IJISET - International Journal of Innovative Science, Engineering & Technology, Vol. 5 Issue 1, January 2018
ISSN (Online) 2348 – 7968 | Impact Factor (2016) – 5.264
www.ijiset.com

An Approach to Bipolar Vague Group and its Properties

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Abstract

The study of fuzzy algebraic theory and the development of fuzzy algebraic structures is an important area of study. The study of vague algebraic structures and its properties has been extensively studied in literature. However, there are no studies that deal with vague algebraic theory in a bipolar setting. As such, this paper aims to initiate the study of the group theory for bipolar vague sets. The notion of bipolar vague groups, and bipolar vague normal subgroups are introduced, and the structural characteristics and properties of these structures are studied.

Keywords: *Bipolar vague set, Vague group, Normal vague subgroup, Conjugate bipolar vague group.*

1 Introduction

The study of fuzzy algebraic theory began with the introduction of the notion of a fuzzy subgroup of a group by Rosenfeld [1]. Since its inception, the study of fuzzy algebraic theory has been actively studied. However, the single-valued membership structure of the fuzzy set model [2] makes it incapable to capture the hesitancy faced by the users, and also makes it incapable of expressing the evidence for and against an element effectively. This and other problems that are inherent in fuzzy sets has led to the expansion of algebraic theory in other fuzzy based and soft set [3] based settings. The fuzzy algebraic framework has since been extended to other settings which include soft set setting [4–8], fuzzy soft setting [9–13], intuitionistic fuzzy setting [14–16], vague setting [17–21], and a vague soft set setting [22–24]. Here we are concerned with further developing the fuzzy algebraic theory in a vague set [25] setting using the properties of bipolar-valued fuzzy sets [26]. Bipolar-valued fuzzy sets [26] is an extension of fuzzy sets whose membership degree range is enlarged from the standard unit interval of $[0, 1]$ to the interval of $[-1, 1]$. Previous research on the development of bipolar-valued fuzzy algebraic theory include the study of bipolar-valued fuzzy subgroup of a group [27], bipolar fuzzy subalgebras and closed ideals of BCH-algebra by Jun et. al. [28], bipolar fuzzy subalgebras and bipolar fuzzy ideals in BCK/BCI algebra [29].

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In this paper, we introduce group theory for bipolar vague sets [30, 31]through the introduction of bipolar vague groups and normal subgroups. The properties and structural characteristics of these structures are also studied and verified.

2 Preliminaries

Definition 2.1. *[2] Let X be a nonempty set. A fuzzy set A over X is defined as $A = \{\langle x : \mu_A(x)\rangle : x \in X\}$, where $\mu_A : X \to [0, 1]$ is the membership function of the fuzzy set A.*

Definition 2.2. *[26] Let X be a universal set, and A be a set over X that is defined by a positive membership function μ_A^+ and a negative membership function μ_A^-, where $\mu_A^+ : X \to [0, 1]$ and $\mu_A^- : X \to [-1, 0]$. Then A is called a bipolar-valued fuzzy set over X, and can be written in the form $A = \{\langle x, \mu_A^+(x), \mu_A^-(x)\rangle : x \in X\}$.*

Definition 2.3. *[27] Let G be a group and A be a bipolar-valued fuzzy subsets of G. Then A is called a bipolar-valued subgroup of G(abbr. $BVFSG$) if the following conditions are satisfied.*

(i) $A^+(xy) \geq \min\{A^+(x), A^+(y)\}$

(ii) $A^+(x^{-1}) \geq A^+(x)$

(iii) $A^-(xy) \leq \max\{A^-(x), A^-(y)\}$

(iv) $A^-(x^{-1}) \leq A^-(x)$.

