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RELATIVISTIC TREATMENT OF D-DIMENSIONAL KLEIN-GORDON EQUATION WITH YUKAWA POTENTIAL

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Abstract

We solved the relativistic Klein-Gordon equation with Yukawa potential via the Nikiforov-Uvarov method. In order to overcome the centrifugal barrier, we employed the Greene and Aldrich approximation scheme. The energy eigenvalues for relativistic and non- relativistic and the corresponding normalized wave function were obtained. A special case of Coulomb potential was obtained. The bound state energy eigenvalues expressions and numerical computations agreed with the already existing literature.

Keywords: Schrödinger equation; Yukawa potential; Nikiforov-Uvarov method; Klein-Gordon equation.

1. Introduction

Solutions of fundamental dynamical equations are of great interest in many fields of physics and chemistry. The exact solutions of the Schrödinger equation (SE) for a hydrogen atom and for a harmonic oscillator represent two typical examples in quantum mechanics [1-3]. The Mie- type and pseudoharmonic potentials are also two exactly solvable potentials [4-5]. The Yukawa potential was proposed by Yukawa in 1935[6] as an effective non- relativistic potential describing the strong interactions between nucleons. It takes the form

$$V(r) = -\frac{A_0 e^{-\alpha r}}{r} \tag{1}$$

And can be seen as a standard version of the Coulomb potential if $\alpha = 0$,with A_0 describing the strength of the interaction and $A_0 = aZ$, $a = (137.037)^{-1}$ is the fine-structure constant and Z is the atomic number and α is the screening parameter. This potential is often used to compute bound-state normalizations and energy levels of natural atoms [7-10] which have been studied over the past years. The Klein-Gordon equation containing a four vector linear momentum operator and a rest mass requires introducing the four vector potential $V(r)$ and a space time scalar potential $S(r)$. With the configuration $S(r) = V(r)$ or $S(r) = -V(r)$, it has been shown extensively in literature that the Klein-Gordon equation share the same energy spectrum [11]. While for $S(r) = V(r) = 2V(r)$, gives non-relativistic limits of the equation conforming exactly to that of the Schrödinger equation[12-15].

Chakrabarti and Das presented a perturbative solution of the Riccati equation leading to an analytic super potential for Yukawa potential [16]. Ikhdair and Sever investigated energy levels of neutral atoms by applying an alternative perturbative scheme in solving the Schrödinger equation for the Yukawa potential model with a modified screening parameter [17]. Ikhdair and Sever also studied bound states of the Hellmann potential, which represents the superposition of the attractive Coulomb potential and the Yukawa potential [18]. Gonul et al.[19] presented a new useful technique for solving the bound state problem for Yukawa-type potentials within the frame of the Riccati equation. Karakoc and Boztosun applied the asymptotic iteration method to solve the radial SE for the Yukawa type potentials [20]. Liverts et al.[21] used the quasi- linearization method (QLM) for solving SE with a Yukawa potential. Hamzavi et al., studied approximate solution of the Yukawa potential with arbitrary angular momenta within the frame of generalized parametric of the Nikiforov- Uvarov method [22].

Apart from that, many authors have solved both relativistic and non-relativistic wave equations with different potentials. For instance,

Edet et al.[23] obtained approximate solutions of the Schrödinger equation with the Generalized Morse potential model including a centrifugal term. Nikiforov and Uvarov obtained relativistic and non-relativistic solutions of the inversely quadratic Yukawa potential [24]. Hamzavi, et al.[25] obtained bound state solutions of the SE with Manning-Rosen potential. William et al.[26] obtained bound state solutions of the radial Schrödinger equation by the combination of Hulthén and Hellmann potentials within the framework of Nikiforov-Uvarov (NU) method for any arbitrary ℓ - state, with the Greene-Aldrish approximation in the centrifugal term. Edet et al. [27], obtained bound state solutions of the SE for the modified Kratzer potential plus screened Coulomb potential. Edet et al,[28] obtained any l-state solutions of the SE interacting with Hellmann-Kratzer potential model. The Yukawa potential is greatly important with applications, cutting across nuclear physics and condensed matter physics [29]. In this present paper, we are motivated by the current trend in the study of bound state problems, to investigate the bound state solutions of the Klein-Gordon equation with Yukawa potential which to the best of our knowledge is not in literature.

