GEOMETRIC PROPERTIES OF SOME CLASS OF UNIVALENT FUNCTIONS BY FIXING FINITEY MANY COEFFICIENTS

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ABSTRACT. In this article we defined a new subclass of univalent functions normalized with finitely many fixed points. Certain properties of the defined subclass of univalent functions like coefficients properties, radii of convexity and star likeness, extreme points and integral operators applied to the functions are investigated.

1. INTRODUCTION AND PRELIMINARIES

Let $\mathcal{A}(\mathbb{D}_1(0))$ denote the class of analytic functions in the open unit disk $\mathbb{D}_1(0) = \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{C}$ be the class of all functions $f \in \mathcal{A}(\mathbb{D}_1(0))$ which are of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{D}_1(0). \tag{1}$$

and normalized by $f(0) = f'(0) - 1 = 0$. By $\mathcal{S}$ we denote the class of all functions in $\mathcal{C}$ which are univalent in $\mathbb{D}_1(0)$.

In 1991 Goodman [2, 3] studied the subclass of uniformly convex functions and uniformly starlike functions. In 2004 Murugusundaramoorthy [4] extended the study of the above class of univalent functions by fixing the second coefficient. Recently, Dixit et al. [1], Owa et al. [5] and Varma et al. [6] have defined a new subclass of univalent functions of $\mathcal{S}$ by fixing finite number of coefficients of functions. In this paper we consider a subclass of $\mathcal{S}$ by fixing finitely many coefficients and properties of the functions in the subclass are examined.

Let $\mathfrak{T}$ denotes the subclass of $\mathcal{S}$ containing the functions with negative coefficients i.e., if $f(z) \in \mathfrak{T}$, then

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \; a_n \geq 0.$$

Definition 1.1. *A function $f \in \mathcal{S}$ is in the class of $\mathcal{ST}(\lambda)$ if*

$$\lambda \left| \frac{f(z)}{z} + f'(z) \right| \leq \mathcal{R}e \left\{ f'(z) \right\}.$$

Theorem 1.1. *Let $f \in \mathfrak{T}$. Then $f \in \mathcal{ST}(\lambda)$ if and only if*

$$\sum_{n=2}^{\infty} \big((\lambda + 1)n + \lambda \big) |a_n| \leq 1 - 2\lambda.$$

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Proof. Suppose $\sum_{n=2}^{\infty}[(\lambda+1)n+\lambda]|a_n| \le 1-2\lambda,\ \lambda \ge 0$, then

$$\begin{aligned}\mathcal{R}e\left\{f'(z)\right\}-\lambda\left|f'(z)+\frac{f(z)}{z}\right| &\geq 1-|f'(z)-1|-\lambda\left|f'(z)+\frac{f(z)}{z}\right|\\ &=1-\sum_{n=2}^{\infty}n|a_n|-\lambda\left|f'(z)+\frac{f(z)}{z}\right|\\ &=1-\left[2\lambda+\sum_{n=2}^{\infty}n|a_n|+\lambda\sum_{n=2}^{\infty}na_nz^{n-1}+\sum_{n=2}^{\infty}a_nz^{n-1}\right]\\ &=1-\left[2\lambda+\sum_{n=2}^{\infty}(n+\lambda(n+1))|a_n|\right]\\ &=1-\left[2\lambda+\sum_{n=2}^{\infty}(\lambda+1)n+\lambda|a_n|\right]\geq 0.\end{aligned}$$

Conversely, suppose that

$$\mathcal{R}e\{f'(z)\}-\lambda\left|f'(z)+\frac{f(z)}{z}\right|>0,$$

then

$$1-\mathcal{R}e\left\{1-\sum_{n=2}^{\infty}n|a_n|z^{(n-1)}\right\}-\lambda\left|2-\sum_{n=2}^{\infty}(n+1)|a_n|z^{(n-1)}\right|>0.$$

If $|z|$ tends to 1, we get

$$\begin{aligned}&1-\sum_{n=2}^{\infty}n|a_n|-\lambda|2-\sum_{n=2}^{\infty}(n+1)|a_n|>0\\ \Longrightarrow\ &1-2\lambda+\sum_{n=2}^{\infty}(\lambda+1)n+\lambda)|a_n|>0\\ \Longrightarrow\ &2\lambda+\sum_{n=2}^{\infty}(\lambda+1)n+\lambda)|a_n|\leq 1\quad\text{for}\quad\lambda\geq 0.\end{aligned}$$

