Generalized Offset Fourier-Mellin Transform & Its Analytical Structure

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Abstract

The offset Fourier transform is the space-shifted and frequency-modulated versions of the original Transform. It is more general and flexible than the original one. Since offset Fourier Transform has close relations with the optical system consisting of lenses, free spaces, prisms and shifted lenses. Also, we have acknowledged that Fourier-Mellin transform is most useful integral transform due its shift-scale invariance property which is applicable in various fields of Mathematical Physics, Applied Mathematics and Engineering etc.
In the current paper we have thrash out the generalization of Offset Fourier-Mellin Transform in the distributional sense and provide its Analyticity structure.

Keywords: *Fourier Transform, Mellin Transform, Offset Fourier-Mellin Transform, Generalized function etc.*

1. Introduction

The function transformation method simply means mathematical operation through which a real or complex valued function is transformed into another setting of data in which the original problem can be solved more easily or in which the problems have clear physical meaning. These methods have been used successfully in solving many problems in engineering, mathematical physics and applied mathematics.

Generalized functions are especially useful in making discontinuous function more like smooth functions and describing discrete physical phenomena such as point charges. Gelfand, Shilov [3], Zemanian [1,2] extended number of integral transforms e.g. Laplace, Mellin, Fourier, Hankle, Stieltjes, Whittaker etc. to the spaces of generalized functions.

The Fourier transform (FT) and Mellin transform (MT) are well-known transforms. The Fourier transform is a fundamental mathematic tool widely used in signal analysis, radiology and integral to modern MR image formation [10]. Fourier and Mellin transforms are extensively used for spectrum analysis, signal processing, and optical system analysis. FMT is frequently used in content-based image retrieval and digital image watermarking [12]. The method of egomotion estimation makes use of the Fourier-Mellin Transform for registering radar images in a sequence [9].

The offset Fourier transform (offset FT) is the space-shifted and frequency-modulated versions of the original transforms. They are more general and flexible than the original ones [4]. Offset FTs are similar to the original FTs, except that the kernel e^{-ist} is replaced by $e^{-i(s-\eta)(t-\tau)}$. That is, the kernel is generalized by appending a space-shifted term and a frequency-modulated term; they are useful in optics and especially useful for analyzing optical systems with prisms or shifted lenses [4].

In the present paper we have generalized the Offset Fourier-Mellin Transform in the distributional sense.

Outline of this paper:

Section 1; define the Generalized Offset Fourier-Mellin Transform. Analyticity theorem of Generalized Offset Fourier-Mellin Transform is furnished in section 2. The notation and terminology given as per A. H. Zemanian [1, 2].

2. Generalized Offset Fourier-Mellin Transform (Offset FMT)

In this section we have proved that the kernel of Offset Fourier-Mellin Transform is a member of the space $F^{\tau,\eta}M_{a,b,\alpha}$. Then we have defined Offset Fourier-Mellin Transform in the distributional sense.

2.1 Testing Function Space $F^{\tau,\eta}M_{a,b,\alpha}$

An infinitely differentiable function $\phi(t,x,s,p)$ define over $0<t<\infty, 0<x<\infty$ with parameters s and p is said to belong to $F^{\tau,\eta}M_{a,b,\alpha}$ for $a,b \in R^2$, if

$$\gamma_{a,b,k,q,l}\phi(t,x) = \sup_{I_1}\left|(t-\tau)^k \xi_{a,b}(x) x^{q+1} D_t^l D_x^q \phi(t,x)\right| < \infty \qquad (2.1.1)$$

for each $k,q,l = 0,1,2,3 \dots \dots \dots$

$$\xi_{a,b} = \begin{cases} x^{-a}, & 0<x\le 1 \\ x^{-b}, & 1<x<\infty \end{cases}$$

