IJISET - International Journal of Innovative Science, Engineering & Technology, Vol. 4 Issue 7, July 2017
ISSN (Online) 2348 – 7968 | Impact Factor (2016) – 5.264
www.ijiset.com

SINGLE CHROMATIC TRANSVERSAL DOMATIC NUMBER OF GRAPHS

J. Felix [a], R. Lawrence Joseph Manoharan[b], L. Benedict Michael Raj[c], S.K.Ayyaswamy [c]
[a,c]Department of Mathematics , St.Joseph's College(Autonomous),Trichy, India.
[b]Department of Mathematics, Hindustan College of Arts and Science, Chennai, India.
[c]Department of Mathematics, SASTRA University , Tanjore, India.
Email: felizeesimp@gmail.com [a],rljmano@gmail.com [b],
benedict.mraj@gmail.com [c],sjcayya@yahoo.co.in [c]

***Abstract*—A set $D \subset V$ of vertices in a graph $G = (V, E)$ is called a dominating set if for every $v \in V - D$, there exists a vertex $u \in D$ such that u and v are adjacent. The cardinality of a minimum dominating set is called the domination number of the graph denoted by $\gamma(G)$. A dominating set D is called a single chromatic transversal dominating set if D intersects every member (color class) of some chromatic partition, also called χ-partition, of G.This set is called a std-set. The cardinality of a minimum std-set is called the single chromatic transversal domination number of the graph G denoted by $\gamma_{st}(G)$. The single chromatic domatic number of G is the maximum order of a partition of $V(G)$ into std-sets of G and is denoted by $d_{st}(G)$. In this paper, we obtain some bounds for $d_{st}(G)$ and characterize graphs attaining the bounds $d_{st}(G) \leq p/2$ and $d_{st}(G) + \gamma_{st}(G) \leq p + 1$. We also obtain Nordhaus-Gaddum inequalities for $d_{st}(G)$ and $d_{st}(G^c)$**

***Index Terms*—Domination number,domatic number, single chromatic transversal domination number, std-domatic number.**

I. Introduction

By a graph $G = (V, E)$, we mean a finite, undirected graph with neither loops nor multiple edges. The order and size of G are denoted by p and q respectively. For graph theoretic terminology we refer to Harary[4]

Coloring and domination are two areas in graph theory which have been extensively studied. Graph coloring deals with the fundamental problem of partitioning vertex set into classes according to certain rules. Time tabling, sequencing and scheduling problems in their many terms are basically of this nature. The fundamental parameter in graph coloring is the chromatic number $\chi(G)$ of a graph G which is defined to be the minimum number of colors required to color the vertices of G in such a way that no two adjacent vertices of G receive the same color. A partition of $V(G)$ into $\chi(G)$ independent sets is called a chromatic partition or χ-partition of G

Another fastest growing area in graph theory is the study of domination and related subset problems such as independence, covering and matching. A set $D \subseteq V(G)$ is said to be a dominating set of G if every vertex in $V(G) - D$ is adjacent to a vertex in D. The minimum cardinality of a dominating set is called the domination number of G and is denoted by $\gamma(G)$. A comprehensive treatment of the fundamentals of domination is given in the book by Haynes et al.[5]. A survey of serval advanced topics in domination can be seen in Haynes et al.[6]. Benedict et al.[1]. introduced the concept of chromatic transversal domination using the concept of graph coloring and domination. A dominating set D of a graph G is called a chromatic transversal dominating set (ctd-set) if D is a transversal of every χ-partition of G. That is, D has non-empty intersection with every color class of every χ-partition of G. The minimum cardinality of a ctd-set of G is called chromatic transversal domination number of G and is denoted by $\gamma_{ct}(G)$. Obviously $\chi(G) \leqslant \gamma_{ct}(G)$.

