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Study of a Hypothetical Metric Based on Vacuum Einstein Field Equation

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***Abstract*—** This research supposes existing a hypothetical metric which is arisen from difference between proper times derived from Schwarzschild metric and what is shall be recorded if no gravity affect _flat metric. Primary purpose of this paper is to find out whether this new metric supports our physical rules or not; hence, this imaginary metric would be analyzed and verified according to vacuum Einstein field equation. Result of this analysis and verification is that Einstein tensor vanishes and solution of vacuum field equation satisfies. Although some parts of this paper are based on imaginary equations, it helps researches and students examine compatibility capability of hypothetical equations with regard to Einstein tensor.

***Keywords*—**General Relativity, Einstein field equation, spacetime, time delay

I. Introduction

Special Relativity formulates structure of Minkowski spacetime in a flat vacuum space distant from any massive object with zero cosmological constant and energy-momentum tensor [1]. While in General Relativity, Schwarzschild metric describes existence of gravitational field and its effect on the structure of space around any spherical rotating or non-rotating massive object, again in a vacuum space [2].

It is known to all how gravitational field of the Sun affects and deflects path of a beam of light passing in vicinity of a mass according to the Schwarzschild metric. Moreover, curvature induced due to the gravity causes a delay in time which seems to reduce speed of light examined by Shapiro [3].

What human can observe is that of exists in the real world because of effects of gravitational fields which surrounds him. This means that human could be able to observe different things, or at least at different times sooner or later, if gravitational field subside or become denser. This paper tries to formalize a relationship between these two metrics and analyze it based on Einstein vacuum field equation.

II. New hypothetical metric

A. Basic Relationships; Calculating Time Delay

Looking at the figure below, two beams of light are considered to be emitted from an infinitely distant star passing near the Sun:

i is a real beam of light corresponding to Schwarzschild Metric,

$$d\tau_i^{\ 2} = (1-\frac{2m}{r_i})dt_i^{\ 2} - (1-\frac{2m}{r_i})^{-1}dr_i^{\ 2} - r_i^{\ 2}(d\theta^2 + \sin^2(\theta)d\phi^2) \qquad (1)$$

j is an imaginary beam of light, geodesic-apart from any massive object sweeping a straight line on a flat surface,

$$d\tau_j^{\,2} = dt_j^{\,2} - dr_j^{\,2} - r_j^{\,2}(d\theta^2 + \sin^2(\theta)d\phi^2) \qquad (2)$$

Fig. 1 Schematic of two beams of light passing near the Sun

Conditions: $r\phi$ Plane and $\theta = \dfrac{\pi}{2}$

From the figure above, following relationships could be obtained

$$\begin{cases} r_i^{\,2} = \delta^2 + \lambda^2 \\ r_j^{\,2} = R^2 + \ell^2 \end{cases} \& \frac{r_i}{r_j} = \frac{\lambda}{\ell} = \frac{\delta}{R}$$

$$\cos\phi = \frac{\delta}{r_i} = \frac{R}{r_j} \Rightarrow d\phi = \frac{\delta}{\lambda r_i}dr_i = \frac{R}{\ell r_j}dr_j \qquad (3)$$

Inserting these relationships into the flat metric and rewriting equation (2) in terms of quantities of *i* will give

$$\begin{cases} d\tau_i^{\,2} = (1-\dfrac{2m}{r_i})dt_i^{\,2} - (1-\dfrac{2m}{r_i})^{-1}dr_i^{\,2} - r_i^{\,2}d\phi^2 \\ d\tau_j^{\,2} = dt_j^{\,2} - (\dfrac{R}{\delta})^2 dr_i^{\,2} - (\dfrac{R}{\delta})^2 r_i^{\,2}d\phi^2 \end{cases} \qquad (4)$$

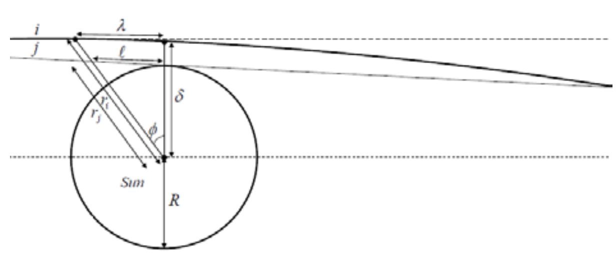
there's no need to prove that [5],

$$(\frac{R}{\delta})^2 = (1-\frac{2m}{r_i}) \qquad (5)$$

this changes equation (4) to

$$\begin{cases} d\tau_i^{\,2} = (1-\dfrac{2m}{r_i})dt_i^{\,2} - (1-\dfrac{2m}{r_i})^{-1}dr_i^{\,2} - r_i^{\,2}d\phi^2 \\ d\tau_j^{\,2} = dt_j^{\,2} - (1-\dfrac{2m}{r_i})dr_i^{\,2} - (1-\dfrac{2m}{r_i})r_i^{\,2}d\phi^2 \end{cases} \qquad (6)$$

