large in M_1 . Thus, $B\cap M_1 = \Theta$ (as $A\cap(M_2\cap M_1) = A\cap\Theta = \Theta$ which implies that Θ is $\cap$-large in Θ . Then, by hypothesis , $M = B \dot{\cup} B^{/}$for some $B^{/}$of M_s and B is a stable retract of M_s . Now , A $= A\cap M_s = A\cap(B\dot{\cup} B^{/}) = B \dot{\cup}(A \cap B^{/})$. Thus , $A\cap B^{/}$is closed in M_s (since $A\cap B^{/}$is closed in A) . Also , $(A\cap B^{/}) \cap M_2 = \Theta$, so by hypothesis $A\cap B^{/}$ is a stable retract of M_s and hence of $B^{/}$ (since $A\cap B^{/} \subseteq B^{/}$) . Thus, $B^{/} = (A\cap B^{/}) \dot{\cup} N$, where N is subact of $B^{/}$. Now, $M_s = B \dot{\cup} B^{/}= B \dot{\cup} ((A \cap B^{/}) \dot{\cup} N) = (B\dot{\cup} (A\cap B^{/})) \dot{\cup} N = A\dot{\cup}N$. It follows that A is a retract of M_s . Let $g:A \longrightarrow M_s$ be any S-homomorphism and $i_1 : B \longrightarrow B \dot{\cup}(A \cap B^{/})$, $i_2 : (A \cap B^{/}) \longrightarrow B \dot{\cup}(A \cap B^{/})$ be the inclusion mappings . let $\pi_1 : M_s\longrightarrow M_1$ and $\pi_2 : M_s\longrightarrow M_2$ be the projection mappings . Put $h_1 = \pi_1\circ g\circ i_1$ and $h_2 = \pi_2\circ g\circ i_2$. Hence $h = h_1\dot{\cup}h_2$ and then $g = g_1\dot{\cup}g_2$. Therefore, $g(A) = g(B \dot{\cup}(A \cap B^{/})) = g_1(B) \dot{\cup} g_2(A \cap B^{/}) \subseteq B \dot{\cup}(A \cap B^{/}) = A$. This implies that $g(A) \subseteq A$ and A is stable retract of M_s . Thus M_s is strongly extending . ■

Theorem(2.19): Let $M_s = \dot{\cup}_{i\in I}M_i$ be an S-act , where M_i are a subacts of M_s for each $i\in I$, where i is finite index set . Then , the following statements are equivalent :

1- M_s is strongly extending ;

2- Each M_i is strongly and each closed subact of M_s is fully invariant ;

3- Each M_i is extending and each closed subact of M_s is fully invariant .

Proof : (1 $\longrightarrow$2) By proposition(2.16) together with proposition(2.3) .

(2 $\longrightarrow$3) It is obvious .

(3 $\longrightarrow$1) Let N be a closed subact of M_s and $\pi_i : M_s\longrightarrow M_i$ be the projection map on M_i for each $i \in I$. Let $x \in N$, so $\pi_i(x)= m_i$,where $m_i \in M_i$. Since N is closed subact of M_s , so by (3) N is fully invariant and hence $\pi_i(N) \subseteq N\cap M_i$. Thus $\pi_i(x)= m_i \in N\cap M_i$ and so $x \in \dot{\cup}_{i\in I}(N\cap M_i)$. Therefore $N \subseteq \dot{\cup}_{i\in I}(N\cap M_i)$ and for other direction , we have $y \in \dot{\cup}_{i\in I}(N\cap M_i)$, then there exists $j \in I =\{1,2,\ldots,n\}$ such that $y \in N\cap M_j$ which implies $y \in N$ and $y \in M_j$ for some $j \in I$, so $\dot{\cup}_{i\in I}(N\cap M_i) \subseteq N$ and then this implies that $N = \dot{\cup}_{i\in I}(N\cap M_i)$. Now, since $N\cap M_i$ is a retract of N , then $N\cap M_i$ is closed in N . But N is closed in M , thus by lemma(2.4) in [10] , $N\cap M_i$ is closed in M_s . As $N\cap M_i \subseteq M_i \subseteq M_s$, thus $N\cap M_i$ is closed in M_i . By extending property of M_i , we have $N\cap M_i$ is a retract of M_i . Thus $N = \dot{\cup}_{i\in I}(N\cap M_i)$ is a retract of $M_s = \dot{\cup}_{i\in I}M_i$. Then N is a fully invariant retract of M_s and so by lemma(2.8) N is stable retract of M_s . Therefore , by proposition (2.3) M_s is strongly extending . ■

3- Fully Principally Extending acts :

Definition (3.1): An S-act M_s is called fully principally extending act (for short FP-extending) if every subact of M_s is P-extending .