Therefore, to obtain approximate solutions, we employ a suitable approximation scheme. It is found that such approximation proposed by Greene and Aldrich [30]

$$\frac{1}{r^2} \approx \frac{\alpha^2}{\left(1-e^{-\alpha r}\right)^2} \tag{2}$$

Is a good approximation to the centrifugal or inverse square term which is valid for α <<1 for a short range potential. The paper is organized as follows: In section 2, the Nikiforov-Uvarov (NU) method is reviewed, in section 3 the bound state energy eigenvalues and corresponding normalized wave function are calculated ,in section 4 the results are discussed. In section 5, conclusion is presented.

2. Review of Nikiforov-Uvarov(NU) method

The NU method was proposed by Nikiforov and Uvarov to transform Schrödinger-like equations into a second-order differential equation via a coordinate transformation $s = s(r)$, of the form [31].

$$\psi''(s)+\frac{\tilde{\tau}(s)}{\sigma(s)}\psi'(s)+\frac{\tilde{\sigma}(s)}{\sigma^2(s)}\psi(s)=0 \tag{3}$$

where $\tilde{\sigma}$(s), and $\sigma(s)$ are polynomials, at most second degree, and $\tilde{\tau}(s)$ is a first-degree polynomial. The exact solution of Eq. (3) can be obtained by using the transformation.

$$\psi(s)=\phi(s)y(s) \tag{4}$$

This transformation reduces Eq.(3) into a hypergeometric-type equation of the form

$$\sigma(s)y''(s)+\tau(s)y'(s)+\lambda y(s)=0 \tag{5}$$

The function $\Phi(s)$ can be defined as the logarithm derivative

$$\frac{\phi'(s)}{\phi(s)}=\frac{\pi(s)}{\sigma(s)} \tag{6}$$

with $\pi(s)$ being at most a first-degree polynomial. The second part of $\psi(s)$ being $y(s)$ in Eq.(5) is the hypergeometric function with its polynomial solution given by Rodrigues relation as

$$y(s)=\frac{B_{nl}(s)}{\rho(s)}\frac{d^n}{ds^n}\left[\sigma^n(s)\rho(s)\right] \tag{7}$$

where B_n is the normalization constant and $\rho(s)$ the weight function which satisfies the condition below;

$$\left(\sigma(s)\rho(s)\right)'=\tau(s)\rho(s) \tag{8}$$

where also

$$\tau(s)=\tilde{\tau}(s)+2\pi(s) \tag{9}$$

For bound solutions, it is required that

$$\frac{d\tau(s)}{ds}<0 \tag{10}$$

The eigenfunction and eigenvalues can be obtained using the definition of the following function $\pi(s)$ and parameter λ, respectively:

$$\pi(s)=\frac{\sigma'(s)-\tilde{\tau}(s)}{2}\pm\sqrt{\left(\frac{\sigma'(s)-\tilde{\tau}(s)}{2}\right)^2-\tilde{\sigma}(s)+k\sigma(s)} \tag{11}$$

and

$$\lambda=k_-+\pi_-'(s) \tag{12}$$

The value of k can be obtained by setting the discriminant in the square root in Eq. (11) equal to zero. As such, the new eigenvalues equation can be given as

$$\lambda+n\tau'(s)+\frac{n(n-1)}{2}\sigma''(s)=0,(n=0,1,2,...) \tag{13}$$

3.0 Bound state solutions of the Klein-Gordon equation with Yukawa potential

The Kelein-Gordon equation for a spinless particle for $\hbar = c = 1$ in D-dimensions is given as [32]

$$\left[-\nabla^2 + \left(M + S(r)\right)^2 + \frac{(D+2l-1)(D+2l-3)}{4r^2}\right]\psi(r,\theta,\varphi) = \left[E - V(r)\right]^2 \psi(r,\theta,\varphi) \tag{14}$$

