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Application of Jordan Standard Form

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***Abstract*—Although a matrix of order n is not necessarily similar to a diagonal matrix, the matrix of order n must be similar to a Jordan standard form which is slightly more complicated in form than diagonal matrix. The theory of Jordan standard form is an extremely important tool in matrices. This paper mainly introduces the applications of Jordan standard form in solving linear differential equations with constant coefficients, calculating matrix polynomials and proving Hamilton-Cayley theorem.**

***Index Terms*—Jordan standard form; matrix; system of linear differential equations; matrix polynomial; characteristic polynomial.**

1 Introduction

Although a matrix of order n is not necessarily similar to a diagonal matrix, the matrix of order n must be similar to a Jordan standard form which is slightly more complicated in form than diagonal matrix. Jordan standard form of matrix is not only the key point but also the difficulty in algebra teaching. It has an inestimable development prospect in other branches of mathematics and disciplines. Therefore, it is of great significance to discuss Jordan standard form for algebra teaching and learning.

The theory of Jordan standard form is an extremely important tool in matrices. This paper mainly introduces the applications of Jordan standard form in solving linear differential equations with constant coefficients, calculating matrix polynomials and proving some problems.

2 Basic theorems and properties of Jordan standard form

Theorem 1. Let A be a linear transformation in the n-dimensional linear space V over the complex field, then there is a set of bases in this linear space, so that the matrix of the linear transformation A under this set of bases is Jordan matrix. This Jordan matrix is uniquely determined by the above linear transformation except for the order of its Jordan blocks. We call it the Jordan standard form of the matrix of linear transformation A.

Theorem 2. Every complex matrix A of order n must be similar to a Jordan matrix, and this Jordan matrix is uniquely determined by matrix A except for the arrangement order of Jordan blocks. We call it Jordan standard form of complex matrix A.

Theorem 3. A complex matrix A is similar to a diagonal matrix if and only if its invariant factors have no multiple roots.

According to the above definition, two properties can be deduced as follows:

Property 1. The Jordan standard form is uniquely determined by the above linear transformation, and all the elements on its principal diagonal are all eigenvalues of the matrix (the multiple roots should be calculated according to the multiplicity). If the Jordan standard form is uniquely determined by the above linear transformation, all the elements on its principal diagonal are all eigenvalues of the matrix (the multiple roots should be calculated according to the multiplicity).

Property 2. The Jordan standard form is uniquely determined by the above linear transformation A, and all elements on its principal diagonal are all roots of the characteristic polynomial of the matrix (the multiple roots should be calculated according to the multiplicity).

3 Application of Jordan standard form

Jordan standard form is closely related to ordinary differential, linear algebra, matrix and so on, which can reduce a lot of calculation. Jordan standard form of matrix is widely used in calculating linear differential equations with constant coefficients, calculating determinant and proving Hamilton-Cayley theorem. Therefore, the calculation method of the standard shape is particularly important.

Next I will focus on the introduction of several commonly used Jordan standard form calculation methods.

3.1 Solving linear differential equations with constant coefficients

The following linear differential equations are given

$$\begin{cases} \frac{dX_1}{dt} = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ \frac{dX_2}{dt} = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \frac{dX_n}{dt} = a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n \end{cases} \quad (1)$$

where $a_{ij}(i = 1,2,\cdots,n)$ are constants。The above equations are written in the form of matrix as follows

$$\frac{dX}{dt} = AX, \quad (2)$$

where

$$A = (a_{ij}) \in C^{n\times n}, X = (x_1(t), x_2(t), \cdots, x_n(t))^T,$$

$$\frac{dX}{dt} = (\frac{dX_1}{dt}, \frac{dX_2}{dt}, \cdots, \frac{dX_n}{dt})^T.$$

Let J be the canonical form of A, then

$$P^{-1}AP = J. \quad (3)$$

Let

$$X = PY, \quad Y = (y_1(t), y_2(t), \cdots, y_n(t))^T, \quad (4)$$

Substituting (4) into (2) gives

$$P\frac{dY}{dt} = APY. \quad (5)$$

Multiply P^{-1} left by both ends of the equation to get

$$\frac{dY}{dt} = P^{-1}APY = JY. \qquad (6)$$

If Y can be obtained by formula (6), then the solution X of the original equation can be obtained by formula (2).