Restricting a dominating set to be a transversal of at least one χ-partition of G, a new domination parameter namely single chromatic transversal domination parameter was defined by Lawrence et al.[8]. Accordingly a dominating set D is called a single chromatic transversal dominating set if D intersects every member (color class) of some χ-partition of G. This set is called a std-set. The cardinality of a minimum std-set is called the single chromatic transversal domination number of a graph G and it is denoted by $\gamma_{st}(G)$. Cockayane and Hedetniemi [3] introduced the concept of domatic number of graph. A partition $\{V_1, V_2, ..., V_k\}$ of $V(G)$ is a domatic partition of G if each V_i is a dominating set. The maximum order of a domatic partition of G is called the domatic number of G and is denoted by $d(G)$. A partition $\{V_1, V_2, ..., V_k\}$ of $V(G)$ is called a std-domatic partition of G if each V_i is an std-set. The number of sets of a std-domatic partition with maximum number of std-sets is called std-domatic number denoted by $d_{st}(G)$. Similarly we can define ctd-domatic partition and ctd-domatic number $d_{ct}(G)$ using ctd-sets.

The following are the exact values of the parameter γ_{st} for some standard graphs found in Lawrence et al.[8].

1. $\gamma_{st}(P_p) = \lceil\frac{p}{3}\rceil$, for all $p \geqslant 4$
2. $\gamma_{st}(C_p) = \lceil\frac{p}{3}\rceil$, for all $p \geqslant 6$
3. $\gamma_{st}(W_p) = \begin{cases} 3 & \text{if } p \text{ is odd,} \\ 4 & \text{if } p \text{ is even,} \end{cases}$ where W_p is a wheel with $(p-1)$ spokes.
4. For the Petersen graph P, $\gamma_{st}(P) = 4$ where as $\gamma_{ct}(P) = 5$.

***Theorem** 1.1:* [8] Let G be a connected bipartite graph G with bipartition (X, Y); $|X| \leqslant |Y|$ and $p \geqslant 3$. Then $\gamma_{st}(G) = \gamma(G)+1$ if and only if every vertex in X has at least two neighbours which are pendant vertices

For a bipartite graph G, $\gamma_{st}(G) = \gamma(G)$ or $\gamma(G)+1$. All connected bipartite graphs for which $\gamma_{st}(G) = \gamma(G)+1$ are called Type 2 graphs and all other bipartite graphs are Type 1 graphs. Note that a disconnected bipartite graph is a Type 1 Graph

***Definition** 1:* A vertex v of G is called a χ-critical vertex, if $\chi(G-v) < \chi(G)$.

***Theorem** 1.2:* [2] Let G be a uniquely colorabe graph. Then $d_{ct}(G) = 1$ if and only if G has a χ- critical vertex.

We note that ,

(i) $\gamma(G) \leq \gamma_{st}(G) \leq \gamma_{ct}(G)$.
(ii) $\gamma_{st}(G).d_{st}(G) \leq p$
(iii) For any graph that has a vertex of full degree or an isolated vertex, $d_{st}(G) = 1$.In particular, $d_{st}(G) = 1$,for star graphs.

***Result** 1:* It can be easily verified that

(i) $1 \leq d_{ct}(G) \leq d_{st}(G) \leq d(G) \leq \delta(G)$
(ii) $1 \leq d_{st}(G) \leq p/\gamma_{st}(G) \leq p/\chi(G) \leq \frac{p}{2}$
(iii) For any tree,$d_{st}(G) \leq 2$.
(iv) For the Petersen graph P, $d_{st}(P) = 2$

***Theorem** 1.3:* For any graph G, $d_{st}(G) + \gamma_{st}(G) \leq p+1$.Equality is attained if and only if $G = K_p$ or K_p^c.