Definition 2.4. *[25] Let X be a space of points (objects) with element of X denoted by x. A vague set V in X is characterized by a truth-membership function $t_V : X \to [0, 1]$ and a false-membership function $f_V : X \to [0, 1]$. The value $t_V(x)$ is a* *lower bound on the grade of membership of x derived from the evidence for x and $f_V(x)$ is a lower bound on the negation of x derived from the evidence against x. The values $t_V(x)$ and $f_V(x)$ both associate a real number in the interval $[0,1]$ with each point in X, where $0 \leq t_V(x) + f_V(x) \leq 1$. This approach bounds the grade of membership of x to a closed subinterval $[t_V(x), 1-f_V(x)]$ of $[0,1]$.*

Next, we present some basic results pertaining to the concepts and operations of vague sets. Let A and B be two vague sets over the universe U, where A and B are as defined below:

$$A = \{\langle u, [t_A(x), 1 - f_A(x)]\rangle : u \in U\},$$

$$B = \{\langle u, [t_B(x), 1 - f_B(x)]\rangle : u \in U\}.$$

Definition 2.5. *[25] The vague value and unit vague set of a vague set A are as defined below:*

(i) The interval $[t_A(x), 1 - f_A(x)]$ is called the vague value of x in A, and is denoted by $V_A(x)$.

(ii) A vague set A of U is called a unit vague set if $t_A(x) = 1$ and $f_A(x) = 0$, for all $x \in U$.

(iii) A vague set A of U is called a null vague set if $t_A(x) = 0$ and $f_A(x) = 1$, for all $x \in U$.

Definition 2.6. *[25] The subset, complement, union and intersection of vague sets are as defined below:*

(i) If for all $x \in U$, $t_A(x) \leq t_B(x)$ and $1 - f_A(x) \leq 1 - f_B(x)$, then A is called a vague subset of B, denoted as $A \subseteq B$.

(ii) The complement of A, denoted as A^c is defined as:

$$A^c = \{(x, [f_A(x), 1 - t_A(x)]) : x \in X\}.$$

(iii) The union of A and B, denoted as $A \cup B$, is a vague set C, defined as $C = \{(x, [\max(t_A(x), t_B(x)), \max(1 - f_A(x), 1 - f_B(x))]) : x \in X\}$.

(iv) The intersection of A and B, denoted as $A \cap B$, is a vague set D, defined as $D = \{(x, [\min(t_A(x), t_B(x)), \min(1 - f_A(x), 1 - f_B(x))]) : x \in X\}$.

Definition 2.7. *[17] Let $(X, *)$ be a group and A be a vague set over X. Then A is called a vague group over X if the following conditions are satisfied:*

(i) $t_A(xy) \geq \min(t_A(x), t_A(y))$ and $1 - f_A(xy) \geq \min(1 - f_A(x), 1 - f_A(y))$;

(ii) $t_A(x^{-1}) \geq t_A(x)$ and $1 - f_A(x^{-1}) \geq 1 - f_A(x)$.

Definition 2.8. *[30] Let X be a universe of discourse, and A be an object over X. Then A is called a bipolar vague set which is of the form: $A = \{\langle x, [t_A^+(x), 1 - f_A^+(x)], [-1 - f_A^-(x), t_A^-(x)]\rangle : x \in X\}$, where $[t_A^+, 1 - f_A^+] : X \to [0,1]$ and $[-1 - f_A^-, t_A^-] : X \to [-1, 0]$ are mappings such that $t_A^+ + f_A^+ \leq 1$ and $-1 \leq t_A^- + f_A^-$. The positive membership degree $[t_A^+(x), 1 - f_A^+(x)]$ denotes the interval of satisfaction of an element x to the property corresponding to a bipolar-valued fuzzy set A, and the negative membership degree $[-1 - f_A^-(x), t_A^-(x)]$ denotes the interval of satisfaction of x to some implicit counter property of A.*

For the sake of simplicity, the notation $v_A^+ = [t_A^+, 1 - f_A^+]$ and $v_A^- = [-1 - f_A^-, t_A^-]$ will be used to denote a bipolar vague set.

3 Bipolar Vague Groups

Definition 3.1. *Let $(X, *)$ be a group and A be a bipolar vague set over X. Then A is called a bipolar vague group of X if it satisfies the following conditions:*

(i) $t_A^+(xy) \geq \min(t_A^+(x), t_A^+(y))$ and $1 - f_A^+(xy) \geq \min(1 - f_A^+(x), 1 - f_A^+(y))$;

(ii) $t_A^+(x^{-1}) \geq t_A^+(x)$ and $1 - f_A^+(x^{-1}) \geq 1 - f_A^+(x)$;

(iii) $t_A^-(xy) \leq \max(t_A^-(x), t_A^-(y))$ and $-1 - f_A^-(xy) \leq \max(-1 - f_A^-(x), -1 - f_A^-(y))$;

(iv) $t_A^-(x^{-1}) \leq t_A^-(x)$ and $-1 - f_A^-(x^{-1}) \leq -1 - f_A^-(x)$.

