Bipolar Pythagorean Fuzzy A-ideals of BCI-algebra

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Abstract

The notion of bipolar Pythagorean Fuzzy A-ideals of BCI-algebra is introduced and their properties are investigated. Also Relationship between bipolar Pythagorean fuzzy subalgebra, bipolar Pythagorean fuzzy ideal, and bipolar Pythagorean fuzzy A-ideals are analyzed.

Keywords: BCI-algebra, bipolar Pythagorean fuzzy ideal, bipolar Pythagorean fuzzy subalgebra, bipolar Pythagorean fuzzy A-ideal, bipolar Pythagorean fuzzy level-cut.

1 Introduction

Fuzzy sets were introduced by Zadeh [14] and he discussed only membership function. After the extensions of fuzzy set theory, Atanassov [1] generalized this concept and introduced a new concept called intuitionistic fuzzy set (IFS). Yager [10] familiarized the model of Pythagorean fuzzy set. IFS has its greatest use in practical multiple attribute decision making (MADM) problems, and the academic research have achieved great development [10,11,13]. However, in the some practical problems, the sum of membership degree and non–membership degree to which an alternative satisfying attribute provided by decision maker (DM) may be bigger than 1, but their square sum is less than or equal to 1. Y.B. Jun [4] introduced the notion of closed fuzzy ideals in BCH-algebras and discussed their properties.

Bosc and Pivert [2] said that "Bipolarity refers to the propensity of the human mind to reason and make decisions on the basis of positive and negative effects. Positive information states what is possible, satisfactory, permitted, desired, or considered as being acceptable. On the other hand, negative statements express what is impossible, rejected, or forbidden. Negative preferences correspond to constraints, since they specify which values or objects have to be rejected (i.e., those that do not satisfy the constraints), while positive preferences correspond to wishes, as they specify which objects are more desirable than others (i.e., satisfy user wishes) without rejecting those that do not meet the wishes". Therefore, Lee [5,6] introduced the concept of bipolar fuzzy sets which is an generalization of the fuzzy sets. Recently bipolar fuzzy models have been studied by many authors on algebraic structures such as; Chen et. al. [3] studied of m-polar fuzzy set. Then, they examined many results which are related to those concepts can be generalized to the case of m-polar fuzzy sets. They also proposed numerical examples to show how to apply m-polar fuzzy sets in the real world problems.

Lee [5] discussed subalgebras and ideals of BCK / BCI-algebras. Liu and Meng [7] introduced an extension the notion of q-ideals and a-ideals in BCI-algebras.

In this paper, we apply the concept of bipolar Pythagorean fuzzy A-ideals to BCI-algebras and investigate its properties. Also we discuss the relationship between the bipolar Pythagorean subalgebras, bipolar Pythagorean fuzzy ideals, bipolar Pythagorean fuzzy A-ideal.

2 Preliminaries

Definition 2.1. [5] An algebra $(X; *, 0)$ of type (2,0) is called a BCI-algebra if it satisfies the following conditions:

i. $((x*y)*(x*z))*(z*y)=0$,

ii. $(x*(x*y))*y=0$,

iii. $x*x=0$,

iv. $x*y=0, y*x=0 \Rightarrow x=y$ for all $x, y \in X$.

We can define a partial order $' \leq '$ on X by $x \leq y$ if and only if $x*y=0$. Any BCI-algebra X has the following properties:

1. $x*0=x$,
2. $(x*y)*z=(x*z)*y$,
3. $x \leq y \Rightarrow x*z \leq y*z, z*y \leq z*x$, for all $x, y, z \in X$.

Definition 2.2. [14] Let X be a nonempty set. A fuzzy set A drawn from X is defined as $A=\{(x:\mu_A(x)):x \in X\}$, where $\mu_A: X \to [0,1]$ is the membership function of the fuzzy set A.

Definition 2.3. [5] Let X be the universe. Then a bipolar fuzzy sets, A on X is defined by positive membership function μ_A^+, that is $\mu_A^+: X \to [0,1]$, and a negative membership function μ_A^-, that is $\mu_A^-: X \to [-1,0]$.For the sake of simplicity, we shall use the symbol $A=\{(x,\mu_A^+(x),\mu_A^-(x)):x \in X\}$.