Proof: For any graph G, $d_{st}(G).\gamma_{st}(G) \leq p$. When $d_{st}(G), \gamma_{st}(G) \geq 2$, we have $d_{st}(G)+\gamma_{st}(G) \leq p$. So the result is true in this case. If $d_{st}(G) = 1$, then $d_{st}(G) + \gamma_{st}(G) \leq p+1$. If $\gamma_{st}(G) = 1$, then G becomes a trivial graph so that $d_{st}(G) = 1$ and $d_{st}(G) + \gamma_{st}(G) = 2 = p+1$

We prove now the equality case. Let $d_{st}(G)+\gamma_{st}(G) = p+1$. Since $d_{st}(G).\gamma_{st}(G) \leq p$, either $\gamma_{st}(G) = 1$ and $d_{st}(G) = p$ or $d_{st}(G) = 1$ and $\gamma_{st}(G) = p$. In the former case $G = K_1$, in the latter case $\gamma_{st}(G) = p$ which implies that $G = K_p$ or K_p^c. The converse is obvious. ■

II. Bipartite and Unicylic Graphs

***Notation** 1:* If the vertex set of a graph is divided into two independent sets X and Y(X or Y but not both may be empty) such that every vertex in X is adjacent to every other vertices of Y and vice-versa, then this graph is denoted by $K_{m,n}$,where $|X| = m, |Y| = n$ and $m, n \geq 0$. If $m, n \geq 1$,then $K_{m,n}$ is a complete bipartite graph. $K_{m,0}, K_{0,m}$ are empty graphs with m isolated vertices. $K_{0,0}$ is meaningless.

***Notation** 2:* $K_{m,n}^k$ is a graph obtained by joining the centres of $K_{1,m}$ and $K_{1,n}$ by a path of length k, where m, n and k are non-negative integers.Denote these centres by x_0 and y_0 respectively. In $K_{m.n}^k$ if every vertex in $N(x_0)$ is adjacent to every vertex of $N(y_0)$, then such a graph is denoted by $H_{m,n}^k$. When k is odd $H_{m,n}$ is a bipartite graph.$K_{5,6}^3, K_{5,6}^0, K_{0,0}^3$ are given in Figure 1

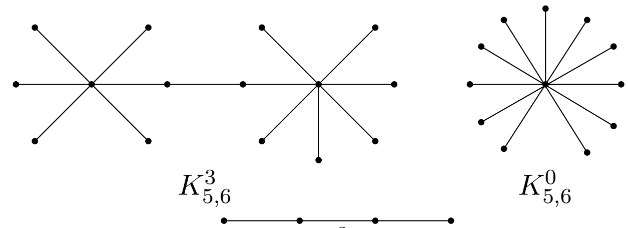

Figure 1

***Lemma** 1:* If G is a non-trivial graph, then $\gamma_{st}(G) = 2$ if and only if G is a bipartite graph that contains $K_{m,n}^k$ as a spanning subgraph, with $k = 0, 1$ or 3 or G is a union of two graphs $K_{m,1}$ and $K_{n,1}$ where $m, n \geq 0$; m, n and k are not identically zero.

Proof: If $\gamma_st(G) = 2$, then G is a bipartite graph with at most two components. Let $D = \{x, y\}$ be a γ_{st}-set of G and (X, Y) be a bipartition of G. Without loss of generality, let $x \in X$ and $y \in Y$. If G has two components,then $N(x) = Y - y$ and $N(y) = X - x$. So G becomes a disjoint union of two graphs namely $K_{1,m}$ and $K_{1,n}$, where m and n are non-negative integers both not identically zero. Let us assume that G be connected. In case x and y are adjacent, G will contain a $K_{m,n}^1$ as a spanning subgraph. If x and y are non adjacent, then $d(x, y) = 3$ and so G contains $K_{m,n}^3$ as a spanning subgraph.

Conversely, suppose G is a union of $K_{m,1}$ and $K_{n,1}$, where $m, n \geq 0$. Then the result is true. Let G be a bipartite graph that contains $K^k_{m,n}$ as a spanning subgraph (m, n and k are not identically zero and $k = 0, 1$ or 3). When either $k = 0$ or $k = 1$ and m or n equal to zero, G becomes a star graph, so that $\gamma_{st}(G) = 2$. When $k = 3$ or $k = 1$ and $m, n > 0$, G is a Type I graph and hence $\gamma_{st}(G) = \gamma(G) = 2$. ■

***Theorem** 2.1:* For a non-trivial graph G, $d_{st}(G) = \frac{p}{2}$ if and only if G is any one of the graphs $\overline{K_2}, 2K_2$ or $H^k_{n,m}$, where $k = 1$ or $k = 3$.

