Rationale of Mathematical Properties of Method of Calculating Material's Thermal Conductivity Coefficient

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Abstract

This work studies one-dimensional problem of heat propagation in matter. The measured values of soil temperature and near ground air temperature are set. Iteration method is proposed for defining overall heat exchange coefficient of multilayer material. Realization of the method should give a solution, converging to the solution of differential problem. Sustainability of the method is confirmed by the proof of monotony of minimized functional and by boundedness of solution.

Keywords: *heat emission coefficient, finite-differential scheme, iteration, primal and conjugate problems, functional, initial boundary conditions*

1. Introduction

Studying of heat and mass transfer laws in nature has always played a major role in development of tech and natural science. Many important problems of soil science, agronomy, geology, environmental preservation, geophysics, energetics, use of natural resources, constructing and other branches are mathematically described by non-linear equation of heat conductivity.

Nowadays many methods of separate and complex definition of thermo-physical characteristics of materials are generally accepted. All those methods are widely used in engineering practices, scientific researches.

Definition of thermo-physical characteristics is strictly related to the study of heat propagation in matter or a specimen, i.e. solving initial boundary problem of heat conductivity equation, that are often called reverse coefficient problems.

Solution of such problems demands a set of correct input data. Determination of the correct input data for the solution of inverse coefficient problems in most cases were of a theoretical nature or difficult to realize in practice. Therefore, the development of new methods for solving the inverse problem of nonlinear differential equations is always a difficult problem.

Matters of sustainability and precision of computational models are checked numerically. However, the check of sustainability and precision of computational models with the use of only computer modeling does not involve the property of completeness of the research. Therefore, the grounding of mathematical properties of iterative processes of calculation is a major part of developing methods of solving reverse problems.

2. Review

Development and grounding of mathematical models of physical processes is solidly linked to solving reverse problems for differential equation.

Equation, describing the process of changing thermo-physical parameters, for which the reverse problem is set, is a mathematical model of a real soil mass. The model corresponding to a given information about the real active soil layer should be chosen from a given set of mathematical models.

There are many works studying thermo-physical parameters of non-homogenious material. Some soil characteristics, i.e. parameters of functions were empirically composed in an eventual formula [1, 2, 3]

Influence of mass transfer on coefficient of heat conductivity of soil is empirically studied by scientists [4, 5, 6, 7, 8].

Drawback of above mentioned methods is a definition of soil characteristics only for separately studied soil strips, which is a restriction for its wide application.

Methods of defining thermo-physical characteristics of materials, based on Fourier's law of heat conductivity for stationary heat flow and non-stationary methods, based of theory of solving thermal conductivity equation with non-stationary heat flow are stated in works [9, 10, 11, 12] and others.

The above mentioned methods are purposed for solving so-called linear initial boundary problem of heat conductivity equation for regular shapes (sphere, cylinder, parallelepiped and others) and with certain simplifications for input parameters.

Reverse problems of mathematical physics are frequently set in a wrong way. This is the main reason of the difficulties of building efficient calculation algorithms in solving reverse problems.

Mathematical features of practically applied methods of calculating thermo-physical parameters are not studied. This is why the work contains development of new methods of proving the restrictions of values of one of the thermo-physical characteristic of soil: a coefficient of soil heat conductivity.

During empirical calculations, there is usually built a functional of misfit between observed and calculated values of target parameters. Thermo-physical characteristics of soil are defined through numerical check of functional monotony. In this work, the limitation of heat conductivity of soil, the monotony of functional and convergence of iteration process are proven.

In the future, it is supposed to confirm the validity of theoretical assumption with the use of numerical calculations.

3. Problem formulation

Let us consider area $Q=(0,H)\times(0,T)$, where heat propagation under the influence of environmental temperature occurs. It is proven by numerous experiments that heat propagation in matter is described by heat conductivity equation [13-17]

$$\gamma_0 c\frac{\partial\theta}{\partial t}=\frac{\partial}{\partial z}\left(\lambda\frac{\partial\theta}{\partial z}\right), \tag{1}$$

where γ_0 is specific weight of the material $\frac{kg}{m^3}$; c is the heat capacity of the material $\frac{kcal}{kg\cdot\deg}$; λ is material thermal conductivity $\frac{\text{kcal}}{m\cdot hour\cdot\deg}$.

On the outer edge of the material the law of conservation of energy is valid

$$\lambda\frac{\partial\theta}{\partial z}\bigg|_{z=H}+\alpha\left(\theta\big|_{z=H}-T_b\right)=0, \tag{2}$$

where α is heat transfer coefficient of the material in the environment.