Definition 3.2. *Let $A = (X; V_A^+, V_A^-)$ be a bipolar vague group over X and $H = \{x \in G / V_A^+(x) = V_A^+(e)$ and $V_A^-(x) = V_A^-(e)\}$, then $O(A)$, order of A is defined as $O(A) = O(H)$.*

Definition 3.3. *Let $A = (X; V_A^+, V_A^-)$ and $B = (X; V_B^+, V_B^-)$ be two bipolar vague groups over group X. Then A and B are said to be conjugate bipolar vague groups in X if for some $g \in G$,*

$$V_A^+(x) = V_B^+(g^{-1}xg)$$

and

$$V_A^-(x) = V_B^-(g^{-1}xg)$$

for every $x \in X$.

Example 3.4. *Let $G = \{1, \omega, \omega^2\}$ where ω is the cubic root of unity with the binary operation defined as below:*

$*$	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	1
ω^2	ω^2	1	ω

Let $A = (X; V_A^+, V_A^-)$ be a bipolar vague set in X which is as defined below:

	1	ω	ω^2
V_A^+	$[0.9, 0.9]$	$[0.6, 0.8]$	$[0.6, 0.8]$
V_A^-	$[-0.4, -0.1]$	$[-0.4, -0.1]$	$[-0.4, -0.1]$

Then $A = (X; V_A^+, V_A^-)$ is a bipolar vague group of the group X.

Theorem 3.5. *Let $A = (X; V_A^+, V_A^-)$ be a bipolar vague group over group. Then the following properties hold true for each $x \in X$:*

(i) $V_A^+(x^{-1}) = V_A^+(x)$, *i.e.* $t_A^+(x^{-1}) = t_A^+(x)$ *and* $1 - f_A^+(x^{-1}) = 1 - f_A^+(x)$;

(ii) $V_A^-(x^{-1}) = V_A^-(x)$, *i.e.* $t_A^-(x^{-1}) = t_A^-(x)$ *and* $-1 - f_A^-(x^{-1}) = -1 - f_A^-(x)$;

(iii) $V_A^+(e) \geq V_A^+(x)$, *i.e.* $t_A^+(e) \geq t_A^+(x)$ *and* $1 - f_A^+(e) \geq 1 - f_A^+(x)$;

(iv) $V_A^-(e) \leq V_A^-(x)$, *i.e.* $t_A^-(e) \leq t_A^-(x)$ *and* $-1 - f_A^-(e) \leq -1 - f_A^-(x)$,

where e is the identity element of X.

Proof. Let $x \in X$. Then

(i)

$$V_A^+(x) = V_A^+[(x^{-1})^{-1}] \geq V_A^+(x^{-1}) \geq V_A^+(x)$$

implies that

$$V_A^+(x^{-1}) = V_A^+(x)$$

and

$$V_A^-(x) = V_A^-[(x^{-1})^{-1}] \leq V_A^-(x^{-1}) \leq V_A^-(x)$$

implies that

$$V_A^-(x^{-1}) = V_A^-(x)$$

(ii)

$$\begin{aligned} V_A^+(e) &= V_A^+(xx^{-1}) \\ &\geq \min\{V_A^+(x), V_A^+(x^{-1})\} \\ &= V_A^+(x) \end{aligned}$$

implies that

$$V_A^+(e) \geq V_A^+(x)$$

and

$$\begin{aligned} V_A^-(e) &= V_A^-(xx^{-1}) \\ &\leq \max\{V_A^-(x), V_A^-(x^{-1})\} \\ &= V_A^-(x) \end{aligned}$$

implies that

$$V_A^-(e) \leq V_A^-(x).$$

Theorem 3.6. *Let $A = (X; V_A^+, V_A^-)$ be a bipolar vague group in a group X if and only if*

$$V_A^+(xy^{-1}) \geq \min\{V_A^+(x), V_A^+(y)\}$$

and

$$V_A^-(xy^{-1}) \leq \max\{V_A^-(x), V_A^-(y)\}$$

for each $x, y \in X$.