2.2 Distributional Generalized Two Dimensional Fourier-Mellin Transform

For $f(t,x) \in F^{\tau,\eta}M^*_{a,b,\alpha}$, where $F^{\tau,\eta}M^*_{a,b,\alpha}$ is the dual space of $F^{\tau,\eta}M_{a,b,\alpha}$ and $a < Re\ p < b, s > 0$. The distributional Offset Fourier-Mellin transform is define as,

$$F^{\tau,\eta}M\{f(t,x)\} = F(s,p) = \langle f(t,x), e^{-i(s-\eta)(t-\tau)}x^{p-1}\rangle, \qquad (2.2.1)$$

Where $\phi(t,x,s,p) = e^{-i(s-\eta)(t-\tau)}x^{p-1}$ and for each fixed $t(0<t<\infty), x(0<x<\infty)$. The right hand side of (2.1.2) is meaningful i.e. $e^{-i(s-\eta)(t-\tau)}x^{p-1} \in F^{\tau,\eta}M_{a,b,\alpha}$ and $f(t,x) \in F^{\tau,\eta}M^*_{a,b,\alpha}$.

3. Analyticity Theorem

In this section we have discussed the analyticity theorem for distributional two dimensional Fourier-Mellin transform.

3.1 Theorem

If $F(s,p) = \langle f(t,x),\ \phi(t,x,s,p)\rangle$ that is $F(s,p) = \langle f(t,x), e^{-i(s-\eta)(t-\tau)}x^{p-1}\rangle$. Then $F(s,p)$ is analytic for some fixed $s>0,\ p>0$ and

$$\frac{\partial}{\partial s}F(s,p) = \langle f(t,x),\ \frac{\partial}{\partial s}\phi(t,x,s,p)\rangle \qquad (3.1.1)$$

and

$$\frac{\partial}{\partial p}F(s,p) = \langle f(t,x),\ \frac{\partial}{\partial p}\phi(t,x,s,p)\rangle \qquad (3.1.2)$$

where $\phi(t,x,s,p) = e^{-i(s-\eta)(t-\tau)}x^{p-1}$.

Proof: Let s and p be an arbitrary but fixed. Choose the real positive numbers a_1, b_1 and r such that $\sigma_1 < a_1 < s-r < s+r < b_1 < \sigma_2$. Also, let Δs be a complex increment such that $0 < \Delta s < r$.

For $\Delta s \neq 0$, we write

$$\frac{F(s+\Delta s,p)-F(s,p)}{\Delta s} - \langle f(t,x),\ \frac{\partial}{\partial s}e^{-i(s-\eta)(t-\tau)}x^{p-1}\rangle$$

$$= \langle f(t,x),\ \frac{x^{p-1}}{\Delta s}\left[e^{-i[(s+\Delta s)-\eta](t-\tau)} - e^{-i(s-\eta)(t-\tau)}\right]\rangle - \langle f(t,x), \frac{\partial}{\partial s}e^{-i(s-\eta)(t-\tau)}x^{p-1}\rangle$$

$$= \langle f(t,x),\ \frac{x^{p-1}}{\Delta s}\left[e^{-i[(s+\Delta s)-\eta](t-\tau)} - e^{-i(s-\eta)(t-\tau)}\right] - \frac{\partial}{\partial s}e^{-i(s-\eta)(t-\tau)}x^{p-1}\rangle$$

$$= \langle f(t,x),\ \psi_{\Delta s}(t,x)\rangle, \qquad (3.1.3)$$

where $\psi_{\Delta s}(t,x) = \frac{x^{p-1}}{\Delta s}\left[e^{-i[(s+\Delta s)-\eta](t-\tau)} - e^{-i(s-\eta)(t-\tau)}\right] - \frac{\partial}{\partial s}e^{-i(s-\eta)(t-\tau)}x^{p-1}$.

To prove $\psi_{\Delta s}(t,x) \in F^{\tau,\eta}M_{a,b,\alpha}$, we shall show that as $|\Delta s| \to 0$, $\psi_{\Delta s}(t,x)$ converges in $F^{\tau,\eta}M_{a,b,\alpha}$ to zero.