where ∇^2 is the Laplacian, M is the reduced mass, E is the energy spectrum and n and l are the radial and orbital angular momentum quantum numbers respectively or vibration-rotation quantum number in quantum chemistry. It is a common practice that for the wavefunction to satisfy the boundary conditions it can be rewritten as

$$\psi(r,\theta,\varphi) = \frac{R_{nl}}{r} Y_{lm}(\theta,\varphi) \tag{15}$$

The angular component of the wavefunction could be separated leaving only the radial part as shown below

$$\frac{d^2R}{dr^2} + \left[\left(E^2 - M^2\right) + V^2(r) - S^2(r) - 2\left(EV(r) + MS(r)\right) - \frac{(D+2l-1)(D+2l-3)}{4r^2}\right]R(r) = 0 \tag{16}$$

Thus, for equal vector and scalar potentials $V(r) = S(r) = 2V(r)$ then Eq.(16) becomes

$$\frac{d^2R}{dr^2} + \left[\left(E^2 - M^2\right) - 2V(r)\left(E + M\right) - \frac{(D+2l-1)(D+2l-3)}{4r^2}\right]R(r) = 0 \tag{17}$$

We substitute Eq.(1) into Eq.(17) and obtain

$$\frac{d^2R}{dr^2} + \left[\left(E^2 - M^2\right) + \frac{2A_0 e^{-\alpha r}}{r}\left(E_{nl} + M\right) - \frac{(D+2l-1)(D+2l-3)}{4r^2}\right]R(r) = 0 \tag{18}$$

We transform the coordinate of Eq.(18) by setting

$$s = e^{-\alpha r} \tag{19}$$

Differentiating Eq.(19) and simplifying we obtain

$$\frac{d^2R}{dr^2} = \alpha^2 s^2 \frac{d^2R}{ds^2} + \alpha^2 s \frac{dR}{ds} \tag{20}$$

Substituting Eqs. (2),(19) and (20) into Eq.(18) we obtain

$$\frac{d^2R(s)}{ds^2} + \frac{(1-s)}{s(1-s)}\frac{dR(s)}{ds} + \frac{1}{s^2(1-s)^2}\left[-\varepsilon(1-s)^2 + \beta\left(s - s^2\right) - \gamma\right]R(s) = 0, \tag{21}$$

where

$$\left.\begin{aligned} -\varepsilon &= \frac{E_{nl}^2 - M^2}{\alpha^2} \\ \beta &= \frac{2A_0\left(E_{nl} + M\right)}{\alpha} \\ \gamma &= \frac{\left(D+2l-1\right)\left(D+2l-3\right)}{4} \end{aligned}\right\} \tag{22}$$

Expanding the square bracket of Eq.(21) we obtain

$$\frac{d^2R(s)}{ds^2} + \frac{(1-s)}{s(1-s)}\frac{dR(s)}{ds} + \frac{1}{s^2(1-s)^2}\left[-(\varepsilon+\beta)s^2 + (2\varepsilon+\beta)s - (\varepsilon+\gamma)\right]R(s) = 0 \tag{23}$$

Comparing Eq.(23) with Eq.(3) we obtain the following parameters

$$\left.\begin{aligned} \tilde{\tau}(s) &= 1-s \\ \sigma(s) &= s(1-s) \\ \sigma'(s) &= 1-2s \\ \tilde{\sigma}(s) &= -(\varepsilon+\beta)s^2 + (2\varepsilon+\beta)s - (\varepsilon+\gamma) \end{aligned}\right\} \tag{24}$$

Substituting Eq.(24) into Eq.(11) we have

$$\pi(s) = -\frac{s}{2} \pm \sqrt{(A-k)s^2 + (B+k)s + C} \tag{25}$$

where

$$\left. A = \frac{1}{4} + \varepsilon + \beta,\; B = -(2\varepsilon+\beta),\; C = \varepsilon + \gamma \right\} \tag{26}$$

To find the constant, k , the discriminant of the expression under the square root of Eq.(25) must be equal to zero. As such, we have that

$$k = -(B+2C) - 2\sqrt{C}\sqrt{C+B+A} \tag{27}$$

Substituting Eq.(26) into Eq.(27) we have

$$k_{-} = \beta - 2\gamma - 2\sqrt{\varepsilon+\gamma}\sqrt{\frac{1}{4}+\gamma} \tag{28}$$