On the inner side of the fabric set the boundary condition

$$\theta(0,t)=T_1=const. \tag{3}$$

Note that the axis $0z$ is directed vertically upwards. At the initial time, the temperature distribution in the material is given, ie,

$$\theta(z,0)=\theta_0(z),\quad 0\le z\le H \tag{4}$$

Let us consider the case, when z changes from $z=0$ to $z=H$ and the material consists of three layers. In the transition from one layer to another, the temperature and its flow remain continuous:

$$[\theta(z,t)]_{h_k}=0,\ \left[\lambda\frac{\partial\theta}{\partial z}\right]_{h_k}=0,\qquad k=1,2. \tag{5}$$

where h_k is border coordinate transition from one layer to another layer.

In order to determine the thermal conductivity of the material the temperature at the outer edge of the material is set

$$\theta(H,t)=\theta_g(t),\ \ 0\le t\le T. \tag{6}$$

In the process of finding solution for the system (1)-(6) it is necessary to define the coefficient of heat conductivity of multilayer material.

In [18] the conjugate problem is obtained from the system (1) - (6)

$$\gamma_0 c\frac{\partial\psi}{\partial t}+\frac{\partial}{\partial z}\left(\lambda\frac{\partial\psi}{\partial z}\right)=0, \tag{7}$$

$$\lambda\frac{\partial\psi}{\partial z}\bigg|_{z=H}+\alpha\psi\bigg|_{z=H}=-2\left(\theta(H,t)-\theta_g(t)\right),$$
$$\psi(z,T)=0,\ \ \psi(0,t)=0, \tag{8}$$

$$[\psi]_{h_k}=0,\ \left[\lambda\frac{\partial\psi}{\partial z}\right]_{h_k}=0. \tag{9}$$

and the integral inequality is obtained as well:

$$2\int_0^T\delta\theta\left(\theta(H,t)-\theta_g(t)\right)dt=\int_0^T\int_0^H\delta\lambda\frac{\partial\theta^{n+1}}{\partial z}\frac{\partial\psi}{\partial z}dzdt. \tag{10}$$

4. The iterative process

Let us consider iterative process where the initial value $\lambda_n(z)$ is specified. The next approximation $\lambda_{n+1}(z)$ is defined by the formula

$$\lambda_{n+1}-\lambda_n=-\beta_n\int_0^T\frac{\partial\theta^n}{\partial z}\frac{\partial\psi^n}{\partial z}dt. \tag{11}$$

Lemma 1. If $\theta_0(z) \in L_2(0,H)$, $T_b(t) \in L_2(0,T)$, then for the solution of problem (1) - (5) we have the estimate

$$\frac{1}{2}\int_0^H \gamma_0 c\theta^2 dz + \int_0^t\int_0^H \lambda_n\left(\frac{\partial\theta}{\partial z}\right)^2 dzd\tau + \frac{\alpha}{2}\int_0^t \theta^2(H,\tau)d\tau \le C_1,$$

where $C_1 = \frac{1}{2}\int_0^H \gamma_0 c\theta_0^2(z)dz + \frac{\alpha}{2}\int_0^t T_b^2(\tau)d\tau$.

Lemma 2. If $\theta_0(z) \in L_2(0,H)$, $T_b(t), \mathrm{T_g}(t) \in L_2(0,T)$, then for the solution of the problem (1)-(5) we have the estimate

$$\frac{1}{2}\int_0^H \gamma_0 c\psi^2 dz + \int_t^T\int_0^H \lambda_n\left(\frac{\partial\psi}{\partial z}\right)^2 dzd\tau + \frac{\alpha}{2}\int_t^T \psi^2(H,\tau)d\tau \le C_2 .$$

At this point $C_2 = \frac{4}{\alpha}\int_0^T T_g^2(\tau)d\tau + \frac{8}{\alpha^2}C_1$.

In each homogeneous layer of the multilayer soil $\lambda_{n+1}(z)$=const. Therefore, integrating (11) over z from 0 to h_1, we get

$$\lambda_{n+1}(0) - \lambda_n(0) = -\beta_n(0)\frac{1}{h_1}\int_0^{h_1} dz\int_0^T \frac{\partial\theta^n}{\partial z}\frac{\partial\psi^n}{\partial z}dt .$$

This formula is valid for the lower layer of soil. Similarly, for the second and third layers, we obtain the formula

$$\lambda_{n+1}(h_1) - \lambda_n(h_1) = -\beta_n(h_1)\frac{1}{h_2 - h_1}\int_{h_1}^{h_2} dz\int_0^T \frac{\partial\theta^n}{\partial z}\frac{\partial\psi^n}{\partial z}dt .$$

$$\lambda_{n+1}(h_2) - \lambda_n(h_2) = -\beta_n(h_2)\frac{1}{H - h_2}\int_{h_2}^{H} dz\int_0^T \frac{\partial\theta^n}{\partial z}\frac{\partial\psi^n}{\partial z}dt .$$