Example 1. Solving linear differential equations

$$\begin{cases} \frac{dX_1}{dt} = -x_1 - 2x_2 + 6x_3, \\ \frac{dX_2}{dt} = -x_1 + 3x_3, \\ \frac{dX_3}{dt} = -x_1 - x_2 + 4x_3. \end{cases}$$

Solution. Note $X = (x_1, x_2, x_3)^T$, $\frac{dX}{dt} = (\frac{dX_1}{dt}, \frac{dX_2}{dt}, \frac{dX_3}{dt})^T$,then

$$\frac{dX}{dt} = AX,$$

where

$$A = \begin{pmatrix} -1 & -2 & 6 \\ -1 & 0 & 3 \\ -1 & -1 & 4 \end{pmatrix}。$$

First find the elementary factor of A:

$$\lambda E - A = \begin{pmatrix} \lambda+1 & 2 & -6 \\ 1 & \lambda & -3 \\ 1 & 1 & \lambda-4 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & \lambda & -3 \\ \lambda+1 & 2 & -6 \\ 1 & 1 & \lambda-4 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & \lambda & -3 \\ 0 & -\lambda^2-\lambda+2 & 3(\lambda-1) \\ 0 & -\lambda+1 & \lambda-1 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & \lambda & 0 \\ 0 & -(\lambda-1)^2 & 0 \\ 0 & 0 & \lambda-1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda-1 & 0 \\ 0 & 0 & (\lambda-1)^2 \end{pmatrix},$$

Therefore, the primary factor of A is $\lambda - 1, (\lambda - 1)^2$.

So there is an invertible matrix $P = (\alpha_1, \alpha_2, \alpha_3)$, which makes

$$P^{-1}AP = J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since $AP = PJ$,so

$$\begin{cases} A\alpha_1 = \alpha_1, \\ A\alpha_2 = \alpha_2, \\ A\alpha_3 = \alpha_3 + \alpha_2. \end{cases}$$

Thus we have

$$\alpha_1 = (-1,1,0)^T, \alpha_2 = (2,1,1)^T, \alpha_3 = (-1,0,0)^T.$$

Therefore,we obtain

$$P = \begin{pmatrix} -1 & 2 & -1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Let $X = PY$,where $Y = (y_1, y_2, y_3)^T$, then

$$APY = AX = \frac{dX}{dt} = p\frac{dY}{dt}.$$

Therefore, we have

$$\frac{dY}{dt} = P^{-1}APY = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix},$$

i.e.,

$$\frac{dy_1}{dt} = y_1,$$

$$\frac{dy_2}{dt} = y_2 + y_3,$$

$$\frac{dy_3}{dt} = y_3.$$

Hence,we have

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$$Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} c_1 e^t \\ (c_2 + c_3)e^t \\ c_3 e^t \end{pmatrix}.$$

Therefore,we get

$$X = PY = \begin{pmatrix} 2c_3 t + 2c_2 - c_1 - c_3 \\ c_1 + c_2 + c_3 t \\ c_2 + c_3 t \end{pmatrix} e^t ,$$

Where c_1, c_2, c_3 are arbitrary constants.

3.2 Computing matrix polynomials

Given the polynomial $f(\lambda) = a_m \lambda^m + a_{m-1}\lambda^{m-1} + \cdots + a_1 \lambda + a_0$ of $A \in C^{n \times n}$ and variable λ, $f(A) = a_m A^m + a_{m-1} A^{m-1} + \cdots + a_1 A + a_0 E$ is called the polynomial of matrix A.