Although it is known that according to the General Relativity velocity of light is not mandatorily equal to *c* and so, proper time of light is not zero, here only Special relativistic effects is considered, i.e. $d\tau_i = d\tau_j = 0$:



$$\begin{cases} dt_i^{\,2} = (1-\dfrac{2m}{r_i})^{-2} dr_i^{\,2} + (1-\dfrac{2m}{r_i})^{-1} r_i^{\,2} d\phi^2 \\ dt_j^{\,2} = (1-\dfrac{2m}{r_i}) dr_i^{\,2} + (1-\dfrac{2m}{r_i}) r_i^{\,2} d\phi^2 \end{cases}$$

inserting (3) quantities and rewriting above relationships based on determining result of division of dt_i on dt_j gives rise to

$$(\frac{dt_i}{dt_j})^2 = \frac{(1-\frac{2m}{r_i})^{-2} + (1-\frac{2m}{r_i})^{-1}(\frac{\delta}{\lambda})^2}{(1-\frac{2m}{r_i}) + (\frac{R}{\lambda})^2} \tag{7}$$

Based on the overall deviation of light passing near the Sun resulting from the Einstein prediction [2], it is known that

$$\phi = \frac{4M}{R} \tag{8}$$

Assuming that the under-experiment beam of the light passes tangent to the Sun, i.e. $r_i \approx r_j = R$:

$$\sin\phi \approx \phi \Rightarrow \frac{\ell}{R} = \frac{4M}{R} \;\&\; \lambda = (\frac{\delta}{R})\ell$$

considering $k = \dfrac{M}{R}$, and using an important principle in mathematics [3]

$$(1+\varepsilon)^p = 1 + p\varepsilon \tag{9}$$

(7) simplifies to

$$(\frac{dt_i}{dt_j})^2 = \frac{1+4k+20k^2+96k^3}{1+16k^2}$$

ignoring 2nd and 3rd powers of k that are puny,

$$(\frac{dt_i}{dt_j})^2 \approx 1+4k \approx (1-\frac{2M}{R})^{-2}$$

in other words;

$$\frac{dt_i}{dt_j} = (1-\frac{2M}{R})^{-1} = 1+2k$$

or

$$\frac{dt_i}{dt_j} = (1-\frac{2m}{r_i})^{-1} \tag{10}$$

This result will be used in deriving new metric discussed as follows [4].

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B. Deriving the New Metric

The only parameter of *j* metric that remained unchanged with respect to the *i* terms in (6) is dt_j. However it could be rewritten by inserting (10) relationship as follows

$$\begin{cases} d\tau_i^{\,2} = (1-\frac{2m}{r_i})dt_i^{\,2} - (1-\frac{2m}{r_i})^{-1}dr_i^{\,2} - r_i^{\,2}d\phi^2 \\ d\tau_j^{\,2} = (1-\frac{2m}{r_i})^2 dt_i^{\,2} - (1-\frac{2m}{r_i})dr_i^{\,2} - (1-\frac{2m}{r_i})r_i^{\,2}d\phi^2 \end{cases}$$

(11)

As mentioned in previous section, relationship (7) was derived by ignoring the General Relativistic effects of light. Nevertheless, in reality proper time and distance of light in Schwarzschild metric is not zero ($d\tau_{light} \neq 0$). This fact permits to subtract both equations of scalar form of metrics as follows

$$d\tau^2 = d\tau_i^{\,2} - d\tau_j^{\,2}$$

$$= \begin{bmatrix} (1-\frac{2m}{r_i}) \\ -(1-\frac{2m}{r_i})^2 \end{bmatrix} dt_i^{\,2} - \begin{bmatrix} (1-\frac{2m}{r_i})^{-1} \\ -(1-\frac{2m}{r_i}) \end{bmatrix} dr_i^{\,2} - (\frac{2m}{r_i})r_i^{\,2}d\phi^2$$

$$d\tau^2 = (\frac{2m}{r_i})dt_i^{\,2} - (\frac{4m}{r_i})dr_i^{\,2} - (\frac{2m}{r_i})r_i^{\,2}d\phi^2 \qquad (12)$$

This is an assumed equation for subtraction of real Schwarzschild and imaginary flat metrics that is used for discussion of the next section.