Substituting Eq. (27) into Eq. (25) we have

$$\pi(s) = -\frac{s}{2} \pm \left[\left(\sqrt{C} + \sqrt{C+B+A}\right)s - \sqrt{C}\right] \tag{29}$$

Substituting Eq.(26) into Eq.(29) we have

$$\pi_{-}(s) = -\frac{s}{2} - \left[\left(\sqrt{\varepsilon+\gamma} + \sqrt{\frac{1}{4}+\gamma}\right)s - \sqrt{\varepsilon+\gamma}\right] \tag{30}$$

Differentiating Eq.(30) we have

$$\pi_{-}'(s) = -\frac{1}{2} - \left(\sqrt{\varepsilon+\gamma} + \sqrt{\frac{1}{4}+\gamma}\right) \tag{31}$$

By substituting Eqs. (28) and (31) into Eq.(12) we have

$$\lambda = \beta - 2\gamma - 2\sqrt{\varepsilon+\gamma}\sqrt{\frac{1}{4}+\gamma} - \frac{1}{2} - \left(\sqrt{\varepsilon+\gamma} + \sqrt{\frac{1}{4}+\gamma}\right) \tag{32}$$

With $\tau(s)$ being obtained from Eq.(9) as

$$\tau(s) = 1 - 2s + 2\sqrt{\varepsilon+\gamma}\,s - 2\sqrt{\frac{1}{4}+\gamma}\,s + 2\sqrt{\varepsilon+\gamma} \tag{33}$$

Differentiating Eq.(33) yields

$$\tau'(s) = -2 - 2\left(\sqrt{\varepsilon+\gamma} + \sqrt{\frac{1}{4}+\gamma}\right) \tag{34}$$

And also taking the second derivative of $\sigma'(s)$ with respect to s from Eq.(24),we have

$$\sigma''(s) = -2 \tag{35}$$

Substituting Eqs.(34) and (35) into Eq.(13) and simplifying, yields

$$\lambda_n = n^2 + n + 2n\sqrt{\varepsilon+\gamma} + 2n\sqrt{\frac{1}{4}+\gamma} \tag{36}$$

Equating Eqs.(32) and (36) and substituting Eq.(22) yields the energy eigenvalue equation of the Yukawa potential in the relativistic limit as

$$M^2 - E^2 = -\alpha^2\left(\frac{(D+2l-1)(D+2l-3)}{4}\right) + \frac{\alpha^2}{4}\left[\frac{\left(n+\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{(D+2l-1)(D+2l-3)}{4}}\right)^2 - \frac{2A_0(E_{nl}+M)}{\alpha}+\frac{(D+2l-1)(D+2l-3)}{4}}{n+\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{(D+2l-1)(D+2l-3)}{4}}}\right]^2 \quad (37)$$

3.1 Non-relativistic limit

In this section, we consider the non-relativistic limit of Eq.(37). Considering a transformation of the form: $M+E \rightarrow \frac{2\mu}{\hbar^2}$ and $M-E \rightarrow -E$ and substitute it into Eq.(37), we have the non relativistic energy eigenvalue equation as

$$E_{nl} = \frac{\hbar^2\alpha^2}{2\mu}\left(\frac{(D+2l-1)(D+2l-3)}{4}\right) - \frac{\hbar^2\alpha^2}{8\mu}\left[\frac{\left(n+\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{(D+2l-1)(D+2l-3)}{4}}\right)^2 - \frac{4\mu A_0}{\alpha\hbar^2}+\frac{(D+2l-1)(D+2l-3)}{4}}{n+\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{(D+2l-1)(D+2l-3)}{4}}}\right]^2 \quad (38)$$