Definition 2.4. (Pythagorean Fuzzy Set (PF)) [16,18] Let X be a non-empty set and I the unit interval [0,1]. A PF set S is an object having the form $P=\{(x,\mu_P(x),\nu_P(x)):x \in X\}$ where the function $\mu_P: X \to [0,1]$ and $\nu_P: X \to [0,1]$denote respectively the degree of membership and degree of non-membership of each element $x \in X$ to the set P, and $0 \leq (\mu_P(x))^2+(\nu_P(x))^2 \leq 1$ for each $x \in X$.

Definition 2.5. (Bipolar Pythagorean Fuzzy Set) [13] Let X be a non-empty set. A bipolar Pythagorean fuzzy set (BPFS) $A=\{(X,T_A^P,F_A^P,T_A^N,F_A^N):x \in X\}$ where $T_A^P: X \to [0,1]$, $F_A^P: X \to [0,1]$, $T_A^N: X \to [-1,0]$ and $F_A^N: X \to [-1,0]$ are the mappings such that $0 \leq (T_A^P)^2+(F_A^P)^2 \leq 1$ and $-1 \leq -((T_A^N)^2+(F_A^N)^2) \leq 0$ and $T_A^P(x)$ denote the positive membership degree, $F_A^P(x)$ denote the positive non-membership degree, $T_A^N(x)$ denote the negative membership degree, $F_A^N(x)$ denote the negative non-membership degree. The degree of inderminancy

$\pi_A^P(x)=\sqrt{1-(T_A^P(x))^2-(F_A^P(x))^2}$ and

$\pi_A^N(x)=-\sqrt{1-(T_A^N(x))^2-(F_A^N(x))^2}$.

Definition2.6. [13] Let $A=\{(X,T_A^P,F_A^P,T_A^N,F_A^N):x \in X\}$ and $B=\{(X,T_B^P,F_B^P,T_B^N,F_B^N):x \in X\}$ be two BPFNs, then their operations are defined as follows:

(1) $A \cup B = \{(x, max(T_A^P,T_B^P), min(F_A^P,F_B^P), min(T_A^N,T_B^N), max(F_A^N,F_B^N)):x \in X\}$

(2) $A \cap B$ $\{(x, min(T_A^P,T_B^P), max(F_A^P,F_B^P), max(T_A^N,T_B^N), min(F_A^N,F_B^N)):x \in X\}$

(3) $A^C=\{(x,F_A^P,T_A^P,F_A^N,T_A^N):x \in X\}$.

3 Bipolar Pythagorean Fuzzy A-ideal

Definition 3.1. Let $A=\{(X,T_A^P,F_A^P,T_A^N,F_A^N):x \in X\}$ and $B=\{(X,T_B^P,F_B^P,T_B^N,F_B^N):x \in X\}$ be two BPFNs, then $A \subseteq B$ if and only if $T_A^P(x) \leq T_B^P(x)$, $F_A^P(x) \geq F_B^P(x), T_A^N(x) \geq T_B^N(x)$ and $F_A^N(x) \leq F_B^N(x)$.

Definition 3.2. A bipolar Pythagorean fuzzy set $A = (X, T_A^P, F_A^P, T_A^N, F_A^N)$ in X is called a bipolar Pythagorean fuzzy subalgebra of X if it satisfies:

i. $T_A^P(x * y) \geq min\{T_A^P(x), T_A^P(y)\}$
ii. $F_A^P(x * y) \leq max\{F_A^P(x), F_A^P(y)\}$
iii. $T_A^N(x * y) \leq max\{T_A^N(x), T_A^N(y)\}$
iv. $F_A^N(x * y) \geq min\{F_A^N(x), F_A^N(y)\}$ for all $x, y \in X$.

Definition 3.3. A bipolar Pythagorean fuzzy set $A = (X, T_A^P, F_A^P, T_A^N, F_A^N)$ of a BCI algebra X is called a bipolar Pythagorean fuzzy ideal of X if the following conditions are true:

i. $T_A^P(0) \geq T_A^P(x)$ and $F_A^P(0) \leq F_A^P(x)$
ii. $T_A^N(0) \leq T_A^N(x)$ and $F_A^N(0) \geq F_A^N(x)$
iii. $T_A^P(x) \geq min\{T_A^P(x * y), T_A^P(y)\}$
$F_A^P(x) \leq max\{F_A^P(x * y), F_A^P(y)\}$
iv. $T_A^N(x) \leq max\{T_A^N(x * y), T_A^N(y)\}$
$F_A^N(x) \geq min\{F_A^N(x * y), F_A^N(y)\}$

