The p-normed Space

Abubakar Umar Terrang and Musa T. Y. Kadzai (PhD)

[1]Department of Mathematics and Computer Science, Federal University Kashere, Gombe Nigeria

[2]Department of Mathematics, Modibbo Adama University of Technology Yola, Adamawa Nigeria

Abstract

In the work, we investigated, stated and proved some basic properties of quasinormed spaces particularly p-normed spaces that are similar to those of the normed spaces. A quasinormed space is a space which is not a locally convex topological space, but those established on them showed that they have some characteristics of topological structures. To achieve this, we begin with some elementary definitions of the functional (quasinorm) on a given vector space and quasinormed.

Keywords: quasinorm, quasinormed space, p-normed space

Introduction

A quasi-normed space is a pair $(X, \|\cdot\|)$ consisting of a vector space X and non-negative real-valued function $\|\cdot\| : X \to R^+$ such that for all $x, y \in X$, all scalar α and constant $K \geq 1$ the following axioms are satisfied.

$$\|x\| = 0 \ whenever \ x = 0 \quad (1.1)$$

$$\|\alpha x\| = |\alpha|\|x\| \quad (1.2)$$

$$\|x + y\| \leq K(\|x\| + \|y\|) \quad (1.3)$$

The constant K in (1.3) is independent of x or y. The least such constant K satisfying (1.3) is called the modulus of concavity of $\|\cdot\|$.

Suppose we denote by $\mathbb{N}$ the set of natural numbers and suppose still that $0 < p \leq 1$. Then $\|\cdot\|$ is said to be p-convex if there exists a positive integer C such that for all $n \in \mathbb{N}$ and for every choice of vectors $x_1, x_2, x_3, \ldots, x_n \in X$ we have the inequality

$$\|\sum_{k=1}^{n} x_k\| \leq C(\sum_{k=1}^{n}\|x_k\|^p)^{\frac{1}{p}} \quad (1.4)$$

Furthermore, $\|\cdot\|$ is said to be p-subadditive if (1.4) is satisfied for all $n \in \mathbb{N}$ and for all vectors $x_1, x_2, x_3, \ldots, x_n \in X$ with $C = 1$, in which case $\|\cdot\|$ is called p-norm, and the corresponding space X is called p-normed space.

Theorem 1. (Inequality.) If $p \in (0,1)$ and $a, b \geq 0$, then

$$(a + b)^p \leq a^p + b^p, \quad (1.5)$$

with equality if and only if either a or b is zero.

Proof.

To see this we define a function

$$f(t) \coloneqq (1 + t)^p - 1 - t^p, \text{ for } t \geq 0.$$

We observed that $f'(t) = p(1 + t)^{p-1} - pt^{p-1} < 0, for\ t \in (0, \infty)$. Since $f(0) = 0$, it follows that $f(t) < 0, for\ t > 0$. If $a, b \neq 0$, then by substituting $t = \frac{a}{b}$, we have

$\left(1+\frac{a}{b}\right)^p - 1 - \left(\frac{a}{b}\right)^p < 0 \Leftrightarrow \left(\frac{a+b}{b}\right)^p - 1 - \frac{a^p}{b^p} < 0 \Leftrightarrow (a+b)^p - (a^p + b^p) < 0$, from where the inequality follows.The equality criterion follows from the fact that f is strictly decreasing on $t \in [0,\infty)$. ■

Theorem 2. For $0 < p < \infty$, $(L^p(X,\mu), \|\cdot\|_{L^p})$ is a complete quasinormed space.

Proof.

We first verify that it satisfies quasinorm axioms. To this end, we define a distance function on $L^p(X,\mu)$ by

$$d(f,g) \coloneqq \|f-g\|_{L^p}^p = \int_X |f-g|^p d\mu,$$

(i) $d(f,g) = \|f-g\|_{L^p}^p = \int_X |f-g|^p d\mu = 0 \; whenever \; f = g$,

(ii) $(\alpha f, \alpha g) = \|\alpha f - \alpha g\|_{L^p}^p = \|\alpha(f-g)\|_{L^p}^p = |\alpha|\|f-g\|_{L^p}^p = |\alpha| \int_X |f-g|^p d\mu = |\alpha| d(f,g)$,

(iii) Applying Theorem (1) we have,

For all $f,g,h \in L^p(X,\mu)$,

$$d(f,g) + d(g,h) = \int_X (|f-g|^p + |g-h|^p) d\mu \geq \int_X (|f-g| + |g-h|)^p d\mu \geq \int_X |f-h|^p d\mu = d(f,h)$$

Next, we show that d is complete. For this, since $\|f_n - f_m\|_{L^p} \to 0, n, m \to \infty$ by continuity of the maps $x \mapsto x^p$ and $x \mapsto x^{\frac{1}{p}}$, it suffices to show that given a sequence $(f_n)_{n=1}^{\infty}$,

$$\|f_n - f_m\|_{L^p}^p \to 0, n, m \to \infty \Rightarrow f \in L^p, \|f_n - f\|_{L^p}^p \to 0, n \to \infty$$

Let $(f_n)_{n=1}^{\infty}$ be such a sequence. Then we can construct a subsequence $\left(f_{n_k}\right)_{k\in\mathbb{N}}$ such that $\left\|f_{n_k} - f_{n_{k+1}}\right\|_{L^p}^p \leq \frac{1}{2^k}$.

