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Complete set of Genera of Compact Riemann Surfaces with Group of Symmetries of Ammonia (NH_3) Molecule

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Abstract

In this paper we consider the group of symmetries, C_{3v} of the Ammonia molecule NH_3 which is a finite group of order 6 and find the complete set of genera $(g \geq 2)$ of Riemann surfaces on which C_{3v} acts as a group of automorphisms as follows:

C_{3v}, the group of symmetries of the ammonia (NH_3) molecule of order 6 acts as an automorphism group of a compact Riemann surfaces of genus $g \geq 2$ if and only if there are integers λ and μ such that $\lambda \leq 1$ and $\mu \geq 1$ and $g = \lambda + 3\mu\ (\geq 2)$, $\mu \geq |\lambda|$

Keywords: *Symmetries, Fuchsian group, Smooth quotient, Riemann surface, Automorphism group and Genus.*

Introduction:

Every finite group acts as group of automorphisms on a compact Riemann surface of some genus $g \geq 2$. The same group may act as an automorphism group of many surfaces of different genera. The problem of finding minimum g such that there is a surface of genus g admitting a given finite group G as an automorphism group has been studied extensively since 1960s. Finding the complete set of genera of the surfaces admitting a given finite group G as an automorphism group seems to be a difficult problem. The problem has however been solved for some simple classes of groups during the last few years[2,3,4,5]. In this paper we find the solution of the problem for the group of symmetries, C_{3v} of the Ammonia (NH_3) molecule which is a finite group of order 6.

A finite group G acts as a group of automorphism of compact Riemann surface of genus $g \geq 2$ if and only if there is a smooth epimorphism of some Fuchian group Γ to G with surface group of genus g as its kernel.

A Fuchsian group Γ is an infinite group having presentation of the form

$\langle x_1, x_2, \dots x_k; b_1, c_1, b_2, c_2 \dots b_\gamma, c_\gamma\ ; x_1^{m_1} = x_2^{m_2} = \cdots = x_k^{m_k} = \prod_{i=1}^{k} x_i \prod_{j=1}^{\gamma} [b_j, c_j] = 1 \rangle$......(1)

Where $[b_j, c_j] = b_j^{-1} c_j^{-1} b_j c_j$ and $x_1, x_2, x_3, \dots\dots\dots\dots\dots . x_k$ are of finite order generators of order $m_1, m_2, m_3, \dots\dots\dots\dots\dots . m_k$ respectively and

$b_1, c_1, b_2, c_2 \dots\dots\dots .. b_\gamma, c_\gamma$ are of infinite order generators of Γ, and the measure of Γ is given by

$\delta(\Gamma) = 2\gamma - 2 + \sum_{i=1}^{k}\left(1 - \frac{1}{m_i}\right) > 0$.. (2)

Such a Fuchsian is also denoted by $\Delta(\gamma;\ m_1, m_2, , \dots\dots\dots\dots\dots . m_k)$ the non negative integer γ is called the genus Γ and the integer $m_i (\geq 2)$ are called the periods of Γ to C_{3v}

If $\gamma = 0$ then Γ has the signature of the form $\Gamma = \Delta(m_1, m_2, , \dots\dots\dots\dots\dots . m_k)$ and if there is no finite order element except identity in Γ then $\Gamma = (\gamma\ ; 0)$, called a surface group.It is known that if Γ_1 is a subgroup of Γ of finite index then Γ_1 is a Fuchsian group and

$[\,\Gamma : \Gamma_1\,] = \frac{\delta(\Gamma_1)}{\delta(\Gamma)}$..(3)

A homomorphism φ from a Fuchsian group Γ to a finite group G is called smooth if the kernel is a surface subgroup of Γ. The factor group $\Gamma/\ker\varphi (\cong \varphi(\Gamma))$ is called a smooth quotient of genus g. It is found that [5] there is a set a set of necessary and sufficient conditions for the existence of a smooth epimorphism from a Fuschian group Γ to $C_{3v} = \{E, C_3^1, C_3^2, \sigma_v^1, \sigma_v^2, \sigma_v^3\}$ of order 6 which has a presentation

$C_{3v} = \{u, v / u^3 = v^2 = uv^2 = 1, \text{where } u = C_3^1,\ v = \sigma_v^1, E = 1\}$ ....(4)

as follows:

Theorem1: There is a smooth epimorphism $\phi: \Gamma \to C_{3v}$ where Γ and C_{3v} are defined as (1) and (4) respectively if and only if

(1) When $k = 0$ then $\Gamma = \Delta(\gamma; -)$ is a surface group and $\gamma \geq 2$

(2) When $k \neq 0$, $m_i = 2 \; or \; 3$

Moreover,

(i) if all $\phi(x_i) \in < u >$ then $m_i = 3$ and $\gamma \geq 1$

(ii) if all $\phi(x_i) \in C_{3v} -< u >$ then $m_i = 2$, k is even and $\gamma \geq 1$ when $k = 2,4$ and $\gamma \geq 0$ when $k \geq 6$,

(iii) if all $\phi(x_i) \in < u >$ for some i without loss of generality $i \in \{1,2,3, \dots \dots .. s\}$ where $1 \leq s \leq k$ and $\phi(x_{s+j}) \in C_{3v} - < u >$ for $j = 1,2,3 \dots \dots \dots t$ so that $s + t = k$ then t is even and $k \geq 3$; moreover (i) if $k = 3$ so that $s = 1, t = 2$ then $\gamma \geq 1$ (ii) if $k \geq 4$ then $\gamma \geq 0$.