This completes the proof. □

Corollary 1.1. *For $f \in \mathcal{ST}(\lambda)$*

$$a_n \leq \frac{1-2\lambda}{\lambda+(\lambda+1)n},\ \ n\geq 2.$$

We now introduce a new subclass $\mathcal{ST}(\lambda, P_k)$ of $\mathcal{ST}(\lambda)$ which are in the form

$$f(z)=z-\sum_{i=2}^{k}\frac{p_i(1-2\lambda)}{\lambda+(\lambda+1)i}z^i-\sum_{n=k+1}^{\infty}a_nz^n.$$

2. Coefficient Estimates

Theorem 2.1. *A function in the form of*

$$f(z) = z - \sum_{i=2}^{k} \frac{p_i(1-2\lambda)}{\lambda+(\lambda+1)i} z^i - \sum_{n=k+1}^{\infty} a_n z^n$$

is in the class of $\mathcal{ST}(\lambda, P_k)$*, then the necessary and sufficient condition is*

$$\sum_{n=k+1}^{\infty} \left[\frac{\lambda+(\lambda+1)n}{1-2\lambda}\right] a_n \leq 1 - \sum_{i=2}^{k} p_i$$

Proof. From the Corollary 1.1 for the subclass having fixed coefficients

$$a_i = \frac{(1-2\lambda)p_i}{\lambda+(\lambda+1)i}, i = 2,3,4,....k, 0 \leq \sum_{i=}^{k} p_i \leq 1,$$

which implies

$$\sum_{i=2}^{k} p_i + \sum_{n=k+1}^{\infty} \left[\frac{\lambda+(\lambda+1)n}{1-2\lambda}\right] a_n \leq 1.$$

Conversely, suppose that

$$\begin{aligned}
\mathcal{R}e\{f'(z)\} - \lambda\left|f'(z) + \frac{f(z)}{z}\right| \leq & 1 - |f'(z)-1| - \lambda\left|f'(z) + \frac{f(z)}{z}\right| \\
= & 1 - \sum_{n=2}^{\infty} n|a_n| - \lambda|2 - \sum_{n=2}^{\infty}(n+1)|a_n| \\
= & 1 - n\sum_{i=2}^{k} \frac{\lambda+(\lambda+1)n}{(1-2\lambda)n}|a_n| - \sum_{n=k+1}^{\infty} \frac{\lambda+(\lambda+1)n}{n(1-2\lambda)}|a_n| \\
= & 1 - \sum_{i=2}^{k} p_i - \sum_{n=k+1}^{\infty} \frac{\lambda+(\lambda+1)n}{n(1-2\lambda)}|a_n| \geq 0.
\end{aligned}$$

Thus if a function belongs to $\mathcal{ST}(\lambda, p_k)$, then the sharpness of the result follows by taking the function as

$$f(z) = z - \sum_{i=2}^{k} \frac{p_i(1-2\lambda)}{\lambda+(\lambda+1)n} z_i - \frac{(1-\sum_{i=2}^{k} p_i)}{\lambda+(\lambda+1)n}(1-2\lambda)z_n, n \geq 1 \tag{2}$$

□

Corollary 2.1. *If a function* $f(z)$ *is in the class of* $\mathcal{ST}(\lambda, p_k)$ *then the coefficient values are in the form*

$$a_n \leq 1 - \sum_{i=2}^{k} p_i \frac{(1-2\lambda)}{\lambda+(\lambda+1)n}, n \geq k+1. \tag{3}$$

The result is sharp for the function $f(z)$ given in the above theorem.

3. Closure Theorems

Theorem 3.1. *The class $\mathcal{ST}(\lambda, p_k)$ is convex.*

Proof. Let us suppose that the two functions namely $g(z)$, $h(z)$ are two functions belongs to the subclass of function $\mathcal{ST}(\lambda, p_k)$ then these functions can be expressed as

$$g(z) = z - \sum_{i=2}^{k} \frac{p_i(1-2\lambda)}{\lambda+(\lambda+1)n} z^i - \sum_{n=k+1}^{\infty} a_n z^n. \tag{4}$$

$$h(z) = z - \sum_{i=2}^{k} \frac{p_i(1-2\lambda)}{\lambda+(\lambda+1)n} z^i - \sum_{n=k+1}^{\infty} b_n z^n. \tag{5}$$