Definition 3.4. A bipolar Pythagorean fuzzy set $A = (X, T_A^P, F_A^P, T_A^N, F_A^N)$ in X is called a bipolar Pythagorean fuzzy A-ideal of X if it satisfies:

i. $T_A^P(0) \geq T_A^P(x)$ and $F_A^P(0) \leq F_A^P(x)$
ii. $T_A^N(0) \leq T_A^N(x)$ and $F_A^N(0) \geq F_A^N(x)$
iii. $T_A^P(y * x) \geq min\{T_A^P((x * z) * (0 * y)), T_A^P(z)\}$ and
$F_A^N(y * x) \leq max\{F_A^P((x * z) * (0 * y)), F_A^P(z)\}$
iv. $T_A^N(y * x) \leq max\{T_A^N((x * z) * (0 * y)), T_A^N(z)\}$ and
$F_A^N(y * x) \geq min\{F_A^N((x * z) * (0 * y)), F_A^N(z)\}$ for all $x, y, z \in X$.

Example 3.5. Consider a BCI-algebra $X = \{0, a, b, c\}$ with the following Cayley table:

*	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Define a bipolar Pythagorean fuzzy set $A = (X; T_A^P, F_A^P, T_A^N, F_A^N)$ in X by

Then $A = (X; T_A^P, F_A^P, T_A^N, F_A^N)$ is a bipolar

X	0	a	b	c
$[T_A^P, F_A^P$	[0.5,0.7]	[0.5,0.7]	[0.4,0.9]	[0.4,0.9]
$[T_A^N, F_A^N$	$[-0.8, -0$	$[-0.8, -0$	$[-0.6, -0$	$[-0.6, -0$

Pythagorean fuzzy A-ideal of X.

Theorem 3.6. If $A = (X; T_A^P, F_A^P, T_A^N, F_A^N)$ is a bipolar Pythagorean fuzzy A-ideal of X, then $T_A^P(x) = T_A^P(0 * x)$, $F_A^P(x) = F_A^P(0 * x)$, $T_A^N(x) = T_A^N(0 * x)$ and $F_A^N(x) = F_A^N(0 * x)$ for all $x \in X$.

Proof. Let $A = (X; T_A^P, F_A^P, T_A^N, F_A^N)$ be a bipolar Pythagorean fuzzy A-ideal of X. Taking $y = z = 0$ in definition 3.4 and using definition 2.1 (iii) and (i), we get $T_A^N(0 * x) \leq T_A^N(x)$, $F_A^N(0 * x) \geq F_A^N(x)$, $T_A^P(0 * x) \geq T_A^P(x)$ and $F_A^P(0 * x) \leq F_A^P(x)$. Setting $x = z = 0$ in definition 3.4 and using definition 2.1 (iii) and (i), we get $T_A^N(y) = T_A^N(y * 0) \leq T_A^N(0 * (0 * y)) \leq T_A^N(0 * y)$, $F_A^N(y) = F_A^N(y * 0) \geq F_A^N(0 * (0 * y)) \geq F_A^N(0 * y)$, $T_A^P(y) = T_A^P(y * 0) \geq T_A^P(0 * (0 * y)) \geq T_A^P(0 * y)$, $F_A^P(y) = F_A^P(y * 0) \leq F_A^P(0 * (0 * y)) \leq F_A^P(0 * y)$ for all $y \in X$.
Hence, $T_A^P(x) = T_A^P(0 * x)$, $F_A^P(x) = F_A^P(0 * x)$, $T_A^N(x) = T_A^N(0 * x)$ and $F_A^N(x) = F_A^N(0 * x)$ for all $x \in X$.

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Theorem 3.7. Every bipolar Pythagorean fuzzy A-ideal of X is both a bipolar Pythagorean fuzzy subalgebra of X and a bipolar Pythagorean fuzzy ideal of X.

Proof. Let $A = (X; T_A^P, F_A^P, T_A^N, F_A^N)$ be a bipolar Pythagorean fuzzy A-ideal of X. Using definition 3.4 and Theorem 3.6, we have

$T_A^N(x) = T_A^N(0 * x)$
$\leq max\{T_A^N((x * z) * (0 * 0)), T_A^N(z)\}$
$= max\{T_A^N(x * z), T_A^N(z)\}$

$F_A^N(x) = F_A^N(0 * x)$
$\geq min\{F_A^N((x * z) * (0 * 0)), F_A^N(z)\}$
$= min\{F_A^N(x * z), F_A^N(z)\}$

$T_A^P(x) = T_A^P(0 * x)$
$\geq min\{T_A^P((x * z) * (0 * 0)), T_A^P(z)\}$
$= min\{T_A^P(x * z), T_A^P(z)\}$

$F_A^P(x) = F_A^P(0 * x)$
$\leq max\{F_A^P((x * z) * (0 * 0)), F_A^P(z)\}$
$= max\{F_A^P(x * z), F_A^P(z)\}$ for all $x, z \in X$.