We take a special case of Eq.(1) by setting $\alpha = 0$ and $D = 3$ to obtain the energy eigenvalues of Coulomb potential as

$$E_{nl} = -\frac{\mu A_0^2}{2\hbar^2(n+l+1)^2} \quad (38a)$$

To obtain the corresponding wavefunction, we consider Eq. (6) and upon substituting Eqs. (24) and (33) and integrating, we get

$$\phi(s) = s^{\sqrt{\varepsilon+\gamma}}(1-s)^{\frac{1}{2}+\sqrt{\frac{1}{4}+\gamma}} \quad (39)$$

To get the hypergeometric function considering Eq.(4) ,we first determine the weight function

of Eq.(8) , upon differentiating the left hand side we obtain

$$\frac{\rho'(s)}{\rho} = \frac{\tau(s)-\sigma'(s)}{\sigma(s)} \quad (40)$$

Substituting Eqs.(24) and (33) into Eq.(40) and integrating ,thereafter simplify we obtain

$$\rho(s) = s^{2\sqrt{\varepsilon+\gamma}} (1-s)^{2\sqrt{\frac{1}{4}+\gamma}} \tag{41}$$

By substituting Eqs.(24) and (41) into Eq.(7) we obtain the Rodrigue's equation as

$$y_n(s) = B_{nl} s^{-2\sqrt{\varepsilon+\gamma}} (1-s)^{-2\sqrt{\frac{1}{4}+\gamma}} \frac{d^n}{ds^n}\left[s^{n+2\sqrt{\varepsilon+\gamma}} (1-s)^{n+2\sqrt{\frac{1}{4}+\gamma}} \right] \tag{42}$$

where B_{nl}= normalization constant.

Equation (42) is a equivalent to

$$P_n^{\left(2\sqrt{\varepsilon+\gamma},2\sqrt{\frac{1}{4}+\gamma}\right)}(1-2s) \tag{43}$$

where P_n= Jacobi Polynomial

The wave function is given by

$$\psi_{nl}(s) = B_{nl} s^{\sqrt{\varepsilon+\gamma}} (1-s)^{\frac{1}{2}+\sqrt{\frac{1}{4}+\gamma}} P_n^{\left(2\sqrt{\varepsilon+\gamma},2\sqrt{\frac{1}{4}+\gamma}\right)}(1-2s) \tag{44}$$

Using the normalization condition, we obtain the normalization constant as follows

$$\int_0^\infty |\psi_{nl}(r)|^2 dr = 1 \tag{45}$$

By differentiating Eq.(19) with respect to r we have

$$\frac{ds}{dr} = -\alpha e^{-\alpha r} \tag{46}$$

Simplifying Eq.(46) and substituting Eq.(19) ,we obtain

$$dr = -\frac{ds}{\alpha s} \tag{47}$$

Substituting Eq.(47) into Eq.(45) and changing the limit we have,

$$\frac{1}{\alpha}\int_0^1 |\psi_{nl}(s)|^2 \frac{ds}{s} = 1 \tag{48}$$

Let

$$y = 1-2s \tag{49}$$

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We differentiate Eq.(49) and simplify to obtain

$$ds = -\frac{dy}{2} \tag{50}$$

By simplifying Eq. (49) we have,

$$\left. s = \frac{1-y}{2},\ \frac{1}{s} = \frac{2}{1-y} \right\} \tag{51}$$

Substituting Eqs.(50) and (51) into Eq.(48) and changing the limit we obtain,

$$\frac{1}{\alpha}\int_{-1}^{1}\left|\psi_{nl}(y)\right|^2 \frac{dy}{1-y} = 1 \tag{52}$$

By substituting Eqs.(44) and (49) into Eq.(52) and with simple algebra we obtain

$$\frac{B_{nl}^2}{\alpha}\int_{-1}^{1}\left(\frac{1-y}{2}\right)^{2\sqrt{\varepsilon+\gamma}}\left(\frac{1+y}{2}\right)^{1+2\sqrt{\frac{1}{4}+\gamma}}\left[P_n^{\left(2\sqrt{\varepsilon+\gamma},2\sqrt{\frac{1}{4}+\gamma}\right)}y\right]^2 dy = 1 \tag{53}$$

Let

$$\left. \mu = 1+2\sqrt{\frac{1}{4}+\gamma},\ \mu-1 = 2\sqrt{\frac{1}{4}+\gamma},\ u = 2\sqrt{\varepsilon+\gamma} \right\} \tag{54}$$

