

# Maximal Strong Independent Functions

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## Abstract

A subset  $S$  of the vertex set  $V$  of a graph  $G$  is said to be independent, if no two vertices of  $S$  are adjacent. Independent functions and maximal independent functions have been defined and studied already. In this paper, strong independent functions, maximal strong independent functions and basic maximal strong independent functions are defined and a study of these is made.

**Keywords:** Independent Function, Strong Independent Function, Maximal Strong Independent Functions

**Introduction:** A function  $f : V(G) \rightarrow [0, 1]$  is called a strong independent function, if the value of the closed strong neighbourhood of any vertex under  $f$  is 1, if the vertex gets positive value under  $f$ . A function  $f$  is called a maximal strong independent function, if it is a strong independent function and the value of the function on the closed strong neighbourhood of a vertex which gets zero value under  $f$  is greater than or equal to one. A detailed study of these functions is made in the following.

*Definition 0.1:*

Let  $G = (V, E)$  be a simple graph. A function  $f : V(G) \rightarrow [0, 1]$  is called an independent function if for every vertex  $v$  with  $f(v) > 0$ ,  $\sum_{u \in N[v]} f(u) = 1$ .

*Definition 0.2:*

Let  $G = (V, E)$  be a simple graph. An independent function  $f : V(G) \rightarrow [0, 1]$  is called a maximal independent function if for any  $v \in V$  with  $f(v) = 0$ ,

$$\sum_{u \in N[v]} f(u) \geq 1.$$

*Definition 0.3:* Let  $G = (V, E)$  be a simple graph. A function  $f : V(G) \rightarrow [0, 1]$  is called a maximal independent function if  $f$  is an independent function and for any independent function  $g$ ,  $f \leq g \Rightarrow f = g$ .

## I. MAXIMAL STRONG INDEPENDENT FUNCTIONS

*Definition 1.1:* A function  $f : V(G) \rightarrow [0, 1]$  is called a **Strong Independent Function (SIF)** if for any  $u \in V(G)$ ,  $f(u) > 0 \Rightarrow f(N_s[u]) = 1$ , where  $N_s[u] = \{x \in N[u] : \deg x \geq \deg u\}$ .

*Definition 1.2:* A function  $f : V(G) \rightarrow [0, 1]$  is called a **Maximal Strong Independent Function (MSIF)** if for any  $u \in V(G)$ ,

$$f(u) > 0 \Rightarrow f(N_s[u]) = 1 \text{ and}$$

$$f(u) = 0 \Rightarrow f(N_s[u]) \geq 1.$$

*Definition 1.3:*  $P_f = \{v \in V(G) : f(v) > 0\}$  and

$$B_f^s = \{v \in V(G) : f(N_s[v]) = 1\}.$$

*Theorem 1.1:* A function  $f : V(G) \rightarrow [0, 1]$  is a strong independent function if and only if  $P_f \subseteq B_f^s$ .

**Proof:**

Let  $f : V(G) \rightarrow [0, 1]$  be a strong independent function.

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Let  $u \in P_f$ . Then  $f(u) > 0$ .

Since  $f$  is a strong independent function,  $f(N_s[u]) = 1$ .

Therefore,  $u \in B_f^s$ .

Conversely, let  $P_f \subseteq B_f^s$ .

Let  $f(u) > 0$ . Then  $u \in P_f$ .

Therefore,  $u \in B_f^s$ .

Therefore,  $f(N_s[u]) = 1$ .

Therefore,  $f$  is a strong independent function.

*Theorem 1.2:* Every MSIF is a minimal strong dominating function.

**Proof:**

Let  $f : V(G) \rightarrow [0, 1]$  be a MSIF.

Let  $u \in V(G)$ . Then  $f(N_s[u]) \geq 1$ .

Therefore,  $f$  is a strong dominating function.

Let  $g : V(G) \rightarrow [0, 1]$  be a strong dominating function such that  $g \leq f$ .

Suppose there exists  $u \in V(G)$  such that  $g(u) < f(u)$ .

Therefore,  $f(u) > 0$ .

Therefore,  $f(N_s[u]) \geq 1$ .

Since  $g \leq f$ ,  $g(N_s[u]) < 1$ .

Therefore,  $g$  is not a strong dominating function, a contradiction

Therefore,  $g = f$ .

Therefore,  $f$  is a minimal strong dominating function.

*Remark 1.1:* If  $f$  is a maximal strong independent function of  $G$ , then  $B_f^s$  is a strong dominating set of  $G$ .

**Proof:**

Let  $u \in V(G) - B_f^s$ .

Since  $P_f \subseteq B_f^s$ ,  $u \notin P_f$ .

Therefore,  $f(u) = 0$ .

Therefore,  $f(N_s[u]) \geq 1$ .

Therefore, there exists  $v \in N_s(u)$  such that  $f(v) > 0$ .