To proceed, let C denotes the circle with centre at s and radius r_1, where $0 < r < r_1 < \min(s - a_1, b_1 - s)$. We may interchange differentiation on s with differentiation on t.

$$(-D_t)^l \psi_{\Delta s}(t,x) = \frac{x^{p-1}}{\Delta s}\{(-i)^l[(s+\Delta s)-\eta]^l e^{-i[(s+\Delta s)-\eta](t-\tau)} - (-i)^l (s-\eta)^l e^{-i(s-\eta)(t-\tau)}\} - \frac{\partial}{\partial s}(-i)^l (s-\eta)^l e^{-i(s-\eta)(t-\tau)} x^{p-1}$$

Now applying Cauchy's integral formula,

$$(-D_t)^l \psi_{\Delta s}(t,x)$$
$$= \frac{x^{p-1}}{\Delta s}\left\{\frac{1}{2\pi i}\int_C \frac{(-i)^l[z-\eta]^l e^{-i[z-\eta](t-\tau)}}{z-s-\Delta s}dz - \frac{1}{2\pi i}\int_C \frac{(-i)^l(z-\eta)^l e^{-i[z-\eta](t-\tau)}}{z-s}dz\right\} - \frac{1}{2\pi i}\int_C \frac{(-i)^l(z-\eta)^l e^{-i[z-\eta](t-\tau)} x^{p-1}}{(z-s)^2}dz$$
$$= \frac{x^{p-1}}{\Delta s\, 2\pi i}\int_C \left[\frac{1}{z-s-\Delta s} - \frac{1}{z-s}\right](-i)^l[z-\eta]^l e^{-i[z-\eta](t-\tau)}dz - \frac{1}{2\pi i}\int_C \frac{(-i)^l(z-\eta)^l e^{-i[z-\eta](t-\tau)} x^{p-1}}{(z-s)^2}dz$$
$$= \frac{x^{p-1}}{\Delta s\, 2\pi i}\int_C \left[\frac{z-s-z+s+\Delta s}{(z-s-\Delta s)(z-s)}\right](-i)^l[z-\eta]^l e^{-i[z-\eta](t-\tau)}dz - \frac{1}{2\pi i}\int_C \frac{(-i)^l(z-\eta)^l e^{-i[z-\eta](t-\tau)} x^{p-1}}{(z-s)^2}dz$$
$$= \frac{x^{p-1}}{\Delta s\, 2\pi i}\int_C \left[\frac{\Delta s}{(z-s-\Delta s)(z-s)}\right](-i)^l[z-\eta]^l e^{-i[z-\eta](t-\tau)}dz - \frac{1}{2\pi i}\int_C \frac{(-i)^l(z-\eta)^l e^{-i[z-\eta](t-\tau)} x^{p-1}}{(z-s)^2}dz$$
$$= \frac{x^{p-1}}{2\pi i}\left\{\int_C \left[\frac{1}{(z-s-\Delta s)(z-s)} - \frac{1}{(z-s)^2}\right](-i)^l[z-\eta]^l e^{-i[z-\eta](t-\tau)}dz\right\}$$
$$= \frac{x^{p-1}}{2\pi i}\int_C \frac{\Delta s}{(z-s-\Delta s)(z-s)^2}[-i(z-\eta)]^l e^{-i[z-\eta](t-\tau)}dz$$
$$(D_t)^l \psi_{\Delta s}(t,x) = \frac{x^{p-1}\Delta s}{2\pi i}\int_C \frac{1}{(z-s-\Delta s)(z-s)^2}[-i(z-\eta)]^l e^{-i[z-\eta](t-\tau)}dz$$