Substituting Eq.(54) into Eq.(53) we have

$$\frac{B_{nl}^2}{m_D(T)}\int_{-1}^{1}\left(\frac{1-y}{2}\right)^{u}\left(\frac{1+y}{2}\right)^{\mu}\left[P_n^{(2u,\mu-1)}y\right]^2 dy = 1 \tag{55}$$

According to Onate et al.,[33] , integral of the form in Eq,(55) can be expresses as

$$\int_{-1}^{1}\left(\frac{1-p}{2}\right)^{x}\left(\frac{1+p}{2}\right)^{y}\left[P_n^{(2x,2y-1)}p\right]^2 dp = \frac{2\Gamma(x+n+1)\Gamma(y+n+1)}{n!x\Gamma(x+y+n+1)} \tag{56}$$

Hence, comparing Eq.(55) with the standard integral of Eq.(56), we obtain the normalization constant as

$$B_{nl} = \sqrt{\frac{n!u\alpha\Gamma(\mathrm{u}+\mu+\mathrm{n}+1)}{2\Gamma(u+n+1)\Gamma(\mu+n+1)}} \tag{57}$$

4 Results and discussion

TABLE1: The bound state energy eigenvalues (in fm^{-1}) of Yukawa potential in units $\hbar = \mu = 1, D = 3.$ Where $A_0 = \sqrt{2}$ and $\alpha = dA_0$ for comparison with other methods

State	d	Present work	AIM[20]	SUSYQM[26]
1S	0.002	-0.99600	-0.99600	-0.99601
	0.005	-0.99002	-0.99003	-0.99004
	0.010	-0.98014	-0.98014	-0.98015
	0.025	-0.95062	-0.95062	-0.95092
	0.050	-0.90363	-0.90363	-0.90363
2S	0.002	-0.24601	-0.24602	-0.24602
	0.005	-0.24014	-0.24014	-0.24015
	0.010	-0.23049	-0.23058	-0.23059
	0.025	-0.20355	-0.20355	-0.20355
	0.050	-0.16345	-0.16354	-0.16351
2P	0.002	-0.24601	-0.24601	-0.24602
	0.005	-0.24010	-0.24012	-0.24012
	0.010	-0.23048	-0.23049	-0.23049
	0.025	-0.20298	-0.20298	-0.20299
	0.050	-0.16115	-0.16148	-0.16114
3P	0.002	-0.10714	-0.10716	-0.10716
	0.005	-0.10143	-0.10141	-0.10142
	0.010	-0.09231	-0.09230	-0.09231
	0.025	-0.06814	-0.06815	-0.06814
	0.050	-0.03721	-0.03711	-0.03739
3D	0.002	-0.10714	-0.10715	-0.10715
	0.005	-0.10133	-0.10136	-0.10134
	0.010	-0.09201	-0.09212	-0.09202
	0.025	-0.06673	-0.06714	-0.06713
	0.050	-0.03361	-0.03383	-0.03388

4.1 Discussion

We computed the bound state energy eigenvalues of the Yukawa potential using Eq.(38). We note that the energy increases as quantum number increases. The results showed good agreement with the earlier results of Ref.[20] with AIM, and SUSYQM of Ref.[26] .We have plotted the energy eigenvalues with the screening parameter, potential strength and quantum number as shown in Figs.2-4, for various values of quantum number. They plots show an increase in energy eigenvalues as the quantum number increases.