where $0 \le p_i \le 1$, $0 \le \sum_{i=2}^{\infty} p_i \le 1$. Define $f(z) = \alpha g(z) + (1-\alpha)h(z)$. Then

$$\begin{aligned} h(z) =& \alpha \left[z - \sum_{i=2}^{k} \frac{p_i(1-2\lambda}{\lambda+(\lambda+1)n} z^i - \sum_{n=k+1}^{\infty} a_n z^n \right] \\ &+ (1-\alpha) \left[z - \sum_{i=2}^{k} \frac{p_i(1-2\lambda}{\lambda+(\lambda+1)n} - \sum_{n=k+1}^{\infty} b_n z^n \right] \\ =& z - \sum_{i=2}^{k} \frac{p_i(1-2\lambda}{\lambda+(\lambda+1)n} - \sum_{n=k+1}^{\infty} (\alpha a_n + (1-\alpha) b_n) z^n, \end{aligned}$$

it follows that $f(z)$ belongs to the subclass $\mathcal{ST}(\lambda, p_k)$ □

Theorem 3.2. *Let us suppose that*

$$f_k(z) = z - \sum_{i=2}^{k} p_i \left[\frac{1-2\lambda}{\lambda+(\lambda+1)n} \right] z^i$$

and

$$f_n(z) = z - \sum_{i=2}^{k} p_i \left[\frac{1-2\alpha}{\lambda+(\lambda+1)n} \right] z^i - \left[1 - \sum_{i=2}^{k} P_i \right] \frac{1-2\lambda}{\lambda+(\lambda+1)} z^n,$$

then $f(z) \in \mathcal{ST}(\lambda, p_k)$ if and only if $f(z)$ can be expressed in the form

$$f(z) = \sum_{n=k}^{\infty} \alpha_n f_n(z) \tag{6}$$

where $\alpha_n \ge 0 (n \ge qk)$ and $\sum_{n=k}^{\infty} \alpha_n = 1$.

Proof. Suppose $f(z) \in \mathfrak{T}$ then this implies

$$\begin{aligned} f(z) =& z - \sum_{n=2}^{\infty} a_n z^n \\ =& z - \sum_{i=2}^{k} p_i \frac{1-2\lambda}{\lambda+(\lambda+1)n} z^i - \sum_{n=k+1}^{\infty} \alpha_n \left[1 - \sum_{i=2}^{k} p_i \right] \frac{1-2\lambda}{\lambda+(\lambda+1)n} z^n. \end{aligned}$$

Now

$$\begin{aligned}\sum_{n=k+1}^{\infty}\alpha_n\left[1-\sum_{i=2}^{k}p_i\right]\frac{1-2\lambda}{\lambda+(\lambda+1)n}z^n &= \sum_{n=k+1}^{\infty}\alpha_n\left[1-\sum_{i=2}^{k}p_i\right]\\ &=\left[1-\sum_{i=2}^{k}p_i\right]\sum_{n=k+1}^{\infty}\alpha_n\\ &=\left[1-\sum_{i=2}^{k}p_i\right](1-\alpha_n)\\ &\leq 1-\sum_{i=2}^{k}p_i,\end{aligned}$$

which shows that f belongs to $\mathcal{ST}(\lambda, p_k)$.

Conversely for $n \geq k+1$set

$$\alpha_n = \frac{\lambda+(\lambda+1)n}{1-2\lambda}\frac{1}{1-\sum_{i=2}^{k}p_k a_n}, n \geq k+1$$

this implies

$$\alpha_k = 1-\sum_{n=k+1}^{\infty}\alpha_n$$

then $f(z)$ can be represented in the the form in the function

$$f(z) = \sum_{n=k}^{\infty}\alpha_n f_n(z)$$

□

Corollary 3.1. *The extreme points in the subclass $\mathcal{ST}(\lambda, p_k)$ are the functions $f_n(n \geq 1)$ given by*

$$f_k(z) = z-\sum_{i=2}^{k}p_i\left[\frac{1-2\lambda}{\lambda+(\lambda+1)n}\right]z^i$$

and

$$f_n(z) = z-\sum_{i=2}^{k}p_i\left[\frac{1-2\alpha}{\lambda+(\lambda+1)i}\right]z^i-\left[1-\sum_{i=2}^{k}p_i\right]\frac{1-2\lambda}{\lambda+(\lambda+1)n}z^n.$$

4. Integral Operators

The Alexander Operator for the class of functions in the class $\mathcal{A}$ is defined as

$$\mathcal{I}(f) = \int_0^z \frac{f(t)}{t}dt \tag{7}$$

This Operator mapping from the class of starlike univalent functions onto the class of univalent convex functions. The effectness of the above mentioned operator on the functions in the defined class of $\mathcal{ST}(\lambda, p_k)$ is explained in the following theorem.