Proof: Let $d_{st}(G) = p/2$. Clearly p is even. By (ii) of result (iv) $\gamma_{st}(G) = 2$ and G is a bipartite graph stated in lemma 2.1. Since $d_{st}(G) = p/2$ there exists a std-domatic partition of G with $p/2$ std-sets such that each std-set has two vertices.Clearly $m = n$ and G is $\overline{K_2}$ or $2K_2$ or $H^k_{n,n}$, where $k = 1, 3$.The converse is obvious. ■

***Theorem** 2.2:* If G is a connected bipartite graph that is not a star, then $d_{st}(G) \geq 2$.

Proof: Let (X, Y) be the bipartition of G. As G is not a star, $|X|, |Y| \geq 2$. If G is the complete bipartite graph $K_{m,n}$, then $d_{st}(G) = \min(m, n) \geq 2$. If $G \neq K_{m,n}$, then there exists a vertex say $x \in X$ such that $N(x) \neq Y$. Then the set $S_1 = (X - x) \cup N(x)$ and $S_2 = (Y - N(x)) \cup \{x\}$ are std-sets of G. Hence $d_{st}(G) \geq 2$. ■

***Corollary** 1:* If G is a tree that is not a star, then $d_{st}(G) = 2$.

Proof: By Theorem 2.3, $d_{st}(G) \geq 2$. But by (iii) of Result 1.4 , $d_{st}(G) \leq 2$. Hence $d_{st}(G) = 2$. ■

***Theorem** 2.3:* If G is a connected unicyclic graph, then $d_{st}(G) \leq 3$.

Proof: Since G is unicyclic, it contains a unique cycle. Let $G = C_p$. $d_{st}(C_3) = d_{st}(C_5) = 1$ and $d_{st}(C_4) = 2$. Now consider the case when $p \geq 6$. $\gamma_{st}(C_p) = \lceil \frac{p}{3} \rceil \geq \frac{p}{3}$. Hence $d_{st}(G) \leq \dfrac{p}{\gamma_{st}(G)} \leq 3$. In case $G \neq C_p$, G has a pendant vertex and therefore $d_{st}(G) \leq 2$. ■

***Corollary** 2:* For a connected unicyclic graph, $d_{st}(G) = 3$ if and only if $G = C_p$, where $p \equiv 0(mod3)$.

Proof: Let G be a connected unicycle graph. Let $d_{st}(G) = 3$. Clearly $G = C_p$, where $p \geq 6$. Since $\gamma_{st}(G) = \lceil p/3 \rceil$ and $d_{st}(G) \leq \dfrac{p}{\gamma_{st}(G)}$, we have $p \equiv 0(mod3)$.
Conversely, let $G = C_p$, $p \equiv 0(mod3)$ and $p \geq 6$. Let $C_p = (v_1, v_2, ..., v_p)$.Clearly the sets $\{v_1, v_4, ..., v_{p-2}\}$, $\{v_2, v_5, ..., v_{p-1}\}$ and $\{v_3, v_6, ..., v_p\}$ are γ_{st}-sets for some χ-partition of C_p as C_p is not uniquely colourable. Hence $d_{st}(G) = 3$. ■

***Corollary** 3:* For a connected unicyclic graph G, with p vertices.

$$d_{st}(G) = \begin{cases} 1 & \text{if } G \text{ is } C_k,\ k = 3, 5, ..., \\ 3 & \text{if } G \text{ is } C_{3k},\ k = 2, 3, ..., \\ 2 & \text{otherwise.} \end{cases}$$