Theorem 4 If

$$A = J_i(\lambda_i) = \begin{pmatrix} \lambda_i & 1 & & & \\ & \lambda_i & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & \lambda_i \end{pmatrix}_{n_i \times n_i} \qquad (7)$$

is n_i-order Jordan block, then

$$f(J_i) = \begin{pmatrix} f(\lambda_i) & f'(\lambda_i) & \frac{1}{2!} f''(\lambda_i) & \cdots & \frac{1}{(n_i - 1)!} f^{(n_i - 1)}(\lambda_i) \\ & f(\lambda_i) & f'(\lambda_i) & \cdots & \frac{1}{(n_i - 2)!} f^{(n_i - 2)}(\lambda_i) \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & f'(\lambda_i) \\ & & & & f(\lambda_i) \end{pmatrix}_{n_i \times n_i}.$$

Theorem 5 If A is a square matrix of order n, let J be the Jordan standard

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form of matrix A, then

$$A = PJP^{-1} = P\begin{pmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_s \end{pmatrix}P^{-1}, \qquad (8)$$

$$f(A) = P\begin{pmatrix} f(J_1) & & & \\ & f(J_2) & & \\ & & \ddots & \\ & & & f(J_s) \end{pmatrix}P^{-1}. \qquad (9)$$

Example 2: Given the polynomial $f(\lambda) = \lambda^4 - 2\lambda^3 + \lambda - 1$ and matrix

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & -1 & 3 \end{pmatrix}$$

calculate $f(A)$.

Solution. Since

$$\lambda E - A = \begin{pmatrix} \lambda-2 & 0 & 0 \\ -1 & \lambda-1 & -1 \\ -1 & 1 & \lambda-3 \end{pmatrix} \to \begin{pmatrix} \lambda-2 & 0 & 0 \\ \lambda-2 & \lambda-1 & -1 \\ 0 & 1 & \lambda-3 \end{pmatrix}$$
$$\to \begin{pmatrix} \lambda-2 & 0 & 0 \\ 0 & \lambda-1 & -1 \\ 0 & 1 & \lambda-3 \end{pmatrix} \to \begin{pmatrix} \lambda-2 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & (\lambda-2)^2 & \lambda-3 \end{pmatrix}$$
$$\to \begin{pmatrix} \lambda-2 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & (\lambda-2)^2 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda-2 & 0 \\ 0 & 0 & (\lambda-2)^2 \end{pmatrix},$$

So the elementary factors of A are $\lambda - 2, (\lambda - 2)^2$, so the Jordan standard form of matrix A is

$$J = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Transformation matrix $P=\begin{pmatrix}0&1&1\\1&0&0\\1&0&-1\end{pmatrix}$,one can obtain $P^{-1}=\begin{pmatrix}0&1&0\\1&-1&1\\0&1&-1\end{pmatrix}$.

Therefore,we have

$$f(A)=Pf(J)P^{-1}=\begin{pmatrix}0&1&1\\1&0&0\\1&0&-1\end{pmatrix}\begin{pmatrix}f(2)&f'(2)&0\\0&f(2)&f'(2)\\0&0&f(2)\end{pmatrix}\begin{pmatrix}0&1&0\\1&-1&1\\0&1&-1\end{pmatrix}$$

$$=\begin{pmatrix}f(2)&0&0\\f'(2)&f(2)-f'(2)&f'(2)\\f'(2)&-f'(2)&f'(2)+f(2)\end{pmatrix}=\begin{pmatrix}1&0&0\\9&-8&9\\9&-9&10\end{pmatrix}.$$

3.3 The application of Jordan standard form in proof problem

In the process of proof, the use of Jordan standard form theory can greatly simplify the proof process and make the analysis more thorough. Here is an example

Example 3. The application of Jordan standard form in proving Hamilton-Cayley theorem

Theorem 6 (Hamilton-Cayley theorem) If $f(\lambda)=|\lambda E-A|$,then $f(A)=0$, where A is $n\times n$ complex matrix.

Proof. If the canonical form of complex matrix A is J, then there exists invertible matrix P, which satisfies

$$P^{-1}AP=J=\begin{pmatrix}\lambda_1&k_1&&&\\&\lambda_2&k_2&&\\&&\lambda_3&\ddots&\\&&&\ddots&k_{n-1}\\&&&&\lambda_n\end{pmatrix},$$

Where $\lambda_1,\lambda_2,\cdots,\lambda_n$ are the eigenvalues of A, which can be the same; k_i is 1 or 0, $i=1,2,\cdots,n-1$,

$$A=PJP^{-1},$$

so $f(\lambda)=|\lambda E-A|=(\lambda-\lambda_1)(\lambda-\lambda_2)\cdots(\lambda-\lambda_n)$.