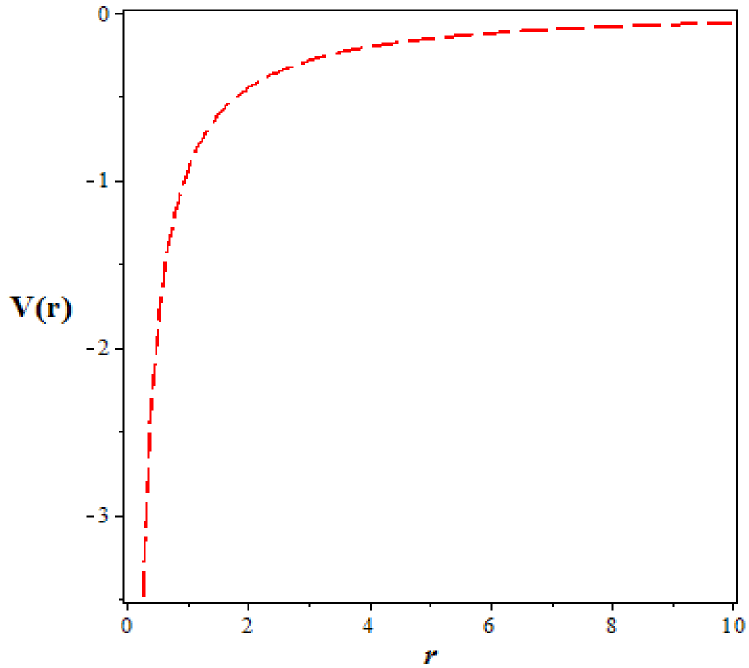

Figure 1: Plots of Yukawa potential with r in (fm^{-1})

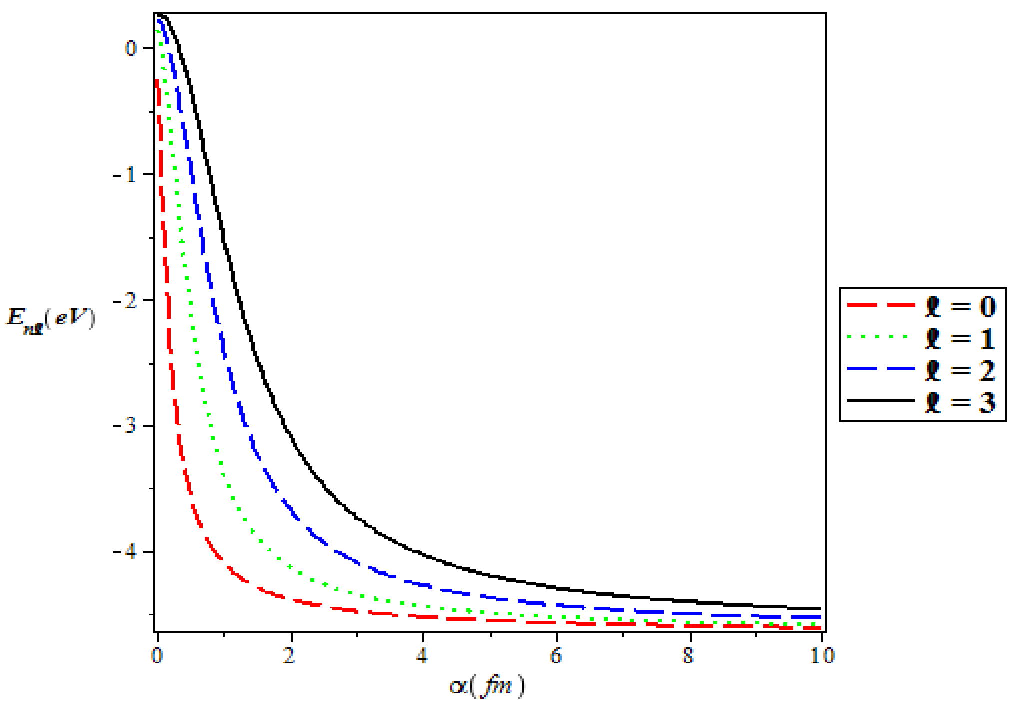

Figure2: Energy eigenvalues variation with screening parameter for various vibrational quantum numbers