Proof: Let G be a connected unicyclic graph. Suppose $G \neq C_p$. Then there exists a pendant vertex in G. Hence $d_{st}(G) \leq 2$. It is always possible to find a std-domatic partition with just two std-sets. Hence $d_{st}(G) = 2$.
Let $G = C_p$. $\gamma_{st}(C_p) = \lceil \frac{p}{3} \rceil$. If $p \equiv 0(mod3)$ and $p \geq 6$, by corollary 2.6 $d_{st}(C_p) = 3$. When $p = 4$, $d_{st}(C_p) = 2$ and when $p = 3$ or 5, $d_{st}(C_p) = 1$. Let $p \geq 7$ and not equal to $3k$, where $k = 3, 4, ...,$. $\gamma_{st}(C_p) = \lceil \frac{p}{3} \rceil > \frac{p}{3}$ and so $d_{st}(C_p) \leq 2$. It is always possible to obtain a std-domatic partition with just two std-sets. Hence $d_{st}(C_p) = 2$. ■

III. Nordhaus-Gaddum type inequalities

We present the Nordhaus-Gaddum type inequalities in the following theorems
Let $\mathcal{G}$ be the collection of graphs $P_4, C_4, C^c_4, K_2, K^c_2$.

***Theorem** 3.1:* For any non-trivial graph G, $2 \leq d_{st}(G) + d_{st}(G^c) \leq p$ and $1 \leq d_{st}(G).d_{st}(G^c) \leq \frac{p^2}{4}$. The bounds are sharp and the upper bounds are attained if and only if $G \in \mathcal{G}$

Proof: For any non-trivial graph, we have $1 \leq d_{st}(G) \leq \dfrac{p}{\gamma_{st}(G)} \leq \dfrac{p}{2}$. Therefore,$1 \leq d_{st}(G) \leq \dfrac{p}{2}$. Similarly $1 \leq d_{st}(G^c) \leq \dfrac{p}{2}$. Adding and multiplying we get,$2 \leq d_{st}(G) + d_{st}(G^c) \leq p$ and $1 \leq d_{st}(G)d_{st}(G^c) \leq \dfrac{p^2}{4}$. Let $d_{st}(G) + d_{st}(G^c) = p$. Since $d_{st}(G)$ and $d_{st}(G^c) \leq \dfrac{p}{2}$, we must have $d_{st}(G) = d_{st}(G^c) = \dfrac{p}{2}$. Hence p is even. If $p = 2$, then $G = K_2$ or K^c_2. Let $p \geq 4$. If $\delta(G) = 0$ or $p-1$, then $d_{st}(G) = d_{st}(G^c) = 1 \neq \dfrac{p}{2}$, a contradiction. Hence $0 < \delta(G), \delta(G^c) < p - 1$. Since $\chi(G) \leq \gamma_{st}(G) \leq \dfrac{p}{d_{st}(G)} = 2$, we have $\chi(G) = 2$. and $\gamma_{st}(G) = 2$. Similarly $\chi(G^c) = 2$ and $\gamma_{st}(G^c) = 2$. Hence G and G^c are both bipartite graphs.
Let (X, Y) be a bipartition of G. If $|X| \geq 3$,then in G^c we get a clique of order 3 or more which is not possible. Hence $|X| \leq 2$ and similarly $|Y| \leq 2$.Therefore $p = 4$ and $|X| = |Y| = 2$. The various possibilities of G are P_4, C_4, C^c_4. If $d_{st}(G).d_{st}(G^c) = \dfrac{p^2}{4}$, then $d_{st}(G) = d_{st}(G^c) = \dfrac{p}{2}$. Hence G is the same as above. ■

***Theorem** 3.2:* Let G be a graph not in $\mathcal{G}$. Then $d_{st}(G) + d_{st}(G^c) \leq p - 1$. The equality is attained if and only if $G \cong K_3, K_3^c, P_3, P_3^c, K_{3,3}, K_{3,3} - X$, where X is matching of $K_{3,3}$.

Proof: In view of Theorem 3.1, $d_{st}(G)+d_{st}(G^c) \leq p - 1$.Assume that $d_{st}(G) + d_{st}(G^c) = p - 1$.