Hence $A = (X; T_A^P, F_A^P, T_A^N, F_A^N)$ is a bipolar Pythagorean fuzzy ideal of X. Now for any $x, y \in X$, we obtain

$T_A^N(x * y) \leq max\{T_A^N((x * y) * x), T_A^N(x)\}$
$= max\{T_A^N(0 * y), T_A^N(x)\}$
$= max\{T_A^N(x), T_A^N(y)\}$ and

$F_A^N(x * y) \geq min\{F_A^N((x * y) * x), F_A^N(x)\}$
$= min\{F_A^N(0 * y), F_A^N(x)\}$
$= min\{F_A^N(x), F_A^N(y)\}$

$T_A^P(x * y) \geq min\{T_A^P((x * y) * x), T_A^P(x)\}$
$= min\{T_A^P(0 * y), T_A^P(x)\}$
$= min\{T_A^P(x), T_A^P(y)\}$

$F_A^P(x * y) \leq max\{F_A^P((x * y) * x), F_A^P(x)\}$
$= max\{F_A^P(0 * y), F_A^P(x)\}$
$= max\{F_A^P(x), F_A^P(y)\}$.

Therefore $A = (X; T_A^P, F_A^P, T_A^N, F_A^N)$ is a bipolar Pythagorean fuzzy subalgebra of X.

The following example shows that the converse of the above need not be true.

Example 3.8. Let $X = \{0, a, b\}$ be a BCI-algebra with the following Cayley table:

*	0	a	b
0	0	b	a
a	a	0	b
b	b	a	0

Define a bipolar Pythagorean fuzzy set $A = (X; T_A^P, F_A^P, T_A^N, F_A^N)$ in X by

X	0	a	b
$[T_A^P, F_A^P]$	[0.4,0.8]	[0.3,0.9]	[0.3,0.9]
$[T_A^N, F_A^N]$	[−0.6, −0.5]	[−0.4, −0.7]	[−0.4, −0.7]

Then $A = (X; T_A^P, F_A^P, T_A^N, F_A^N)$ is both a bipolar Pythagorean fuzzy ideal and a bipolar Pythagorean fuzzy subalgebra of X, but not a bipolar Pythagorean fuzzy A-ideal of X.

Theorem 3.9. Let $A = (X; T_A^P, F_A^P, T_A^N, F_A^N)$ be a bipolar Pythagorean fuzzy ideal of X. If the inequality $x * y \leq z$ holds in X.

Then

$T_A^N(x) \leq max\{T_A^N(y), T_A^N(z)\}$
$F_A^N(x) \geq min\{F_A^N(y), F_A^N(z)\}$
$T_A^P(x) \geq min\{T_A^P(y), T_A^P(z)\}$
$F_A^P(x) \leq max\{F_A^P(y), F_A^P(z)\}$.

Proof. Let $x, y, z \in X$ be such that $x * y \leq z$. Then $(x * y) * z = 0$. and so

$T_A^N(x) \leq max\{T_A^N(x * y), T_A^N(y)\}$
$\leq max\{max\{T_A^N((x * y) * z), T_A^N(z)\}, T_A^N(y)\}$
$= max\{max\{T_A^N(0), T_A^N(z)\}, T_A^N(y)\}$
$= max\{T_A^N(y), T_A^N(z)\}$ and

$F_A^N(x) \geq min\{F_A^N(x * y), F_A^N(y)\}$
$\geq min\{min\{F_A^N((x * y) * z), F_A^N(z)\}, F_A^N(y)\}$
$= min\{min\{F_A^N(0), F_A^N(z)\}, F_A^N(y)\}$
$= min\{F_A^N(y), F_A^N(z)\}$

$T_A^P(x) \geq min\{T_A^P(x*y), T_A^P(y)\}$

$\geq min\{min\{T_A^P((x*y)*z), T_A^P(z)\}, T_A^P(y)\}$

$= min\{min\{T_A^P(0), T_A^P(z)\}, T_A^P(y)\}$

$= min\{T_A^P(y), T_A^P(z)\}$

$F_A^P(x) \leq max\{F_A^P(x*y), F_A^P(y)\}$

$\leq max\{max\{F_A^P((x*y)*z), F_A^P(z)\}, F_A^P(y)\}$

$= max\{max\{F_A^P(0), F_A^P(z)\}, F_A^P(y)\}$

$= max\{F_A^P(y), F_A^P(z)\}.$

Hence the proof.