Now for all $z \in C$ and $0 < t < \infty$, $\sup_{I_1}\left|(t-\tau)^k \xi_{a,b}(x) x^{q+1} x^{p-1}\right| \le K$, where K is a constant independent of z and t. Moreover, $|z-s-\Delta s| > r_1 - r > 0$ and $|z-s| = r_1$,
$C_1 = \max\{|z-\eta|^l, z \in C\}$, consequently

$$\sup_{I_1}\left|(t-\tau)^k \xi_{a,b}(x) x^{q+1} D_t^l \psi_{\Delta s}(t,x)\right|$$
$$= \sup_{I_1}\left|(t-\tau)^k \xi_{a,b}(x) x^{q+1}\frac{x^{p-1}\Delta s}{2\pi i}\int_C \frac{[-i(z-\eta)]^l e^{-i[z-\eta](t-\tau)}}{(z-s-\Delta s)(z-s)^2}dz\right|$$
$$\le \sup_{I_1}\left|(t-\tau)^k \xi_{a,b}(x) x^{q+1}\frac{x^{p-1}}{2\pi}\right||\Delta s|\int_C \frac{|z-\eta|^l\left|e^{-i[z-\eta](t-\tau)}\right|}{|z-s-\Delta s||z-s|^2}|dz|$$
$$\le K\frac{|\Delta s|}{2\pi}\int_C \frac{C_1}{(r_1-r)(r_1)^2}|dz|$$
$$\le \frac{|\Delta s|}{2\pi}\int_C \frac{K C_1}{(r_1-r)(r_1)^2}|dz|$$
$$\le \frac{|\Delta s|}{2\pi}\frac{C_2}{(r_1-r)(r_1)^2}2\pi r_1, \quad \text{where } C_2 = K\,C_1$$
$$\le \frac{|\Delta s| C_2}{(r_1-r)r_1}$$

The right hand side is independent of t and converges to zero as $\Delta s \to 0$. This shows that $\psi_{\Delta s}(t,x)$ converges to zero.

Now, let s, p be an arbitrary but fixed points. Choose the real positive number a_2, b_2 and h such that $\sigma_1' < a_2 < Re\, p - r < Re\, p + r < b_2 < \sigma_2'$. Also let Δp be a complex increment such that $0 < |\Delta p| < r$. For $\Delta p \neq 0$, we write

$$\frac{F(s,p+\Delta p) - F(s,p)}{\Delta p} - \langle f(t,x), \frac{\partial}{\partial p} e^{-i[s-\eta](t-\tau)} x^{p-1} \rangle$$

$$= \langle f(t,x), \frac{e^{-i[s-\eta](t-\tau)}}{\Delta p}\left[x^{(p+\Delta p)-1} - x^{p-1}\right] \rangle - \langle f(t,x), \frac{\partial}{\partial p} e^{-i[s-\eta](t-\tau)} x^{p-1} \rangle$$

$$= \langle f(t,x), \frac{e^{-i[s-\eta](t-\tau)}}{\Delta p}\left[x^{(p+\Delta p)-1} - x^{p-1}\right] - \frac{\partial}{\partial p} e^{-i[s-\eta](t-\tau)} x^{p-1} \rangle$$

$$= \langle f(t,l,x,y), \psi_{\Delta p}(t,x) \rangle,$$

where $\psi_{\Delta p}(t,x) = \frac{e^{-i[s-\eta](t-\tau)}}{\Delta p}\left[x^{(p+\Delta p)-1} - x^{p-1}\right] - \frac{\partial}{\partial p} e^{-i[s-\eta](t-\tau)} x^{p-1}$.

To prove $\psi_{\Delta p}(t,x) \in F^{\tau,\eta} M_{a,b,\alpha}$, we shall show that as $|\Delta p| \to 0$, $\psi_{\Delta p}(t,x)$ converges in $F^{\tau,\eta} M_{a,b,\alpha}$ to zero.

To proceed, let C_1 denote the circle with centre at p and radius r_1, where $0 < r < r_1 < \min(p - a_2, b_2 - p)$. We may interchange differentiation on p with differentiation on x and by using the Cauchy's integral formula,

$$(-D_x)^q \psi_{\Delta p}(t,x)$$

$$= \frac{e^{-i[s-\eta](t-\tau)}}{\Delta p}\left[P(p+\Delta p) x^{(p+\Delta p)-q-1} - P(p) x^{p-q-1}\right] - \frac{\partial}{\partial p} e^{-i[s-\eta](t-\tau)} P(p) x^{p-q-1}$$

where $P(p+\Delta p)$ is polynomial in $p + \Delta p$ and $P(p)$ is polynomial in p.