Theorem 4.1. *Let $f(z)$ defined by the function in the class of $\mathcal{ST}(\lambda, p_k)$.Then $\mathcal{I}$ will be in the class of $\mathcal{ST}(\lambda, q_k)$ where $q_k = \frac{p_k}{k}$*

Proof. We know that

$$\mathcal{I} = z - \sum_{i=2}^{k} \frac{(1-2\lambda)}{\lambda+(\lambda+1)n} q_i z^i - \sum_{n=k+1}^{\infty} \frac{a_n}{n} z^n \tag{8}$$

Now

$$\begin{aligned}\sum_{n=k+1}^{\infty} \left[\frac{\lambda+(\lambda+1)n}{(1-2\lambda)}\right] \frac{a_n}{n} &\leq \frac{1}{k+1} \sum_{n=k+1}^{\infty} \left[\frac{\lambda+(\lambda+1)n}{1-2\lambda}\right] a_n \\ &\leq \frac{1}{k+1}\left[1-\sum_{i=2}^{k} p_i\right] \\ &= \frac{1}{k+1} - \sum_{i=2}^{k} \frac{p_i}{k+1} \\ &< 1 - \sum_{i=2}^{k} \frac{p_i}{i}\end{aligned}$$

Which gives that the Integral Operator $\mathcal{I}$ beongs to the class $\mathcal{ST}(\lambda, P_k)$. Hence the theorem. □

5. RADIUS OF CONVEXITY AND STARLIKENESS

In the present section we discuss the radii resuts of the functions in the defined class $\mathcal{ST}(\lambda, p_k)$ to be the starike or convex of some order β.

Theorem 5.1. *The function $f(z)$ which was defined in the class of univalent function $\mathcal{ST}(\lambda, p_k)$ is starlike of some order β $(0 \leq \beta \leq 1)$ in the disk $B(0, r_1)$ where r_1 is the largest value which satisfies*

$$\sum_{i=2}^{\infty} \frac{(2-i)-\beta)(1-2\lambda)}{\lambda+(\lambda+1)i} p_i r^{i-1} + \frac{((2-n)-\beta)[1-\sum_{i=2}^{k} p_i](1-2\lambda)}{\lambda+(\lambda+1)n} r^{n-1} \leq \beta. \tag{9}$$

Proof.

$$\left|\frac{zf^i(z)}{f(z)} - 1\right| \leq \frac{\sum\limits_{i=2}^{k} [1-i] p_i (1-2\lambda) r^{i-1} \frac{1}{\lambda+(\lambda+1)} - \sum\limits_{n=k+1}^{\infty} (n-1) a_n r^{n-1}}{1 - \sum\limits_{i=2}^{k} \frac{p_i(1-2\lambda)}{\lambda+(\lambda+1)} r^{i-1} - \sum\limits_{n=k+1}^{\infty} a_n r^{n-1}}$$

which is less than or equal to $1-\beta$ for $|z| \leq r$ if and ony if

$$\sum_{i=2}^{k} \frac{((2-i)-\beta)(1-2\lambda)}{\lambda+(\lambda+1)} p_i r^{i-1} + \sum_{n=k+1}^{\infty} (2-n-\beta) a_n r^{n-1} \leq 1-\beta$$

By the corollary discussed above we may be set the coefficient a_n as

$$a_n = \frac{\left[1-\sum\limits_{i=2}^{k} p_i\right](1-2\lambda)}{\lambda+(\lambda+1)i} \alpha_n$$

where $\alpha_n \geq 0, (n \geq k+1$ and $\sum_{n=k+1}^{\infty} \alpha_n \leq 1$ for each fixed r, choosing an integral $n_0 = n_0(r)$, for which

$$\frac{((2-n)-\beta))r^{n-1}(1-2\lambda)}{\lambda+(\lambda+1)n}$$

is a maximum coefficient value for the defined class of function then, we obtain

$$\sum_{n=k+1}^{\infty}(n-\beta)a_n r^{n-1} \leq \frac{((2-n_0)-\beta)\left[1-\sum_{i=2}^{k} p_i\right](1-2\lambda)}{\lambda+(\lambda+1)n_0} r^{n_0-1}$$