Examples(3.2):

1- Every $\cap$-reversible act is FP-extending for example Q as Z-act with multiplication .

2- The converse of (1) is not true in general for example Z_6 as Z-act is FP-extending which is not $\cap$-reversible , since $A =\{\bar{0} , \bar{2} , \bar{4}\}$ is subact of Z_6 but it is not $\cap$-large since there exists non-zero subact $B =\{\bar{0} , \bar{3}\}$ of Z_6 , but the intersection with A is zero.

3- Every FP-extending act is P-extending , but the converse is not true in general .

4- Any subact (and hence retract) of FP-extending act is FP-extending .

Lemma(3.3): Any fully invariant subact of FP-extending act is FP-extending .

Proof : Let N be fully invariant subact of FP-extending act M_s and A be subact of N (and then of M_s), so A is P-extending by definition (3.1). Thus N is FP-extending act. ■

Lemma(3.4): Every duo P-extending S-act is FP-extending . ■

It is known that every stable subact of any S-act is fully invariant [3] , thus we restate above lemma as follows :

Lemma(3.5): Every stable subact of P-extending act is FP-extending . ■

Proposition(3.6): Let M_s be a fully stable S-act . Then M_s is FP-extending if and only if M_s is P-extending . ■

Proposition (3.7): Let $M_s = \dot{\bigcup}_{i\in I} M_i$ be fully stable act and $N \cap M_i$ be an $\cap$-large cyclic subact of M_s , for every $\cap$-large cyclic closed subact N of M_s with $N \cap M_i = \Theta$ or $N \cap M_j = \Theta$ is a retract ,where $i \neq j$ and i , j $\in$ I . Then M_s is FP-extending act if and only if M_i is P-extending for each $i \in I$.

Proof : Assume that M_s is FP-extending act . By FP-extending property , we have M_i is FP-extending acts $\forall$ $i\in I$. Then M_i is P-extending acts . Conversely, let M_i is P-extending acts . By hypothesis and the proof of proposition(3.5) in [10] ,we have M_s is P-extending . Since M_s is fully stable, so by proposition (3.6)M_s is FP-extending .■

4- Relation among Strongly extending S-acts with other Classes of Injectivity :

The following proposition give a condition under which strongly extending and extending are equivalent :

Proposition(4.1): Let M_s be a fully stable S-act . Then the following statements are equivalent :

1- M_s is strongly extending ;

2- M_s is extending act . ■

Corollary(4.2): Let M_s be an S-act such that every retract of M_s is stable . Then M_s is strongly extending if and only if M_s is extending . ■

From above corollary , we observed that the concepts of strongly extending acts and extending acts are equivalent under the condition " every retract is stable ".This lead us to introduce and study the following concept which is a proper generalization of fully stable acts:

Definition(4.3): An S-act M_s is called RS-act if every retract of M_s is stable . A monoid S is right(left) RS-monoid if , S is RS-act as right (left) S-act .

Corollary(4.4): Let M_s be RS-act . Then M_s is strongly extending if and only if M_s is extending . ■

Remarks and Examples(4.5):

1- Every fully stable act is RS-act , while the converse is not true in general for example : Q as Z-act is RS-act which is not fully stable .

2- Every $\cap$-reversible act is RS-act , for example Z-act Z with multiplication is $\cap$-reversible which is not fully stable . In fact if $h : 5Z \longrightarrow Z$ is defined by $h(5x) = 3x$, $\forall x \in Z$. It is obviously that h is Z-homomorphism but $h(5Z) \nsubseteq 5Z$. Thus 5Z is not stable subact of Z .

3- The converse of (2) is not true in general . For example , the Z-act with multiplication Z_6 is RS-act but it is not $\cap$-reversible .

4- Every fully stable acts are duo acts and every duo acts are RS-acts . But the converse is not true in general . For example Q as Z-act is RS-act which is not duo since Z is not fully invariant of Q_Z and Q_Z is not fully stable .