Let $z = 0$ by means of h_0, i.e. $h_0 = 0$, $h_3 = H$. Then all three formulas can be written as follows

$$\lambda_{n+1}(h_k) - \lambda_n(h_k) = -\beta_n(h_k)\frac{1}{h_{k+1} - h_k}\int_{h_k}^{h_{k+1}} dz\int_0^T \frac{\partial\theta^n}{\partial z}\frac{\partial\psi^n}{\partial z}dt \cdot \quad (12)$$

Summing (12) over n from 0 to arbitrary n, i.e.

$$\lambda_{n+1}(h_k) - \lambda_0(h_k) = -\sum_n \beta_n(h_k)\frac{1}{h_{k+1} - h_k}\int_{h_k}^{h_{k+1}} dz\int_0^T \frac{\partial\theta^n}{\partial z}\frac{\partial\psi^n}{\partial z}dt \cdot$$

Let us estimate this equation with the use of Cauchy inequality

$$\lfloor \lambda_{n+1}(h_k) - \lambda_0(h_k)\rfloor \le \sum_n \beta_n(h_k)\frac{1}{h_{k+1} - h_k}\times \int_0^T\left(\int_{h_k}^{h_{k+1}}\lambda_n\left(\frac{\partial\theta^n}{\partial z}\right)^2 dz\right)^{\frac{1}{2}}\left(\int_{h_k}^{h_{k+1}}\lambda_n\left(\frac{\partial\psi^n}{\partial z}\right)^2 dz\right)^{\frac{1}{2}} dt\frac{1}{\lambda_n(h_k)}$$

Once again, apply the Cauchy inequality over the variable t. Then

$$\lfloor \lambda_{n+1}(h_k) - \lambda_0(h_k)\rfloor \le \sum_n \beta_n(h_k)\frac{1}{h_{k+1} - h_k}\times \left(\int_0^T\int_{h_k}^{h_{k+1}}\lambda_n\left(\frac{\partial\theta^n}{\partial z}\right)^2 dzdt\right)^{\frac{1}{2}}\times \left(\int_0^T\int_{h_k}^{h_{k+1}}\lambda_n\left(\frac{\partial\psi^n}{\partial z}\right)^2 dzdt\right)^{\frac{1}{2}}\frac{1}{\lambda_n(h_k)} \quad (13)$$

As follows from lemma 1 $\int_0^T\int_0^H \lambda_n\left(\frac{\partial\theta}{\partial z}\right)^2 dzdt \le C_1$. Hence, in particular

$$\int_0^T\int_{h_k}^{h_{k+1}}\lambda_n\left(\frac{\partial\theta^n}{\partial z}\right)^2 dzdt \le C_1 .$$

The inequality $\int_0^T\int_{h_k}^{h_{k+1}}\lambda_n\left(\frac{\partial\psi^n}{\partial z}\right)^2 dzdt \le C_2$ similarly follows from lemma 2.

In view of these inequalities the ratio (13) can be reduced to the following form

$$\lfloor \lambda_{n+1}(h_k) - \lambda_0(h_k)\rfloor \le C_3\sum_n \beta_n(h_k)\frac{1}{h_{k+1} - h_k}\frac{1}{\lambda_n(h_k)},$$

where $C_3 = \sqrt{C_1 C_2}$.

Let $\beta_n(h_k)\frac{1}{h_{k+1}-h_k}\frac{1}{\lambda_n}=\frac{\beta}{n^{\alpha_0}}$, $\alpha_0>1$. Then

$$\lfloor\lambda_{n+1}(h_k)-\lambda_0(h_k)\rfloor\le C_3\beta\sum_n\frac{1}{n^{\alpha_0}}.$$

But a series $\sum_n\frac{1}{n^{\alpha_0}}$ converges, therefore $\sum_n\frac{1}{n^{\alpha_0}}\le C_4$, then

$$\lfloor\lambda_{n+1}(h_k)-\lambda_0(h_k)\rfloor\le C_5\beta,\ C_5=C_4C_3.$$

Hence

$$\lambda_0(h_k)-C_5\beta\le\lambda_{n+1}(h_k)\le\lambda_0(h_k)+C_5\beta,\quad k=0,1,2.$$

Small quantity β is chosen so that there is inequality

$$\lambda_0(h_k)-C_5\beta\ge C_6>0.$$

Then $C_5\beta\le\lambda_0(h_k)$

$$0<C_6\le\lambda_{n+1}(h_k)\le C_7<\infty.$$

The above proven statement can be formalized as a theorem.

Theorem. If $\theta_0(z)\in L_2(0,H),T_b(t),\theta_1(t)\in L_2(0,T)$, then there is a sufficiently small number β so that (11) implies the inequality

$$0<C_6\le\lambda_{n+1}(h_k)\le C_7<\infty.$$

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