Now applying Cauchy's integral formula, we get

$$= \frac{e^{-i[s-\eta](t-\tau)}}{\Delta p}\left\{\frac{1}{2\pi i}\int_C \frac{P(z)\,x^{z-q-1}}{z-p-\Delta p}\,dz - \frac{1}{2\pi i}\int_C \frac{P(z)\,x^{z-q-1}}{z-p}\,dz\right\} - \frac{1}{2\pi i}\int_C \frac{P(z)\,x^{z-q-1} e^{-i[s-\eta](t-\tau)}}{(z-p)^2}\,dz$$

$$= \frac{e^{-i[s-\eta](t-\tau)}}{\Delta p\, 2\pi i}\int_C \left[\frac{1}{(z-p-\Delta p)} - \frac{1}{z-p}\right] P(z) x^{z-q-1}\,dz - \frac{1}{2\pi i}\int_C \frac{P(z)\,x^{z-q-1} e^{-i[s-\eta](t-\tau)}}{(z-p)^2}\,dz$$

$$= \frac{e^{-i[s-\eta](t-\tau)}}{\Delta p\, 2\pi i}\int_C \left[\frac{z-p-z+p+\Delta p}{(z-p-\Delta p)(z-p)}\right] P(z)\,x^{z-q-1}\,dz - \frac{1}{2\pi i}\int_C \frac{P(z)\,x^{z-q-1} e^{-i[s-\eta](t-\tau)}}{(z-p)^2}\,dz$$

$$= \frac{e^{-i[s-\eta](t-\tau)}}{2\pi i}\int_C \left[\frac{1}{(z-p-\Delta p)(z-p)} - \frac{1}{(z-p)^2}\right] P(z)\,x^{z-q-1}\,dz$$

$$= \frac{e^{-i[s-\eta](t-\tau)}}{2\pi i}\int_C \left[\frac{z-p-z+p+\Delta p}{(z-p-\Delta p)(z-p)^2}\right] P(z)\,x^{z-q-1}\,dz$$

$$D_x^q \psi_{\Delta p}(t,x) = \frac{e^{-i[s-\eta](t-\tau)}}{2\pi i}\int_C \left[\frac{\Delta p}{(z-p-\Delta p)(z-p)^2}\right] P(z)\,x^{z-q-1}\,dz$$

Now for all $z \in C$ and $0 < x < \infty$, $\sup_{I_1}\left|(t-\tau)^k \xi_{a,b}(x) x^{q+1} e^{-i[s-\eta](t-\tau)}\right| \leq K$, where K is a constant independent of z and x. Moreover, $|z - p - \Delta p| > r_1 - r > 0$ and $|z - p| = r_1$, $C_1 = \max\{|P(z)x^z| : z \in C\}$, consequently

$$\sup_{I_1}\left|(t-\tau)^k \xi_{a,b}(x) x^{q+1} D_x^q \psi_{\Delta p}(t,x)\right|$$

$$= \sup_{I_1}\left|(t-\tau)^k \xi_{a,b}(x) x^{q+1} \frac{e^{-i[s-\eta](t-\tau)}}{2\pi i}\int_C \left[\frac{\Delta p}{(z-p-\Delta p)(z-p)^2}\right] P(z)\,x^{z-q-1}\right|$$

$$\leq \sup_{I_1}\left|(t-\tau)^k \xi_{a,b}(x) x^{q+1} \frac{e^{-i[s-\eta](t-\tau)}}{2\pi}\right| |\Delta p| \int_C \frac{|P(z)x^z|}{|z-p-\Delta p||z-p|^2}\,|dz|$$