Therefore,  $v \in P_f$ . Therefore,  $v \in B_f^s$ .

Therefore,  $B_f^s$  strongly dominates  $u$ .

Therefore,  $B_f^s$  is a strong dominating set of  $G$ .

*Remark 1.2:* The convex combination of two strong independent functions need not be a strong independent function.

For example,

let  $G = P_3$ .

Let  $V(P_3) = \{u_1, u_2, u_3\}$  where  $u_1$  and  $u_3$  are pendant vertices.

Let  $f_1, f_2, f_3 : V(G) \rightarrow [0, 1]$  be defined as follows:

$$f_1(u_1) = 0, f_1(u_2) = 1, f_1(u_3) = 0$$

$$f_2(u_1) = f_2(u_3) = 1, f_2(u_2) = 0$$

$$f_3(u_1) = f_3(u_2) = f_3(u_3) = \frac{1}{2}.$$

$$P_{f_1} = \{u_2\},$$

$$B_{f_1} = \{u_1, u_2, u_3\},$$

$$P_{f_2} = \{u_1, u_3\},$$

$$B_{f_2} = \{u_1, u_3\},$$

$$P_{f_3} = \{u_1, u_2, u_3\},$$

$$B_{f_3} = \{u_1, u_3\}.$$

Since  $P_{f_1} \subseteq B_{f_1}$ ,  $P_{f_2} \subseteq B_{f_2}$ ,  $f_1$  and  $f_2$  are strong independent functions.

Here,  $f_3$  is not a strong independent function, since  $P_{f_3} \not\subseteq B_{f_3}$ .

Clearly,  $f_3 = \frac{1}{2}f_1 + \frac{1}{2}f_2$ .

$f_3$  is a convex combination of strong independent functions  $f_1$  and  $f_2$  and  $f_3$  is not a strong independent function.

*Theorem 1.3:* Let  $f_1$  and  $f_2$  be two strong independent functions of  $G$ .

Let  $0 < \lambda < 1$ .

Then  $h_\lambda = \lambda f_1 + (1 - \lambda)f_2$  is a strong independent function if and only if  $P_{f_1} \cup P_{f_2} \subseteq B_{f_1}^s \cap B_{f_2}^s$ .

**Proof:**

Let  $f_1$  and  $f_2$  be strong independent functions of  $G$ . Let  $0 < \lambda < 1$ .

Let  $h_\lambda = \lambda f_1 + (1 - \lambda)f_2$ .

Suppose  $h_\lambda$  is a strong independent function.

Then  $P_{h_\lambda} \subseteq B_{h_\lambda}^s$ .

Let  $u \in P_{f_1} \cup P_{f_2}$ .

Therefore,  $u \in P_{f_1}$  or  $u \in P_{f_2}$ .

If  $u \in P_{f_1}$ , then  $u \in B_{f_1}^s$  (since  $f_1$  is a strong independent function of  $G$ ).

Suppose  $u \in P_{f_2}$ .

Then  $u \in B_{f_2}^s$  (since  $f_2$  is a strong independent function of  $G$ ).

Therefore,  $u \in B_{f_1}^s \cap B_{f_2}^s$ .

Suppose  $u \notin P_{f_2}$ .

Therefore,  $f_2(u) = 0$ .

$$P_{h_\lambda}(u) = \lambda f_1(u) + (1 - \lambda)f_2(u)$$

$$= \lambda f_1(u)$$

Since  $P_{h_\lambda}(u) \subseteq B_{h_\lambda}^s(u)$ ,  $h_\lambda(N[u]) = 1$

That is,  $(\lambda f_1 + (1 - \lambda)f_2)(N[u]) = 1$

$$\lambda f_1(N[u]) + (1 - \lambda)f_2(N[u]) = 1$$

$$\lambda + (1 - \lambda)f_2(N[u]) = 1$$

Therefore,  $(1 - \lambda)f_2(N[u]) = 1 - \lambda$

$$f_2(N[u]) = 1$$

Therefore,  $u \in B_{f_2}^s$ .

Hence  $P_{f_1} \cup P_{f_2} \subseteq B_{f_1}^s \cap B_{f_2}^s$ .

Conversely, let  $P_{f_1} \cup P_{f_2} \subseteq B_{f_1}^s \cap B_{f_2}^s$ .

Let  $u \in P_{h_\lambda}$ . Therefore  $h_\lambda(u) > 0$ .

If  $u \in P_{f_1}$  and  $u \in P_{f_2}$ , then  $u \in B_{f_1}^s \cap B_{f_2}^s$ .

$$f_1(N[u]) = 1, f_2(N[u]) = 1.$$

Therefore,  $\lambda f_1(N[u]) + (1 - \lambda)f_2(N[u])$

$$= \lambda + 1 - \lambda = 1.$$

Therefore,  $u \in B_{h_\lambda}^s$ .