Hence f(z) is a starike of order β in $|z| \leq r_1$, provided

$$\sum_{i=2}^{k}\frac{((2-i)-\beta)(1-2\lambda)}{\lambda+(\lambda+1)i}p_i r^{i-1} + \frac{((2-n_0)-\beta)(1-2\lambda)\left[1-\sum_{i=2}^{k} p_i\right]}{\lambda+(\lambda+1)n_0} r^{n_0-1} \leq 1-\beta$$

We find the value r_0 and the corresponding $n_0(r_0)$, so that

$$\sum_{i=2}^{k}\frac{((2-i)-\beta)(1-2\lambda)}{\lambda+(\lambda+1)i}p_i {r_0}^{i-1} + \frac{((2-n_0)-\beta)(1-2\lambda)\left[1-\sum_{i=2}^{k} p_i\right]}{\lambda+(\lambda+1)n_0} {r_0}^{n_0-1} = 1-\beta$$

which the radius of the starike ness of order β for the function $f(z)$ in the class $S\tau(\lambda, p_k)$. □

Theorem 5.2. *The function $f(z)$ defined in the class $\mathcal{ST}(\lambda, p_k)$ is convex of order $\beta(0 \leq \beta \leq 1)$ in the disk $|z| < r_2$ where r_2 is the largest value which satisfies*

$$\sum_{i=2}^{\infty}\frac{i(i-\beta)(1-2\lambda)}{\lambda+(\lambda+1)}p_i r^{i-1} + \frac{n(n-\beta)\left[1-\sum_{i=2}^{k} p_i\right]}{\lambda+(\lambda+1)n}(1-2\lambda)r^{n-1} \leq \beta$$

Proof.

$$\left|\frac{zf''(z)}{f(z)}\right| \leq \frac{\sum_{i=2}^{k}\frac{i(i-1)p_i(1-2\lambda)}{\lambda+(\lambda+1)i}r^{i-1} + \sum_{n=k+1}^{\infty} n(n-1)a_n r^{n-1}}{1-\sum_{i=2}^{k}\frac{ip_i(1-2\lambda)}{\lambda+(\lambda+1)i} - \sum_{n=k+1}^{\infty} n a_n r^{n-1}}$$

which is less than or equal to $1-\beta$ for $|z| < r$ if and only if

$$\sum_{n=2}^{k}\frac{i(i-\beta)(1-2\lambda)}{\lambda+(\lambda+1)i}r^{i-1} + \sum_{n=k+1}^{\infty} n(n-\beta)a_n r^{n-1} \leq 1-\beta$$

Using the corallary in the second section and for each fixed r , choosing an integer $n_0 = n_0(r)$ for which

$$\frac{n_0(n_0-\beta)r^{n-1}(1-2\lambda)}{\lambda+(\lambda+1)n_0}$$

is a maximum, we get

$$\sum_{n=k+1}^{\infty} n(n-\beta)a_n r^{n-1} \leq \frac{n_0(n_0-\beta)r^{n-1}(1-\sum_{i=2}^{k} p_i)(1-2\lambda)}{\lambda+(\lambda+1)n_0} r_0^{n-1}.$$

Hence the function $f(z)$ is convex of order β in $|z| < r_2$ provided

$$\sum_{n=2}^{k} \frac{i(i-\beta)(1-2\lambda)}{\lambda+(\lambda+1)i} r^{i-1} + \frac{n_0(n_0-\beta)(1-\sum_{i=2}^{k} p_i)}{\lambda+(\lambda+1)n_0} r^{n-1} \leq 1-\beta.$$

We find the values of r_0 and the corresponding $n_0(r_0)$ so that

$$\sum_{n=2}^{k} \frac{i(i-\beta)(1-2\lambda)}{\lambda+(\lambda+1)i} r^{i-1} + \frac{n_0(n_0-\beta)(1-\sum_{i=2}^{k} p_i)}{\lambda+(\lambda+1)n_0} {r_0}^{n-1} \leq 1-\beta$$

which is the radius of convexity of order β for function in the class $\mathcal{ST}(\lambda, p_k)$. □

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