$$\leq K\,\frac{|\Delta p|}{2\pi}\int_C \frac{C_1}{|z-p-\Delta p||z-p|^2}\,|dz|$$

$$\leq \frac{|\Delta p|}{2\pi}\int_c \frac{K\,C_1}{(r_1-r)(r_1)^2}|dz|$$
$$\leq \frac{|\Delta p|}{2\pi}\int_c \frac{C_2}{(r_1-r)(r_1)^2}|dz|,\quad \text{where } C_2 = K\,C_1$$
$$\leq \frac{|\Delta p|}{2\pi}\frac{C_2}{(r_1-r)(r_1)^2}2\pi r_1$$
$$\leq \frac{|\Delta p| C_2}{(r_1-r)r_1}$$

The right hand side is independent of x and converges to zero as $|\Delta p| \to 0$. This shows that $\psi_{\Delta p}(t,x)$ converges to zero in $F^{\tau,\eta}M_{a,b,\alpha}$ as $|\Delta p| \to 0$, which ends the proof.

4. Conclusion:

It is well known that Offset Fourier Transform has close relations with the optical system consisting of lenses, free spaces, prisms and shifted lenses; so in the specified paper we have generalized Offset Fourier-Mellin transform in the distributional sense and proved its analytical structure.

References:

[1] Zemanian A. H., "Distribution Theory and Transform Analysis", Mc-Graw Hill, NewYork, 1965.
[2] Zemanian A. H., "Generalized Integral Transformation", Inter-Science Publishers,New York, 1968.
[3] Gelfand I. M. and Shilov G. E., "Generalized Functions", Vol. III, Academic Press, New York, 1968.
[4] Soo-Chang Pei and Jian-Jiun Ding: "Eigenfunctions of the offset Fourier, fractional Fourier, and linear canonical transforms", J. Opt. Soc. Am., Vol. 20, No. 3,March 2003.
[5] Sharma V. D. and Thakare M. M.: "Offset Fractional Fourier Transform and Its Operators", 5th International Conference in Recent Trends in Engineering, Science and Management, 9th-10th Dec. 2016, ISBN: 978-93-86171-12-2.
[6] Thakare P. S., Sonatakke P. K. & Gudadhe A. S.: "Generalized Offset Fractional Hartley Transform & It's Analyticity Theo em", International Journal of Advances in Science Engineering and Technology, Special Issue-1, June-2015.
[7] Sharma V. D. and Dolas P. D.: "Analytical Structure of Distributional Two Dimensional Fourier-Mellin Transform", International Journal in Physical and Applied Sciences, Vol.04 Issue-06, June 2017.
[8] Haiye Huo: "Uncertainty Principles for the Offset Linear Canonical Transform", Circuits Syst Signal Process, 25 May 2018.
[9] Franck Gérossier, Paul Checchin, Christophe Blanc, Roland Chapuis and Laurent Trassoudaine: "Trajectory-oriented EKF-SLAM using the Fourier-Mellin Transform applied to Microwave Radar Images", The 2009 IEEE/RSJ International Conference on Intelligent Robots and Systems October 11-15, 2009 St. Louis, USA.
[10] Thomas A. Gallagher, Alexander J. Nemeth, Lotfi Hacein-Bey: "An Introduction to the Fourier Transform: Relationship to MRI", American Roentgen Ray Society, 0361–803X,Nov. 2007.
[11] Soo-Chang Pei and Jian-Jiun Ding: "Generalized Eigenvectors and Fractionalization of Offset DFTs and DCTs", IEEE TRANSACTIONS ON SIGNAL PROCESSING, VOL. 52, NO. 7, JULY 2004.
[12] Xiaoxin Guo, Zhiwen Xu, Yinan Lu, Yunjie Pang: "An Application of Fourier-Mellin Transform in Image Registration", Proceedings of the 2005 The Fifth International Conference on Computer and Information Technology (CIT'05), 0-7695-2432-X/05, 2005.

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