Therefore,

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$$f(A) = (A - \lambda_1 E)(A - \lambda_2 E)\cdots(A - \lambda_n E)$$
$$= (PJP^{-1} - \lambda_1 E)(PJP^{-1} - \lambda_2 E)\cdots(PJP^{-1} - \lambda_n E)$$
$$= P\left[(J - \lambda_1 E)(J - \lambda_2 E)\cdots(J - \lambda_n E)\right]P^{-1}$$

$$= P\begin{pmatrix} 0 & k_1 & & & \\ & \lambda_2 - \lambda_1 & k_2 & & \\ & & \ddots & \ddots & \\ & & & \ddots & k_{n-1} \\ & & & & \lambda_n - \lambda_1 \end{pmatrix} \cdot \begin{pmatrix} \lambda_2 - \lambda_1 & k_1 & & & \\ & 0 & k_2 & & \\ & & \ddots & & \\ & & & \ddots & k_{n-1} \\ & & & & \lambda_n - \lambda_2 \end{pmatrix} \cdot \begin{pmatrix} \lambda_1 - \lambda_n & k_1 & & & \\ & \lambda_2 - \lambda_n & k_2 & & \\ & & \ddots & & \\ & & & \lambda_{n-1} - \lambda_n \ \ddots & k_{n-1} \\ & & & & 0 \end{pmatrix} P^{-1}$$

$$= P\begin{pmatrix} 0 & 0 & * & \cdots & * \\ 0 & 0 & * & \cdots & * \\ \vdots & \vdots & * & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & & & * \end{pmatrix} \cdot \begin{pmatrix} \lambda_1 - \lambda_3 & k_1 & & & \\ & \lambda_2 - \lambda_3 & k_2 & & \\ & & \ddots & & \\ & & \ddots & & \\ & & & \lambda_{n-1} - \lambda_3 & k_{n-1} \\ & & & & \lambda_n - \lambda_3 \end{pmatrix} \cdot \begin{pmatrix} \lambda_1 - \lambda_n & k_1 & & & \\ & \lambda_2 - \lambda_2 & k_2 & & \\ & & \ddots & \ddots & \\ & & & \lambda_{n-1} - \lambda_n & k_{n-1} \\ & & & & 0 \end{pmatrix} P^{-1}$$

=0.

Example 4. Let $A^k = 0$, wehere k is the natural number, prove $|A + E| = 1$.

Proof. From $A^k = 0$, we know that the eigenvalues of A are all zero. Therefore, in the Jordan standard form of A, all the principal diagonal elements are zero. If the canonical form of A is J, that is $P^{-1}AP = J$ or $A = PJP^{-1}$, then $|A + E| = |PJP^{-1} + E| = |P||J + E||P^{-1}| = 1$.

Example 5. Let A be a matrix wei==ith order n, Prove the necessary and sufficient condition of $A^m = 0$ is that all eigenvalues of A are 0.

Proof. Let λ be the eigenvalue of A. Since $A^m = 0$, So $\lambda^m = 0$, thus $\lambda = 0$.

On the contrary, if $P^{-1}AP = J$ is the Jordan standard form of A, then

$A = PJP^{-1}$,

$$J = \begin{pmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_s \end{pmatrix},$$

$$J_i = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & 0 \end{pmatrix},$$

Assume that J_i is m_i-order Jordan block，then $J_i^{m_i} = 0$. Take $m = \max\{m_i\}$，then

$$J = \begin{pmatrix} J_1^m & & & \\ & J_2^m & & \\ & & \ddots & \\ & & & J_s^m \end{pmatrix} = 0,$$

Therefore $A^m = PJ^mP^{-1} = 0$.

4 Conclusions

Through the above research on the application of Jordan standard form, we can find that if the canonical form can simplify the calculation to a great extent when solving linear differential equations with constant coefficients, calculating matrix polynomials and solving problems related to matrix, we can see the importance of Jordan standard form.

Based on the introduction of Jordan standard form theory, this paper summarizes the applications of Jordan standard form in solving linear differential equations with constant coefficients, computing matrix polynomials and proving Hamilton-Cayley theorem. Through the previous introduction, we can find that if we transform the general matrix into Jordan standard form, we can simplify the problem.

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