Theorem 3.10. Let $A = (X; T_A^P, F_A^P, T_A^N, F_A^N)$ be a bipolar Pythagorean fuzzy ideal of X. Then the following are equivalent.

i. $A = (X; T_A^P, F_A^P, T_A^N, F_A^N)$ is a bipolar Pythagorean fuzzy A-ideal of X.

ii. $A = (X; T_A^P, F_A^P, T_A^N, F_A^N)$ satisfies the following conditions:

$T_A^N(y*(x*z)) \leq T_A^N((x*z)*(0*y))$

$F_A^N(y*(x*z)) \geq F_A^N((x*z)*(0*y))$

$T_A^P(y*(x*z)) \geq T_A^P((x*z)*(0*y))$

$F_A^P(y*(x*z)) \leq F_A^P((x*z)*(0*y))$

for all $x, y, z \in X$.

iii. $A = (X; T_A^P, F_A^P, T_A^N, F_A^N)$ satisfies the following conditions:

$T_A^N(y*x) \leq T_A^N(x*(0*y))$

$F_A^N(y*x) \geq F_A^N(x*(0*y))$

$T_A^P(y*x) \geq T_A^P(x*(0*y))$

$F_A^P(y*x) \leq F_A^P(x*(0*y))$ for all $x, y \in X$.

Proof. (i) ⇒ (ii) Assume that $A = (X; T_A^P, F_A^P, T_A^N, F_A^N)$ is a bipolar Pythagorean fuzzy A-ideal of X and let $x, y, z \in X$ using Definition 3.4, we get

$T_A^N(y*(x*z)) \leq max\left\{T_A^N\left(((x*z)*0)*(0*y)\right), T_A^N(0)\right\}$

$= T_A^N((x*z)*(0*y))$ and

$F_A^N(y*(x*z)) \geq min\left\{F_A^N\left(((x*z)*0)*(0*y)\right), F_A^N(0)\right\}$

$= F_A^N((x*z)*(0*y))$

$T_A^P(y*(x*z)) \geq min\left\{T_A^P\left(((x*z)*0)*(0*y)\right), T_A^P(0)\right\}$

$= T_A^P((x*z)*(0*y))$

$F_A^P(y*(x*z)) \leq max\left\{F_A^P\left(((x*z)*0)*(0*y)\right), F_A^P(0)\right\}$

$= F_A^P((x*z)*(0*y)).$

(ii) ⇒ (iii) Taking $z = 0$ in (ii) and using (1) induce (iii)

(iii) ⇒ (i) Note that $(x*(0*y))*((x*z)*(0*y)) \leq x*(x*z) \leq z$ for all $x, y, z \in X$. It follows from (iii) and Theorem 3.9. that

$T_A^N(y*x) \leq T_A^N(x*(0*y))$

$\leq max\{T_A^N((x*z)*(0*y)), T_A^N(z)\},$

$F_A^N(y*x) \geq F_A^N(x*(0*y))$

$\geq min\{F_A^N((x*z)*(0*y)), F_A^N(z)\}$

$T_A^P(y*x) \geq T_A^P(x*(0*y))$

$\geq min\{T_A^P((x*z)*(0*y)), T_A^P(z)\}$

$F_A^P(y*x) \leq F_A^P(x*(0*y))$

$\leq max\{F_A^P((x*z)*(0*y)), F_A^P(z)\}.$

Hence $A = (X; T_A^P, F_A^P, T_A^N, F_A^N)$ is a bipolar Pythagorean fuzzy A-ideal of X.

Theorem 3.11. Assume that X is associate, i.e., X satisfies the following identity: $(x*y)*z = x*(y*z)$, for all $x, y, z \in X$. Then every bipolar Pythagorean fuzzy ideal is a bipolar Pythagorean fuzzy A-ideal of X.