Define

$$f = f_{n_1} + \sum_{k=1}^{\infty} \left(f_{n_{k+1}} - f_{n_k}\right)$$

Since

$$\left\|\sum_{k=1}^{N} \left(f_{n_{k+1}} - f_{n_k}\right)\right\|_{L^p}^p \leq \sum_{k=1}^{N} \left\|f_{n_{k+1}} - f_{n_k}\right\|_{L^p}^p \leq \sum_{k=1}^{N} \frac{1}{2^k} \leq 1 \; \forall N \in \mathbb{N}$$

It follows from monotone convergence theorem, $\left|f_{n_1}\right| + \sum_{k=1}^{\infty} \left|f_{n_{k+1}} - f_{n_k}\right| \in L^p(X,\mu)$. Hence by Lebesgue dominated convergence theorem, $f \in L^p(X,\mu)$.

Hence, $(f_n)_{n=1}^{\infty}$ is Cauchy with convergent subsequence and so $\|f_n - f\|_{L^p}^p \to 0, n \to \infty$ ■

If $\mathfrak{T}$ denotes the topology on X by $\|\cdot\|$ then $\|\cdot\|$ is $\mathfrak{T}$-continuous at 0 by the definition of $\mathfrak{T}$. For a p-norm we have the following proposition.

Proposition 3. If X is a p-normed space, then

$$\left|\|x\|^p - \|y\|^p\right| \leq \|x-y\|^p, x, y \in X. \qquad (1.6)$$

Proof.

To prove (1.6), we start with the fact that

$$\|x\|^p = \|(x-y)+y\|^p$$
$$\leq \|x-y\|^p + \|y\|^p$$
$$\therefore \ \|x\|^p - \|y\|^p \leq \|x-y\|^p \qquad (1.7)$$

Applying (1.7) we have

$$-(\|x\|^p - \|y\|^p) = \|y\|^p - \|x\|^p = \|y-x\|^p$$
$$\|y-x\|^p = \|(-1)(x-y)\|^p = |-1|\|x-y\|^p = \|x-y\|^p \qquad (1.8)$$

On combining (1.7) and (1.8), result follows immediately. ∎

Proposition 4. If X is a p-normed space, then $\|\cdot\|_p : X \to R^+$ is a continuous function on X.

Proof.

To show that $\|\cdot\|_p$ is a continuous function on X, we make use of the fact that if $x_n \to x$ implies $\|x_n\| \to \|x\|$. Applying Theorem 1, we have

$$\|x_n\|^p - \|x\|^p \leq \|x_n - x\|^p \qquad (1.9)$$

Since $x_n \to x$ given $\varepsilon > 0$, there exists a $\mathbb{N}$ such that

$$\|x_n - x\|^p < \varepsilon \text{ for all } n \geq \mathbb{N} \qquad (1.10)$$

Using (1.10) in (1.9), we have

$\|x_n\|^p - \|x\|^p < \varepsilon$ for all $n \geq \mathbb{N}$ so that $\|x_n\|^p \to \|x\|^p$ which shows that p-norm is a continuous function on X. ∎

Proposition 5. The operations of addition and scalar multiplication of p-norm in X are jointly continuous.

Proof.

To prove the continuity of operations of addition and scalar multiplication we show that if $x_n \to x \wedge y_n \to y \Rightarrow x_n + y_n \to x + y$ and if $\alpha_n \to \alpha \ \Rightarrow \alpha_n x_n \to \alpha x$.

Now since $x_n \to x, y_n \to y \ \wedge \alpha_n \to \alpha$ we have

$$\|x_n - x\|, \|y_n - y\| \ \wedge \ |\alpha_n - \alpha| \to 0 \ as \ n \to \infty \qquad (1.11)$$

and the sequence (α_n) is bounded. Consequently,

$$\|(x_n + y_n) - (x+y)\|^p = \|(x_n - x) + (y_n - y)\|^p$$
$$\leq \|x_n - x\|^p + \|y_n - y\|^p \to 0 \ as \ n \to \infty \quad (1.12)$$

We observe that by using (1.11), (1.12) $\to 0 \ as \ n \to \infty$ so that

$$\|(x_n + y_n) - (x+y)\|^p \to 0 \ as \ n \to \infty$$

Thus

$x_n + y_n \to x + y \ as \ n \to \infty$, showing that the addition is a continuous operation.

For the scalar multiplication, we work thus

$$\|\alpha_n x_n - \alpha x\|^p = \|\alpha_n x_n - \alpha_n x + \alpha_n x - \alpha x\|^p$$
$$= \|\alpha_n (x_n - x) + (\alpha_n - \alpha) x\|^p$$

$$\leq |\alpha_n| \|x_n - x\|^p + |\alpha_n - \alpha| \|x\|^p \quad (1.13)$$

$$\to 0 \; as \; n \to \infty \qquad \text{(from 1.11).}$$

Thus;

$$\alpha_n x_n \to \alpha x \; as \; n \to \infty$$

This shows that scalar multiplication is continuous.

Consequently, the addition of vectors and scalar multiplication of the p-norm in X are jointly continuous.

2. Conclusion. From the investigation, we see that though p −normed spaces are not locally convex linear topological convex spaces, but still exhibit some topological vector spaces properties evident which is continuity in the $\|\cdot\|_p$.

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