Proof. Let A be a bipolar vague group on X. Then we have

$$\begin{aligned} V_A^+(xy^{-1}) &\geq \min\{V_A^+(x), V_A^+(y^{-1})\} \\ &= \min\{V_A^+(x), V_A^+(y)\} \end{aligned}$$

and

$$\begin{aligned} V_A^-(xy^{-1}) &\leq \max\{V_A^-(x), V_A^-(y^{-1})\} \\ &= \max\{V_A^-(x), V_A^-(y)\} \end{aligned}$$

for each $x, y \in X$. Conversely suppose that

$$V_A^+(xy^{-1}) \geq \min\{V_A^+(x), V_A^+(y)\}$$

and

$$V_A^-(xy^{-1}) \leq \max\{V_A^-(x), V_A^-(y)\}.$$

Let $x = y$ to obtain

$$V_A^+(e) \geq V_A^+(x)$$

and

$$V_A^-(e) \leq V_A^-(x)$$

for all $x \in X$.

Hence

$$\begin{aligned} V_A^+(y^{-1}) &= V_A^+(ey^{-1}) \\ &\geq \min\{V_A^+(e), V_A^+(y)\} \end{aligned}$$

and it follows that

$$\begin{aligned} V_A^+(xy) &= V_A^+(x(y^{-1})^{-1}) \\ &\geq \min\{V_A^+(x), V_A^+(y^{-1})\} \\ &= \min\{V_A^+(x), V_A^+(y)\} \end{aligned}$$

and

$$\begin{aligned} V_A^-(y^{-1}) &= V_A^-(ey^{-1}) \\ &\leq \min\{V_A^-(e), V_A^-(y)\} \\ &= V_A^-(y) \end{aligned}$$

and it follows that

$$\begin{aligned} V_A^-(xy) &= V_A^-(x(y^{-1})^{-1}) \\ &\leq \max\{V_A^-(x), V_A^-(y^{-1})\} \\ &= \max\{V_A^-(x), V_A^-(y)\}. \end{aligned}$$

Theorem 3.7. *Let $A = (X; V_A^+, V_A^-)$ be a bipolar vague group over group X. If*

$$V_A^+(xy^{-1}) = V_A^+(e)$$

and

$$V_A^-(xy^{-1}) = V_A^-(e)$$

for any $x, y \in X$ then

$$V_A^+(x) = V_A^+(y)$$

and

$$V_A^-(x) = V_A^-(y).$$

Proof.

$$\begin{aligned} V_A^+(x) &= V_A^+[(xy^{-1})y] \\ &\geq \min\{V_A^+(xy^{-1}), V_A^+(y)\} \\ &= \min\{V_A^+(e), V_A^+(y)\} \\ &= V_A^+(y) \\ &= V_A^+[(yx^{-1})x] \\ &\geq \min\{V_A^+(e), V_A^+(x)\} \\ &= V_A^+(x) \end{aligned}$$

and

$$\begin{aligned} V_A^-(x) &= V_A^-[(xy^{-1})y] \\ &\leq \max\{V_A^-(xy^{-1}), V_A^-(y)\} \\ &= \max\{V_A^-(e), V_A^-(y)\} \\ &= V_A^-(y) \\ &= V_A^-[(yx^{-1})x] \\ &\leq \max\{V_A^-(e), V_A^-(x)\} \\ &= V_A^-(x). \end{aligned}$$

Theorem 3.8. *Let $A = (X; V_A^+, V_A^-)$ be a bipolar vague group in a group X and let $x \in X$. Then*

$$V_A^+(xy) = V_A^+(y)$$

and

$$V_A^-(xy) = V_A^-(y)$$

for each $y \in X$ if and only if $V_A^+(x) = V_A^+(e)$ and $V_A^-(x) = V_A^-(e)$.

Proof. Let $V_A^+(xy) = V_A^+(y)$ for each $y \in X$. Then we have

$$V_A^+(x) = V_A^+(xe) = V_A^+(e)$$

and

$$V_A^-(xy) = V_A^-(y)$$

for each $y \in X$ so we have

$$V_A^-(x) = V_A^-(xe) = V_A^-(e).$$

Conversely, let

$$V_A^+(x) = V_A^+(e)$$

and

$$V_A^-(x) = V_A^-(e)$$

by theorem 3.5 we have $V_A^+(y) \leq V_A^+(x)$ for each $y \in X$ since A is bipolar vague group over X we have

$$\begin{aligned} V_A^+(xy) &\geq \min\{V_A^+(x), V_A^+(y)\} \\ &= V_A^+(y) \\ i.e., V_A^+(xy) &\geq V_A^+(y) \end{aligned}$$

and

$$\begin{aligned} V_A^-(xy) &\leq \max\{V_A^-(x), V_A^-(y)\} \\ &= V_A^-(y) \\ i.e., V_A^-(xy) &\leq V_A^-(y). \end{aligned}$$