Complete Set of Genera

In this section we first find all the values of $g \geq 2$ for which a compact Riemann surface of genus g admits C_{3v} as an automorphism group. The condition of the theorem(1) above enables us to do so as follows:

Theorem: Let C_{3v} be the group of symmetries of the ammonia (NH_3) molecule of order 6. Then C_{3v} acts as an automorphism group of a compact Riemann surface of genus $g \geq 2$ if and only if there are integer λ and μ such that $\lambda \leq 1$ and $\mu \geq 1$ and $g = \lambda + 3\mu (\geq 2)$. Furthermore $\mu \geq |\lambda|$

Proof: Let g be the genus of a compact Riemann surface on which C_{3v} acts as an automorphism group. Then there exist a Fuschian group

$\Gamma = \langle x_1, x_2, \dots x_k; b_1, c_1, b_2, c_2 \dots b_\gamma, c_\gamma \; ; x_1^{m_1} = x_2^{m_2} = \cdots = x_k^{m_k} = \prod_{i=1}^{k} x_i \prod_{j=1}^{\gamma} [b_j, c_j] = 1 \rangle$ and an epimorphism $\phi: \Gamma \to C_{3v}$ such that the kernel of ϕ is a surface group of genus g.

Now

$O(C_{3v}) = O(\Gamma / \ker\phi) = \frac{\delta(\ker\phi)}{\delta(\Gamma)}$

i.e. $2.3 = \frac{2(g-1)}{\delta(\Gamma)}$

i.e. $g = 1 + 3\delta(\Gamma)$

We know that for the existence of a smooth epimorphism $\phi: \Gamma \to C_{3v}$, the periods of Γ and the genus of Γ must satisfy the conditions of the theorem(1) therefore the periods of Γ can take the values 2 and 3 only.

Let the number of periods equal to 2 be r and that of periods equal to 3 be s then it is clear from the conditions of the theorem(1) that r must be even. Also from (2) it follows that

$$g = 1 + 3[\, 2(\gamma - 1) + s\left(1 - \tfrac{1}{3}\right) + \tfrac{1}{2}\mathrm{r}\,]$$

i.e. $g = 1 - s + 3[2\gamma - 2 + s + \frac{1}{2}\mathrm{r}\,]$

Let us put $\lambda = 1 - s$ and $\mu = 2\gamma - 2 + s + \frac{1}{2}\mathrm{r}$ then $g = \lambda + 3\mu$ so that $\lambda \leq 1,\ \mu \geq 1$ because subject to the conditions mentioned in the previous theorem on γ, s and r we get $\lambda \leq 1,\ \mu \geq 1$. Now we see that under those conditions $g \geq 2$. It is observe that $g \geq 2$ for $\gamma \geq 2$. If $\gamma = 0$ then $r \neq 0,2,4$ when $s = 0$, which gives $g \geq 2$. If $\gamma = 1$ then r and s are not zero simultaneously. This also give $g \geq 2$.Thus it follows that $g \geq 2$ for all possible (admissible) values of γ, r and s under the conditions of the theorem(1). It is also noted that in the above cases $\lambda = 1 - s$ and $\mu = 2\gamma - 2 + s + \frac{1}{2}\mathrm{r}$, $\mu \geq |\lambda\,|$ also. Thus the conditions are necessary.

Now we show that the conditions are sufficient:

Let g be an integer ≥ 2 such that $g = \lambda + 3\mu$ where λ and μ are integers with $\lambda \leq 1,\ \mu \geq 1$ and $\mu \geq |\lambda\,|$. It is sufficient to show that there is a Fuschian group Γ satisfies the conditions of theorem(1) and that there is a smooth epimorphism $\phi\colon \Gamma \to C_{3v}$ such that kerϕ has genus $g \geq 2$.

Set

$$\lambda = 1 - l \cdots \cdots \cdots \cdots \cdots \cdots \cdots \quad (1)$$

$$\mu = 2\gamma + \frac{m}{2} + \mathrm{l} - 2 \cdots \cdots \cdots \cdots \ (2)$$

For any given integer $\lambda \leq 1$ we can find at least one non-zero positive integral value l_0 of l which satisfy the equation (1). Putting the value of l in (2) we get an equation

$$\mu = 2\gamma + \frac{m}{2} + l_0 - 2$$

This equation in μ, γ and m has a solution (μ_0, γ_0, m_0) where $\gamma_0$and m_0 satisfy the conditions of the previous Theorem and μ_0 is the integer ≥ 1 . Therefore we can always find a set of values l_0, m_0, γ_0 of $l,\ m$ and γ respectively which give $\lambda \leq 1,\ \mu \geq 1$ so that $\Gamma = \Delta\,(\gamma_0\colon \underbrace{3,3 \ldots \ldots 3}_{l_0}, \underbrace{2,2, \ldots \ldots 2}_{m_0})$ is a Fuschian group satisfying conditions of the previous Theorem. Hence there is a smooth epimorphism from Γ to C_{3v} and consequently C_{3v} acts as a group of automorphisms of a compact Riemann surface of genus g, given by

$$g = 1 + 3[\, 2\gamma_0 - 2 + l_0\left(1 - \tfrac{1}{3}\right) + m_0(1 - \tfrac{1}{2})\,]$$

$$= 1 - l_0 + 3[\, 2\gamma_0 + l_0 + \tfrac{m_0}{2} - 2\,]$$

$$= \lambda + 3\mu \quad \text{from (1) and (2)}$$

Thus C_{3v} acts as an automorphism group of a compact Riemann surface of genus g.

This completes the proof.

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