***Case** 1:* Let p be even. Then obviously $p \geq 4$. $d_{st}(G)+d_{st}(G^c) = p-1$, which implies that $d_{st}(G) = \frac{p}{2}$ and $d_{st}(G^c) = \frac{p}{2}-1$ or $d_{st}(G) = \frac{p}{2}-1$ and $d_{st}(G^c) = \frac{p}{2}$. Consider the case $d_{st}(G) = \frac{p}{2}$ and $d_{st}(G^c) = \frac{p}{2} - 1$.The other choice can be similarly dealt with. As $p \geq 4$ and $d_{st}(G) = \frac{p}{2}$, we have $d_{st}(G) \neq 1$. Hence $0 < \delta(G) < p - 1$. This implies that $\chi(G) \geq 2$. Also $\chi(G) \leq \gamma_{st}(G) \leq \frac{p}{d_{st}(G)} = 2$. So $\chi(G) = \gamma_{st}(G) = 2$. Hence G is a bipartite graph with bipartition (X, Y). As $d_{st}(G) = \frac{p}{2}$ and $\gamma_{st}(G) = 2$, we have $|X| = |Y|$. Since $|X| = \frac{p}{2}$, we have $\chi(G^c) \geq 2$. This implies that $\gamma_{st}(G^c) \geq \frac{p}{2}$. Hence $d_{st}(G) = 1$ or 2. So $p = 4$ or 6. No bipartite graph on four vertices satisfies the given conditions. Hence $p = 6$. In this case G is isomorphic to $K_{3,3}$ or $K_{3,3} - X$, where X is a matching of $K_{3,3}$.

***Case** 2:* Let p be odd. Since $d_{st}(G) + d_{st}(G^c) = p - 1$, we have $d_{st}(G) = d_{st}(G^c) = \frac{p-1}{2}$. When $p = 3, G \cong K_3, K_3^c, P_3, P_3^c$. Take $p \geq 5$. If $\delta(G) = 0$ or $p - 1$, then $d_{st}(G) = d_{st}(G^c) = 1 \neq \frac{(p-1)}{2}$, a contradiction. Hence $0 < \delta(G), \delta(G^c) < p - 1$. This implies that $\chi(G), \chi(G^c) \geq 2$. Also $\chi(G) \leq \gamma_{st}(G) \leq \frac{p}{d_{st}(G)} = \frac{2p}{p-1} < 3$. So $\chi(G) \leq \gamma_{st}(G) \leq 2$. Hence $\chi(G) = \gamma_{st}(G) = 2$.Similarly we have $\gamma_{st}(G^c) = \chi(G^c) = 2$. Let (X, Y) be a bipartition of G. Since $\gamma_{st}(G) = 2$ and $d_{st}(G) = \frac{p-1}{2}$, $|X|, |Y| \geq \frac{p-1}{2}$. Also $|X| + |Y| = p$ and p is odd. Hence $|X| \neq |Y|$. Without loss of generality, take $|X| > |Y| \geq \frac{p-1}{2}$. This implies that $\chi(G^c) \geq \frac{p+1}{2}$. So $\gamma_{st}(G) \geq \frac{p+1}{2} > \frac{p}{2}$ and $d_{st}(G^c) = 1$, which is again a contradiction, since $d_{st}(G^c) = \frac{p-1}{2} \neq 1$. Hence $p \geq 5$ does not arise. The converse is obvious. ■

The following are some interesting open problems.

1.Characterize graphs G for which

(i) $d_{st}(G) = d_{ct}(G)$
(ii) $d_{st}(G) = d(G)$
(iii) $d_{ct}(G) = d_{st}(G) = d(G)$
(iv) $d_{st}(G) + \gamma_{st}(G) = p$
(v) $d_{st}(G).\gamma_{st}(G) = p$

We have characterized all bipartite and unicyclic graphs for which $d_{st}(G) = 1$ and 2. Hence it is natural to ask the following.

(i) Characterize all graphs for which $d_{st}(G) = 1$ or 2,where G is neither a bipartite graph nor a unicyclic graph.

(ii) Given any three integers a, b, c, $a \leq b \leq c$, is it possible to find a graph G for which $d_{ct}(G) = a, d_{st}(G) = b$ and $d(G) = c$?

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