By Theorem 3.6, we have

$$\begin{aligned} V_A^+(y) &= V_A^+(x^{-1}xy) \\ &\geq \min\{V_A^+(x), V_A^+(xy)\} \\ &= V_A^+(xy) \end{aligned}$$

and

$$\begin{aligned} V_A^-(y) &= V_A^-(x^{-1}xy) \\ &\leq \max\{V_A^-(x), V_A^-(xy)\} \\ &= V_A^-(xy). \end{aligned}$$

Hence it can be concluded that

$$V_A^+(xy) = V_A^+(y)$$

and

$$V_A^-(xy) = V_A^-(y)$$

for each $y \in X$.

Theorem 3.9. *Let $\psi : X \to Y$ be a group homomorphism and let V be a bipolar vague set in Y. If V is a bipolar vague group over Y, then $\psi^{-1}(V)$ is a bipolar vague group over X.*

Proof. Let $x, y \in G$. Then we have the following:

$$\begin{aligned} V^+_{\psi^{-1}(B)}(xy) &= \psi^{-1}(V_B^+)(xy) \\ &= (V_B^+)\psi(xy) \\ &= V_B^+[\psi(x)\psi(y)] \\ &\geq \min\{V_B^+(\psi(x)), V_B^+(\psi(y))\} \\ &= \min\{\psi^{-1}(V_A^+)(x), \psi^{-1}(V_A^+)(y)\} \end{aligned}$$

and

$$\begin{aligned} V^-_{\psi^{-1}(B)}(xy) &= \psi^{-1}(V_B^-)(xy) \\ &= (V_B^-)\psi(xy) \\ &= V_B^-[\psi(x)\psi(y)] \\ &\leq \max\{V_B^-(\psi(x)), V_B^-(\psi(y))\} \\ &= \max\{\psi^{-1}(V_A^-)(x), \psi^{-1}(V_A^-)(y)\}. \end{aligned}$$

Let $x \in X$. Then we have the following:

$$\begin{aligned} V^+_{\psi^{-1}(B)}(x^{-1}) &= \psi^{-1}(V^+_B)(x^{-1}) \\ &= (V^+_B)\psi(x^{-1}) \\ &= V^+_B[(\psi(x))^{-1}] \\ &\geq V^+_B\psi(x) \\ &= V^+_{\psi^{-1}(B)}(x) \end{aligned}$$

and

$$\begin{aligned} V^+_{\psi^{-1}(B)}(x^{-1}) &= \psi^{-1}(V^-_B)(x^{-1}) \\ &= (V^-_B)\psi(x^{-1}) \\ &= V^-_B[(\psi(x))^{-1}] \\ &\leq V^+_B\psi(x) \\ &= V^-_{\psi^{-1}(B)}(x). \end{aligned}$$

Definition 3.10. *Let $A = (X; V^+_A, V^-_A)$ be a bipolar vague set over group in G. Let $\theta : G \to G$ be a map, and we define the maps $V^{\theta+}_A : G \to [0,1]$ and $V^{\theta-}_A : G \to [-1,0]$ which are as given below, respectively:*

(i) $V^{\theta+}_A(g) = V^+_A(\theta(g)), \forall g \in G$, and

(ii) $V^{\theta-}_A(g) = V^-_A(\theta(g)), \forall g \in G$.

Theorem 3.11. *If A is a bipolar vague group over group G and θ is a homomorphism of G, then the bipolar vague set A^θ over G is given by*

$$A^\theta = \{\langle g, V^{\theta+}_A, V^{\theta-}_A\rangle : g \in G\}$$

is also a bipolar vague group of G.