Proof. Let $A = (X; T_A^P, F_A^P, T_A^N, F_A^N)$ be a bipolar Pythagorean fuzzy ideal of an associative BCI-algebra X. Since $0*x = x$ for all $x \in X$, that $y*x = (0*y)*x = (0*x)*y = x*y = x*(0*y)$ for all $x, y \in X$. Therefore $T_A^N(y*x) =$

$T_A^N(x*(0*y))$, $F_A^N(y*x) = F_A^N(x*(0*y))$, $T_A^P(y*x) = T_A^P(x*(0*y))$, $F_A^P(y*x) = F_A^P(x*(0*y))$, by Theorem 3.10, we conclude that $A = (X; T_A^P, F_A^P, T_A^N, F_A^N)$ is a bipolar Pythagorean fuzzy A-ideal of X.

Theorem 3.12. Let $A = (X; T_A^P, F_A^P, T_A^N, F_A^N)$ be a bipolar Pythagorean fuzzy A-ideal of X. Then the set $\Omega = \{x \in X | T_A^N(x) = T_A^N(0), F_A^N(x) = F_A^N(0), T_A^P(x) = T_A^P(0), F_A^P(x) = F_A^P(0)\}$ is an A-ideal of X.

Proof. Obviously, $0 \in \Omega$. Let $x, y, z \in X$ be such that $(x*z)*(0*y) \in \Omega$ and $z \in \Omega$. Then

$$T_A^N(0) \le T_A^N(y*x) \le max\{T_A^N((x*z)*(0*y)), T_A^N(z)\} = T_A^N(0),$$

$$F_A^N(0) \ge F_A^N(y*x) \ge min\{F_A^N((x*z)*(0*y)), F_A^N(z)\} = F_A^N(0),$$

$$T_A^P(0) \ge T_A^P(y*x) \ge min\{T_A^P((x*z)*(0*y)), T_A^P(z)\} = T_A^P(0),$$

$$F_A^P(0) \le F_A^P(y*x) \le max\{F_A^P((x*z)*(0*y)), F_A^P(z)\} = F_A^P(0)$$

by using Definition 3.4. it follows that $T_A^N(y*x) = T_A^N(0)$, $F_A^N(y*x) = F_A^N(0)$, $T_A^P(y*x) = T_A^P(0)$, and $F_A^P(y*x) = F_A^P(0)$. That is, $y*x \in \Omega$. Therefore Ω is an A-ideal of X.

Theorem 3.13. If $A = (X; T_A^P, F_A^P, T_A^N, F_A^N)$ and $B = (X; T_B^P, F_B^P, T_B^N, F_B^N)$be a bipolar Pythagorean fuzzy A-ideal of BCI-algebra X, then $A \cap B$ is a bipolar Pythagorean fuzzy A-ideal of X.

Proof. $T_A^N(0) \le T_A^N(x), F_A^N(0) \ge F_A^N(x)$ and $T_B^N(0) \le T_B^N(x), F_B^N(0) \ge F_B^N(x)$ for all $x \in X$. $max\{T_A^N(0), T_B^N(0)\} \le max\{T_A^N(x), T_B^N(x)\} = T_{A\cap B}^N(0) \le T_{A\cap B}^N(0)$ and $min\{F_A^N(0), F_B^N(0)\} \ge min\{F_A^N(x), F_B^N(x)\} = F_{A\cap B}^N(0) \ge F_{A\cap B}^N(0)$ for all $x \in X$.

To verify second condition,

$$T_A^N(y*x) \le max\{T_A^N((x*z)*(0*y)), T_A^N(z)\},$$
$$T_B^N(y*x) \le max\{T_B^N((x*z)*(0*y)), T_B^N(z)\},$$
$$F_A^N(y*x) \ge min\{F_A^N((x*z)*(0*y)), F_A^N(z)\}$$
$$F_B^N(y*x) \ge min\{F_B^N((x*z)*(0*y)), F_B^N(z)\}.$$

$$max\{T_A^N(y*x), T_B^N(y*x)\} \le max\left\{max\{T_A^N((x*z)*(0*y)), T_A^N(z)\}, max\{T_B^N((x*z)*(0*y)), T_B^N(z)\}\right\},$$

$$min\{F_A^N(y*x), F_B^N(y*x)\} \ge min\left\{min\{F_A^N((x*z)*(0*y)), F_A^N(z)\}, min\{F_B^N((x*z)*(0*y)), F_B^N(z)\}\right\}.$$

$$T_{A\cap B}^N \le max\{T_{A\cap B}^N((x*z)*(0*y)), T_{A\cap B}^N(z)\},$$
$$F_{A\cap B}^N \ge min\{F_{A\cap B}^N((x*z)*(0*y)), F_{A\cap B}^N(z)\}.$$