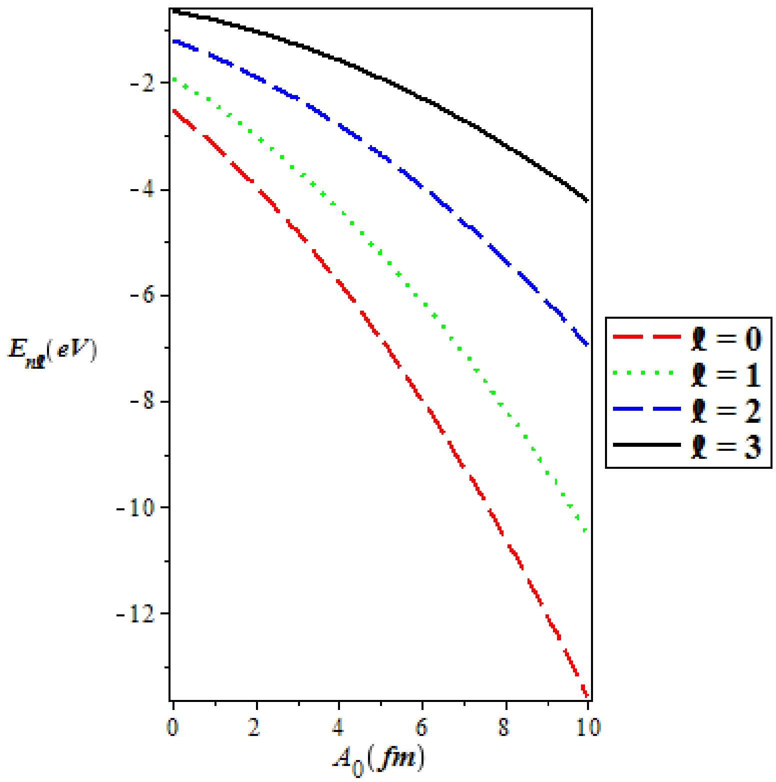

Figure3: Energy eigenvalues variation with potential parameter A_0 for various vibrational quantum numbers

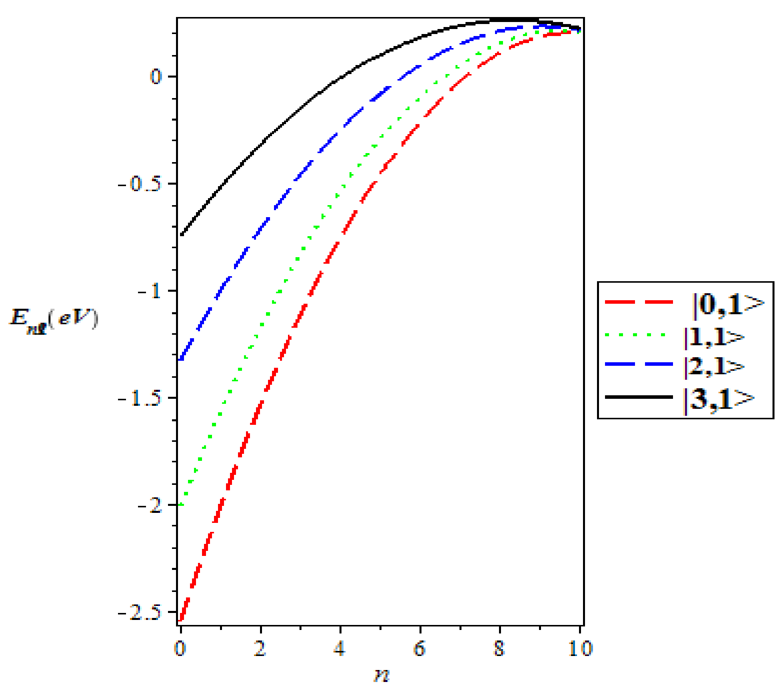

Figure 4: Energy eigenvalues variation with quantum number for various vibrational quantum number

5 Conclusion

In this work, we have obtained the bound state solutions of the Klein-Gordon equation for the Yukawa potential using the Nikiforov-Uvarov method. The energy eigenvalues are obtained both in the relativistic and non-relativistic regime and corresponding normalized eigenfunction. We obtain a special case of Coulomb potential which is in agreement with Ref. [27] and Ref. [28] when $D = 3$. In table 1, we have numerically obtain the energy eigenvalues for this potential and comparison were made with other works has proven the success of the formalism. The variation in the energy eigenvalues with screening parameter, potential strength and quantum number for various values of quantum number was also plotted. It was observed that the energy eigenvalues increases as the parameter increases.

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