Proof. Let $x, y \in G$, then

$$\begin{aligned} V^{\theta+}_A(xy) &= V^+_A(\theta(xy)) \\ &= V^+_A(\theta(x)\theta(y)) \\ &\geq \min\{V^+_A(\theta(x), V^+_A(\theta(y)\} \end{aligned}$$

and

$$\begin{aligned} V^{\theta-}_A(xy) &= V^-_A(\theta(xy)) \\ &= V^-_A(\theta(x)\theta(y)) \\ &\leq \max\{V^-_A(\theta(x), V^-_A(\theta(y)\}. \end{aligned}$$

In addition, for all $x \in G$ we have the following

$$\begin{aligned} V^{\theta+}_A(x^{-1}) &= V^+_A(\theta(x^{-1})) \\ &= V^+_A((\theta(x))^{-1}) \\ &\geq V^+_A\theta(x) \\ &\geq V^{\theta+}_A(x) \end{aligned}$$

and

$$\begin{aligned} V^{\theta-}_A(x^{-1}) &= V^-_A(\theta(x^{-1})) \\ &= V^-_A((\theta(x))^{-1}) \\ &\leq V^-_A(\theta(x)) \\ &\leq V^{\theta-}_A(x). \end{aligned}$$

Hence it can be concluded that, A^θ is a bipolar vague group over G.

Theorem 3.12. *Let $A = (X; V^+_A, V^-_A)$ and $B = (X; V^+_B, V^-_B)$ be two bipolar vague subsets of a Abelian group G. Then A and B are conjugate bipolar vague subsets of G if and only if $A = B$.*

Proof. ($\Rightarrow$) Let A and B be conjugate bipolar vague subsets of group G, Then for some $y \in G$ we have

$$\begin{aligned} V^+_A(x) &= V^+_B(y^{-1}xy) \\ &= V^+_B(y^{-1}yx) \\ &= V^+_B(ex) \\ &= V^+_B(x). \end{aligned}$$

and

$$\begin{aligned} V^-_A(x) &= V^-_B(y^{-1}xy) \\ &= V^-_B(y^{-1}yx) \\ &= V^-_B(ex) \\ &= V^-_B(x). \end{aligned}$$

That shows that $A = B$.
($\Leftarrow$) Conversely if $A = B$, then for the identity element e of group G, we have $V^+_A(x) = V^+_B(e^{-1}xe)$ and $V^-_A(x) = V^-_B(e^{-1}xe)$ for every $x \in G$. Hence A and B are conjugate bipolar vague subsets of G.

Theorem 3.13. *If $A = (X; V^+_A, V^-_A)$ and $B = (X; V^+_B, V^-_B)$ are conjugate bipolar vague groups over G, then $O(A) = O(B)$.*

Proof. Let A and B are conjugate bipolar vague groups over G. Thus it follows that:

$$\begin{aligned} O(A) &= \text{order of } \{x \in G / V^+_A(x) = V^+_A(e) \\ &\quad \text{and } V^-_A(x) = V^-_A(e)\} \\ &= \text{order of } \{x \in G / V^+_B(y^{-1}xy) \\ &= V^+_B(y^{-1}ey) \text{ and} \\ &\quad V^-_B(y^{-1}xy) = V^-_B(y^{-1}ey)\} \\ &= \text{order of } \{x \in G / V^+_B(x) = V^+_B(e) \\ &\quad \text{and } V^-_B(x) = V^-_B(e)\} \\ &= O(B). \end{aligned}$$

Therefore, the result follows.

Definition 3.14. *Let X be a group and A be a bipolar vague set on X.*
Then A is called a bipolar vague normal subgroup over X , if

$$V^+_A(xyx^{-1}) \geq V^+_A(y)$$

and

$$V^-_A(xyx^{-1}) \leq V^-_A(y)$$

for all $x, y \in X$. The set of all bipolar vague normal subgroups on X are denoted by $BVNS(X)$.

Theorem 3.15. *Let X be a group. If $A, B \in BVNS(X)$ then $A \cap B \in BVNS(X)$.*

Proof. Let $A = \{\langle [t_A^+(x), 1 - f_A^+(x)][-1 - f_A^-(x), t_A^-(x)]\rangle / x \in X\}$,
and $B = \{\langle [t_B^+(x), 1 - f_B^+(x)][-1 - f_B^-(x), t_B^-(x)]\rangle / x \in X\}$.
Then the intersection of A and B is as given below:

$$\begin{aligned} A \cap B = \{\langle x, &\min\{t_A^+(x), t_B^+(x)\}\ , \\ &\min\{1 - f_A^+(x), 1 - f_B^+(x)\}, \\ &\max\{-1 - f_A^+(x), -1 - f_B^+(x)\}, \\ &\max\{t_A^-(x), t_B^-(x)\}\rangle / x \in X\}\,. \end{aligned}$$