and $T_A^P(0) \ge T_A^P(x), F_A^P(0) \le F_A^P(x)$ and $T_B^P(0) \ge T_B^P(x), F_B^P(0) \le F_B^P(x)$ for all $x \in X$. $min\{T_A^P(0), T_B^P(0)\} \ge min\{T_A^P(x), T_B^P(x)\} = T_{A\cap B}^P(0) \ge T_{A\cap B}^P(0)$ and $max\{F_A^P(0), F_B^P(0)\} \le max\{F_A^P(x), F_B^P(x)\} = F_{A\cap B}^P(0) \le F_{A\cap B}^P(0)$ for all $x \in X$. To verify second condition,

$$T_A^P(y*x) \ge min\{T_A^P((x*z)*(0*y)), T_A^P(z)\},$$
$$T_B^P(y*x) \ge min\{T_B^P((x*z)*(0*y)), T_B^P(z)\},$$
$$F_A^P(y*x) \le max\{F_A^P((x*z)*(0*y)), F_A^P(z)\}$$

and

$$F_B^P(y*x) \le max\{F_B^P((x*z)*(0*y)), F_B^P(z)\}.$$

$$min\{T_A^P(y*x), T_B^P(y*x)\} \ge min\left\{min\{T_A^P((x*z)*(0*y)), T_A^P(z)\}, min\{T_B^P((x*z)*(0*y)), T_B^P(z)\}\right\},$$

$max\{F_A^P(y*x), F_B^P(y*x)\}$

$\leq max\left\{max\{F_A^P((x*z)*(0*y)), F_A^P(z)\}, max\{F_B^P((x*z)*(0*y)), F_B^P(z)\}\right\}.$

$T_{A\cap B}^P \geq min\{T_{A\cap B}^P((x*z)*(0*y)), T_{A\cap B}^P(z)\},$

$F_{A\cap B}^P \leq max\{F_{A\cap B}^P((x*z)*(0*y)), F_{A\cap B}^P(z)\}.$

for all $x, y, z \in X$.

Hence $A \cap B$ is a bipolar Pythagorean fuzzy A-ideal of X.

Definition 3.14. For a bipolar Pythagorean fuzzy set $A = (X; T_A^P, F_A^P, T_A^N, F_A^N)$ in X and $(\alpha, \beta) \in [0,1]$ and $(\mu, \gamma) \in [-1,0]$, the positive (α, β)-cut and negative (μ, γ)-cut are denoted by $A_{(\alpha,\beta)}^P$ and $A_{(\mu,\gamma)}^P$, and are defined as follows:

$A_{(\alpha,\beta)}^P = \{x \in X | T_A^P(x) \geq \alpha \text{ and } F_A^P(x) \leq \beta\}$ and

$A_{(\mu,\gamma)}^N = \{x \in X | T_A^N(x) \leq \mu \text{ and } F_A^N(x) \geq \gamma\}$ with $\alpha + \beta \leq 1$ and $\mu + \gamma \geq -1$

respectively.

The bipolar Pythagorean fuzzy level cut of $A = (X; T_A^P, F_A^P, T_A^N, F_A^N)$ denoted by A_{cut} is denoted to be the set $A_{cut} = (A_{(\alpha,\beta)}^P, A_{(\mu,\gamma)}^N)$.

Theorem 3.15. A bipolar Pythagorean fuzzy set $A = (X; T_A^P, F_A^P, T_A^N, F_A^N)$ in X is a bipolar Pythagorean fuzzy A-ideal of X if and only if for all $(\alpha, \beta) \in [0,1]$ and $(\mu, \gamma) \in [-1,0]$, the non-empty positive (α, β)-cut and the non-empty negative (μ, γ)-cut are bipolar Pythagorean fuzzy A-ideals of X.