On the other hand _there is no need to prove that_ in reality, proper time of a beam of light in the flat metric is zero. Although this fact contradicts the condition above ($d\tau_{light} \neq 0$), based on the mathematics it does not interfere in process of obtaining (12). This hint results to

$$\begin{aligned} &d\tau_j^{\,2} = (1-\frac{2m}{r_i})^2 dt_i^{\,2} - (1-\frac{2m}{r_i})dr_i^{\,2} - (1-\frac{2m}{r_i})r_i^{\,2}d\phi^2 = 0 \\ &\Rightarrow (1-\frac{2m}{r_i})dt_i^{\,2} = dr_i^{\,2} + r_i^{\,2}d\phi^2 \end{aligned} \qquad (13)$$

Inserting this result in $d\tau_i$ equation:

$$d\tau_i^{\,2} = (1-\frac{2m}{r_i})dt_i^{\,2} - (1-\frac{2m}{r_i})^{-1}dr_i^{\,2} - r_i^{\,2}d\phi^2$$

$$= dr_i^{\,2} + r_i^{\,2}d\phi^2 - (1-\frac{2m}{r_i})^{-1}dr_i^{\,2} - r_i^{\,2}d\phi^2$$

$$d\tau_i^{\,2} \approx -\frac{2m}{r_i}dr_i^{\,2} \qquad (14)$$

since $d\tau_j = 0$, equation (12) is the same for $d\tau_i$. Inserting (14) in (12):

$$-\frac{2m}{r_i}dr_i^{\,2} = (\frac{2m}{r_i})dt_i^{\,2} - (\frac{4m}{r_i})dr_i^{\,2} - (\frac{2m}{r_i})r_i^{\,2}d\phi^2$$

simplifying will give

$$(\frac{2m}{r_i})dt_i^{\,2} - (\frac{2m}{r_i})dr_i^{\,2} - (\frac{2m}{r_i})r_i^{\,2}d\phi^2 = 0 \qquad (15)$$

$$\Rightarrow dt_i^{\,2} - dr_i^{\,2} - r_i^{\,2}d\phi^2 = 0$$

that is the quantity of the proper time of a beam of light in a flat surface. Result (15) confirms validity of the equation (12) that is deductive frame of the next section.

C. Vacuum Einstein Field Equation

First purpose of this section is to find Einstein tensor of equation (12) as a hypothetical space metric. As it was derived,

$$d\tau^2 = (\frac{2m}{r})dt^2 - (\frac{4m}{r})dr^2 - (\frac{2m}{r})r^2 d\phi^2$$

Here only 1st and 2nd dimensions of space metric (g) will be verified and other degrees are assumed negligible for simplification:

$$g_{00} = \frac{2m}{r}, dx^0 = dt$$

$$g_{11} = -\frac{4m}{r}, dx^1 = dr$$

$$g_{22} = 0, dx^2 = d\theta$$

$$g_{33} = 0, dx^3 = d\phi$$

It is known that

$$g^{ij} = (g_{ij})^{-1} \qquad (16)$$

and the final relationship to be examined is [7]

$$G_{ab} = R_{ab} - \frac{1}{2}Rg_{ab} \qquad (17)$$

where G_{ab} is the Einstein tensor, R_{ab} Ricci tensor and R scalar curvature.

Here Christoffel symbols are used to derive Ricci tensor and scalar curvature [7]. Since the purpose is to find R_{00} & R_{11}, and

$$R_{ab} = R^e{}_{aeb} \qquad (18)$$

where $R^e{}_{aeb}$ is Riemann (curvature) tensor. Hence

$$\begin{cases} R_{11} = R^0{}_{101} \\ R_{00} = R^1{}_{010} \end{cases}$$

On the other hand

$$R^k{}_{hij} = \Gamma^k_{hj,i} - \Gamma^k_{hi,j} + \Gamma^k_{li}\Gamma^l_{hj} - \Gamma^k_{lj}\Gamma^l_{hi}$$

$$R^k{}_{hij} = \frac{1}{2} g^{ke}(g_{ej,hi} - g_{ei,hj} + g_{hi,ej} - g_{hj,ei}) \qquad (19)$$

$$+ \Gamma^k_{li}\Gamma^l_{hj} - \Gamma^k_{lj}\Gamma^l_{hi}$$

and

$$\Gamma^k_{ij} = \frac{1}{2} g^{kh}(g_{jh,i} + g_{hi,j} - g_{ij,h}) \qquad (20)$$

where

$$g_{ij,h} = \frac{\partial g_{ij}}{\partial x^h}$$

$$g_{ij,hk} = \frac{\partial^2 g_{ij}}{\partial x^h \partial x^k} \qquad (21)$$

Pondering on equation (19), as the purpose was to obtain R_{00} & R_{11} it could be understood that