Let $\alpha, \beta, \gamma, \delta$ denote the following expressions
$\alpha_{A\cap B} = \min\{t_A^+(x), t_B^+(x)\}$,
$\beta_{A\cap B} = \min\{1 - f_A^+(x), 1 - f_B^+(x)\}$,
$\gamma_{A\cap B} = \max\{-1 - f_A^-(x), -1 - f_B^-(x)\}$,
$\delta_{A\cap B} = \max\{t_A^-(x), t_B^-(x)\}$.
Then for any arbitrary $x, y \in X$, and $A, B \in BVNS(X)$, we have the following:

$$\begin{aligned} \alpha_{A\cap B}(xyx^{-1}) &= \min\{t_A^+(xyx^{-1}), t_B^+(xyx^{-1})\} \\ &\geq \min\{t_A^+(y), t_B^+(y)\} \\ &= \alpha_{A\cap B}(y), \\ \beta_{A\cap B}(xyx^{-1}) &= \min\{1 - f_A^+(xyx^{-1}), \\ &\quad 1 - f_B^+(xyx^{-1})\} \\ &\geq \min\{1 - f_A^+(y), 1 - f_B^+(y)\} \\ &= \beta_{A\cap B}(y), \\ \gamma_{A\cap B}(xyx^{-1}) &= \max\{-1 - f_A^-(xyx^{-1}), \\ &\quad -1 - f_B^-(xyx^{-1})\} \\ &\leq \max\{-1 - f_A^-(y), -1 - f_B^-(y)\} \\ &= \gamma_{A\cap B}(y), \\ \delta_{A\cap B}(xyx^{-1}) &= \max\{t_A^-(xyx^{-1}), t_B^-(xyx^{-1})\} \\ &\leq \max\{t_A^-(y), t_B^-(y)\} \\ &= \delta_{A\cap B}(y). \end{aligned}$$

Hence it can be concluded that

$$A \cap B \in BVNS(X).$$

Theorem 3.16. *Let $A = (X; V_A^+, V_A^-)$ be a bipolar vague subset of X. Then the following conditions are equivalent:*

(i) $A \in BVNS(X)$;

(ii) $A(xyx^{-1}) = A(y)$ *for all* $x, y \in X$;

(iii) $A(xy) = A(yx)$ *for all* $x, y \in X$.

Proof.
$(i) \Rightarrow (ii)$
$(\Rightarrow)$ Let
$A = \{\langle [t_A^+(x), 1 - f_A^+(x)][-1 - f_A^-(x), t_A^-(x)]\rangle / x \in X\}$.

Since $A \in BVNS(X)$. Then by Definition 3.14, for arbitrary $x, y \in X$, we have

$$V_A^+(xyx^{-1}) \geq V_A^+(y)$$

and

$$V_A^-(xyx^{-1}) \leq V_A^-(y)$$

for all $x, y \in X$. Thus, by taking for any arbitrary x, we obtain

$$V_A^+(x^{-1}yx) = V_A^+(x^{-1}y(x^{-1})^{-1}) \geq V_A^+(y).$$

Therefore it follows that:

$$V_A^+(y) = V_A^+(x^{-1}(xy(x^{-1})x)) \geq V_A^+(xyx^{-1})$$

i.e., $V_A^+(xyx^{-1}) = V_A^+(y)$ and

$$V_A^-(x^{-1}yx) = V_A^-(x^{-1}y(x^{-1})^{-1}) \leq V_A^-(y).$$

Thus we have

$$V_A^-(y) = V_A^-(x^{-1}(xy(x^{-1})x)) \leq V_A^-(xyx^{-1}) \leq V_A^-(y)$$

i.e.,$V_A^-(xyx^{-1}) = V_A^-(y)$.
Hence for all $x, y \in X$, $A(xyx^{-1}) = A(y)$ is proved.
$(ii) \Rightarrow (iii)$
By using y for yx in (ii), we are able to prove (iii) easily.
$(iii) \Rightarrow (i)$
From $A(yx) = A(xy)$, we obtain
$A(xyx^{-1}) = A(yxx^{-1}) = A(y) \geq A(y)$.

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