5- Every strongly extending act is RS-act , since in strongly extending S-act M_s every closed subact is a stable retract , then every retract is stable .

6- If M_s is extending S-act , then the following conditions are equivalent :

a- M_s is RS-act;

b- Every closed subact of M_s is stable ;

c- Every closed subact of M_s is fully invariant .

Proof: (a ⟶b) Let N be closed subact of M_s . By extending property N is a retract of M_s . By(a) N is stable .

(b ⟶c) It is obvious , since every stable subact is fully invariant .

(c ⟶a) Let N be a retract of M_s . Then N is closed . By(c) N is fully invariant and by lemma(2.8) N is stable . ■

7- We can restate corollary (4.2) as follows : If an S-act M_s is RS-act , then M_s is strongly extending if and only if M_s is extending .

8- From (5) and (7) , we have an S-act M_s is strongly extending if and only if M_s is extending and RS-act .

From (4) , we noticed that : every fully stable act is duo and then RS-act . The following lemma explain that these concepts are equivalent under strongly extending act and semisimple conditions :

Lemma(4.6): Let M_s be semisimple and strongly extending act . The following concepts are equivalent :

1- M_s is RS-act ;

2- M_s is duo ;

3- M_s is fully stable .

Proof : (1⟶2) As M_s is semisimple , so every subact is retract and since M_s is RS-act , so every retract is stable . Since every stable is fully invariant , so every subact is fully invariant . Then M_s is duo act .

(2⟶3) By corollary(2.9) .

(3⟶1) By remarks and examples(4.5) . ■

Proposition(4.7): Every multiplication act is RS-act .

Proof : Let N be a retract of multiplication S-act M_s and f : N ⟶M_s be any S-homomorphism . Since M_s is multiplication , then N = MI for some ideal I of S . But N is retract of M_s . Thus f can be extended to S-endomorphism $g:M_s \longrightarrow M_s$. Now, f(N)=g(N) =g(MI)=g(M)I ⊆ MI = N . Thus , N is stable of M_s . Therefore M_s is RS-act.■

Corollary(4.8): Every cyclic act over commutative monoid is RS-act . ■

Corollary(4.9): Every commutative monoid is RS-monoid . ■

Corollary(4.10): If S is commutative monoid , then S is strongly extending monoid if and only if S is extending monoid . ■

Proposition(4.11): If A ⊆ N ⊆ M and B is a complement (respectively ⋂-large complement) of A in M then B⋂N is a complement (respectively ⋂-large complement) of A in N.

Proof : Let M_s = A$\dot{\cup}$B and Θ ≠P ⊆ N . This implies that P ⊆ M_s , so P ⋂(A$\dot{\cup}$B) ≠ Θ , then there exists Θ ≠ x ∈ P ⋂(A$\dot{\cup}$B) . Thus , we have x ∈ P and x ∈A$\dot{\cup}$B (this means x ∈A or x ∈B) . Since x ∈ P and P ⊆ N , so x ∈ N . Thus, we have x ∈ B⋂N . Then , we have x ∈ A$\dot{\cup}$(B⋂N) .

Proposition(4.12): Let M_s be indecomposable S-act . Then, the following statements are equivalent :

1- M_s is FP-extending act ;

2- M_s is ∩-reversible;

3- M_s is strongly extending ;

4- M_s is extending ;

5- M_s is P-extending .

Proof : (1⟶ 2) Let N (≠ Θ) be subact of M_s . Then there exists x(≠ Θ) ∈ N . Since M_s is FP-extending act so M_s is P-extending . Thus , there exists a retract H of M_s such that xS is ∩-large in H . But M_s is indecomposable , so M_s and Θ are the only retract of M_s . Thus H = M_s and xS is ∩-large in M_s . Therefore M_s is ∩-reversible .

(2⟶ 3) and (3⟶ 4) by remarks and examples(2.2) (3) and (1) respectively .

(4⟶ 5) It is obvious .

(5⟶ 1) Let N (≠ Θ) be subact of M_s . Then, there exists x(≠ Θ) ∈ N such that xS is a subact of N (and then of M_s) . Since M_s is P-extending act by(5) so , there exists a retract H of M_s such that xS is ∩-large in H . But M_s is indecomposable, so M_s and Θ are the only retract of M_s and then by proposition(4.11) they are retract of N . Thus H= N = M_s and xS is ∩-large in N.Therefore N is P-extending and M_s is FP-extending act . ■

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