$$\Gamma^1_{00} = \frac{1}{2} g^{11}(-g_{00,1})$$

$$\Gamma^1_{11} = \frac{1}{2} g^{11}(g_{11,1})$$

$$\Gamma^0_{01} = \Gamma^0_{10} = \frac{1}{2} g^{00}(g_{00,1})$$

also

$$g^{00} = \frac{r}{2m}$$

$$g^{11} = \frac{-r}{4m}$$

and

$$g_{00,1} = \frac{\partial g_{00}}{\partial r} = \frac{-2m}{r^2}$$

$$g_{11,1} = \frac{\partial g_{11}}{\partial r} = \frac{4m}{r^2}$$

so

$$\Gamma^1_{00} = \frac{-1}{4r}$$

$$\Gamma^1_{11} = \frac{-1}{2r} \qquad (22)$$

$$\Gamma^0_{01} = \Gamma^0_{10} = \frac{-1}{2r}$$

Next step is to find values of $R^0{}_{101}$ & $R^1{}_{010}$:

$$R^0{}_{110} = \frac{1}{2} g^{00}(g_{00,11}) + \Gamma^0_{10}\Gamma^0_{01}$$

$$R^1{}_{010} = \frac{1}{2} g^{11}(-g_{00,11}) + \Gamma^1_{00}\Gamma^1_{11}$$

where

$$g_{00,11} = \frac{\partial}{\partial r}(g_{00,1}) = \frac{4m}{r^3}$$

then

$$R^0{}_{110} = \frac{5}{4r^2} \qquad R^1{}_{010} = \frac{5}{8r^2} \tag{23}$$

It is known that

$$\begin{cases} R^0{}_{110} = -R^0{}_{101} \\ R^1{}_{010} = -R^1{}_{001} \end{cases} \tag{24}$$

hence

$$\begin{cases} R_{11} = R^0{}_{101} = \dfrac{-5}{4r^2} \\ R_{00} = R^1{}_{010} = \dfrac{5}{8r^2} \end{cases} \tag{25}$$

Now dealing with Einstein tensor, equation (19) is rewritten

$$G_{ab} = R_{ab} - \frac{1}{2} R g_{ab}$$

where

$$R = g^{ab} R_{ab} \qquad R = \frac{5}{8mr} \tag{26}$$

So

$$G_{ab} = 0 \tag{27}$$

Therefore Einstein tensor vanishes. This result satisfies solution of the vacuum Einstein equation [8]

$$0 = (R_{ab} - \frac{1}{2} R g_{ab}) + \Lambda g_{ab} + \mathrm{K} T_{ab} \tag{28}$$

when Λ is cosmological constant, $\mathrm{K} = \frac{8\pi G}{c^2}$, and T_{ab} energy-momentum tensor.

III. DISCUSSIONS

Equation (11) that is difference of two times, one which is real time calculated by Schwarzschild metric and another obtained from flat metric, i.e. time delay is different from that of known as Shapiro time delay due to General Relativistic effects. Former quantity gives us what could be observed if no gravity existed. If there exists only the Sun in space, a beam of light which comes towards the Earth reaches $2k_i t_j$ seconds later than what would be without existence of the Sun. This demonstrates that amount of delay shall be more because a lot of massive objects exert gravitational field rather than Sun.

Einstein tensor of derived hypothetical metric was calculated to be zero. When there exists no cosmological constant,

$$R_{ab} = \mathrm{K} T_{ab}$$

However, while Ricci tensor is negligible Einstein tensor vanishes, and subsequently energy momentum tensor vanishes too. This corresponds to the vacuum Einstein field equation, saying that the metric under study could be either flat corresponding to the Minkoswki spacetime, or Schwarzschild. It even might possess an intermediate form. It must be denoted that, the metric derived is simplified form of complete metric. Also as mentioned before, tensors were solved by considering only 1st and 2nd dimensions. Solving by entering all dimensions will give similar results certainly.

IV. Conclusion

Entire this paper is based upon a hypothesis namely difference of two Schwarzschild and flat metrics as a new metric. It is clear that this metric does not exist in reality; however, it was created in order to answer a question: How much time is delayed due to existence of gravitational field of massive objects of the universe? Or: how many seconds, minutes or hours human are back from collecting information from other galaxies? This is a really controversial question and its answer is not exact; but effect of the Sun could be assessed.

Time required for a beam of light emitted from an outer source to reach Earth could be easily calculated by knowing the distance; however, what is more important is to find out how many seconds it will take to reach an earthy observer if gravity does not exist. Time delay which is calculated in this paper is due to existence of Sun alone.

It was understood that subtracting real Schwarzschild and flat equations of metrics of proper time of light, the result would be that of exists for Schwarzschild metric ($d\tau_i$), of course with different format since proper time of flat metric is zero. This difference is treated with as a hypothetical new metric and subsequently, as a space metric Einstein tensor and vacuum field equation derived and examined. Solving tensors in terms of hypothetical space metric, Einstein tensor was equal to zero. Therefore in case of zero cosmological constant, energy-momentum tensor vanishes too, that satisfies solution of vacuum field equation.

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