Proof. Let $A = (X; T_A^P, F_A^P, T_A^N, F_A^N)$ be a bipolar Pythagorean fuzzy A-ideal of X and assume that $A_{(\alpha,\beta)}^P$ and $A_{(\mu,\gamma)}^N$ are non-empty for $(\alpha, \beta) \in [0,1]$ and $(\mu, \gamma) \in [-1,0]$. Obviously, $0 \in A_{(\alpha,\beta)}^P \cap A_{(\mu,\gamma)}^N$. Let for all $x, y, z \in X$ be such that $T_A^N((x*z)*(0*y)) \in A_{(\mu,\gamma)}^N$ and $T_A^N(z) \in A_{(\mu,\gamma)}^N$, also $F_A^N((x*z)*(0*y)) \in A_{(\mu,\gamma)}^N$ and $F_A^N(z) \in A_{(\mu,\gamma)}^N$. Then $T_A^N((x*z)*(0*y)) \leq \mu, T_A^N(z) \leq \mu$, $F_A^N((x*z)*(0*y)) \geq \gamma \text{ and } F_A^N(z) \geq \gamma$. It follows from Definition 3.4 that $T_A^N(y*x) \leq max\{T_A^N((x*z)*(0*y)), T_A^N(z)\} \leq \mu$ and $F_A^N(y*x) \geq min\{F_A^N((x*z)*(0*y)), F_A^N(z)\} \geq \gamma$ so that $y*x \in A_{(\mu,\gamma)}^N$. Now assume that $T_A^P((x*z)*(0*y)) \in A_{(\alpha,\beta)}^P$ and $T_A^P(z) \in A_{(\alpha,\beta)}^P$, also $F_A^P((x*z)*(0*y)) \in A_{(\alpha,\beta)}^P$ and $F_A^P(z) \in A_{(\alpha,\beta)}^P$. Then $T_A^P((x*z)*(0*y)) \geq \alpha$, $T_A^P(z) \geq \alpha$, $F_A^P((x*z)*(0*y)) \leq \beta \text{ and } F_A^P(z) \leq \beta$. It follows from Definition 3.4 that $T_A^P(y*x) \geq min\{T_A^P((x*z)*(0*y)), T_A^P(z)\} \geq \alpha$ and $F_A^P(y*x) \leq max\{F_A^P((x*z)*(0*y)), F_A^P(z)\} \leq \beta$ so that $y*x \in A_{(\alpha,\beta)}^P$. Therefore $A_{(\alpha,\beta)}^P$ and $A_{(\mu,\gamma)}^N$ are A-ideal of X.

Conversely, suppose that the non-empty negative (μ, γ)-cut and the non-empty positive (α, β)-cut are A-ideals of X for every $(\alpha, \beta) \in [0,1]$ and $(\mu, \gamma) \in [-1,0]$. If $T_A^N(0) > T_A^N(a), F_A^N(0) < F_A^N(a)$ or $T_A^P(0) < T_A^P(b), F_A^P(0) > F_A^P(b)$ for some $a, b \in X$, then $0 \notin A_{(T_A^N(a), F_A^N(a))}^N$ or $0 \notin A_{(T_A^P(a), F_A^P(a))}^P$. This is a contradiction. Thus $T_A^N(0) \leq T_A^N(x), F_A^N(0) \geq F_A^N(x)$ and $T_A^P(0) \geq T_A^P(x), F_A^P(0) \leq F_A^P(x)$ for all $x \in X$. Assume that $T_A^N(b*a) > max\{T_A^N((a*c)*(0*b)), T_A^N(c)\} = \mu$ and $F_A^N(b*a) < min\{F_A^N((a*c)*(0*b)), F_A^N(c)\} = \gamma$ for some $a, b, c \in X$. Then $(a*c)*(0*b) \in A_{(\mu,\gamma)}^N$ and $c \in A_{(\mu,\gamma)}^N$ but $b*a \notin A_{(\mu,\gamma)}^N$. This is impossible, and thus $T_A^N(y*x) \leq max\{T_A^N((x*z)*(0*y)), T_A^N(z)\} = \mu$ and $F_A^N(y*x) \geq min\{F_A^N((x*z)*(0*y)), F_A^N(z)\} = \gamma$ for all $x, y, z \in X$. If $T_A^P(b*a) < min\{T_A^P((a*c)*(0*b)), T_A^P(c)\} = \alpha$ and $F_A^P(b*a) > max\{F_A^P((a*c)*(0*b)), F_A^P(c)\} = \beta$ for some

$a, b, c \in X$. Then $(a * c) * (0 * b) \in A^P_{(\alpha,\beta)}$ and $c \in A^P_{(\alpha,\beta)}$ but $b * a \notin A^P_{(\alpha,\beta)}$. This is impossible, and thus $T^P_A(y * x) \geq min\{T^P_A((x * z) * (0 * y)), T^P_A(z)\} = \alpha$ and $F^P_A(y * x) \leq max\{F^P_A((x * z) * (0 * y)), F^P_A(z)\} = \beta$ for all $x, y, z \in X$. Consequently $A = (X; T^P_A, F^P_A, T^N_A, F^N_A)$ is a bipolar Pythagorean fuzzy A-ideal of X.

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