Since  $h_\lambda(u) > 0$ ,  $\lambda f_1(u) + (1 - \lambda)f_2(u) > 0$ .

At least one of  $f_1(u)$ ,  $f_2(u)$  is  $> 0$ .

Suppose  $u \in P_{f_1}$  and  $u \notin P_{f_2}$ .

(similar proof for  $u \notin P_{f_1}$  and  $u \in P_{f_2}$ ).

Then  $f_1(N[u]) = 1$ .

$$\lambda f_1(N[u]) + (1 - \lambda)f_2(N[u]) = \lambda + (1 - \lambda)f_2(N[u]) \quad \longrightarrow \quad (i)$$

Since  $u \in P_{f_1} \cup P_{f_2}$ ,  $u \in B_{f_1}^s \cap B_{f_2}^s$ .

Therefore,  $u \in B_{f_2}^s$ .  $f_2(N[u]) = 1$ .

Therefore, (i) gives  $h_\lambda(N[u]) = \lambda + 1 - \lambda = 1$ .

Therefore,  $u \in B_{h_\lambda}^s$ .

That is,  $P_{h_\lambda} \subseteq B_{h_\lambda}^s$ .

Hence  $h_\lambda$  is a strong independent function.

*Remark 1.3:* Let  $f$  and  $g$  be two strong independent functions. If  $\lambda f + (1 - \lambda)g$  is a strong independent function for some  $\lambda$ ,  $0 < \lambda < 1$ , then any convex combination of  $f$  and  $g$  is strong independent.

For, since  $\lambda f + (1 - \lambda)g$  is a strong independent function,  $P_f \cup P_g \subseteq B_f^s \cap B_g^s$ . Since this is independent of  $\lambda$ , any convex combination of  $f$  and  $g$  is also strong independent.

*Remark 1.4:* If  $f$  and  $g$  are strong independent functions, then either no convex combination of  $f$  and  $g$  is strong independent or every convex combination of  $f$  and  $g$  is strong independent.

*Theorem 1.4:* Let  $f$  and  $g$  be two maximal strong independent functions. Then either all convex combinations of  $f$  and  $g$  are maximal strong independent functions or no one of them is a maximal strong independent function.

**Proof:**

Let  $f$  and  $g$  be two maximal strong independent functions.

Let  $0 < \lambda < 1$ .

Let  $h_\lambda = \lambda f + (1 - \lambda)g$ . Then  $h_\lambda$  is strong independent if and only if  $P_f \cup P_g \subseteq B_f^s \cap B_g^s$ .

Let  $u \in V(G)$ . Suppose  $h_\lambda(u) = 0$ .

Therefore,  $\lambda f(u) + (1 - \lambda)g(u) = 0$ .

Therefore,  $f(u) = 0$  if and only if  $g(u) = 0$ .

Since at least one of  $f(u)$  or  $g(u)$  is zero, we get that both  $f(u)$  and  $g(u)$  are equal to zero.

Therefore,  $f(N[u]) \geq 1$  and  $g(N[u]) \geq 1$  (since  $f$  and  $g$  are maximal strong independent functions).

$$\begin{aligned} \text{Therefore, } \lambda f(N[u]) + (1 - \lambda)g(N[u]) &\geq \lambda + (1 - \lambda) \\ &= 1. \end{aligned}$$

Therefore,  $h_\lambda(N[u]) \geq 1$ .

Therefore,  $h_\lambda$  is a maximal strong independent function if and only if

$$P_f \cup P_g \subseteq B_f^s \cap B_g^s.$$

Hence the theorem.

*Remark 1.5:* Let  $f$  and  $g$  be two MSIF<sup>s</sup>. If  $h_\lambda = \lambda f + (1 - \lambda)g$  is strong independent, then  $h_\lambda$  is a MSIF.

*Definition 1.4:* Let  $f$  be a MSIF.  $f$  is said to be **Universal Maximal Strong Independent Function (UMSIF)**, if the convex combinaiton of  $f$  with any other maximal strong independent function is a MSIF.

*Remark 1.6:* A MSIF  $f$  of a graph  $G$  is universal if and only if  $P_f \cup P_g \subseteq B_f^s \cap B_g^s$ , for any MSIF  $g$ .

For example,

Let  $G = K_3$ .

Let  $f : V(G) \rightarrow [0, 1]$  be defined by

$$f(v_1) = 1, f(v_2) = f(v_3) = 0, \text{ where } V(K_3) = \{v_1, v_2, v_3\}.$$

$$f(N[v_i]) = 1, \forall i, 1 \leq i \leq 3$$

Therefore,  $f$  is a MSIF.

Let  $g$  be any MSIF on  $K_3$ .

$v_i \in P_f \cup P_g$  if and only if  $f(v_i) > 0$  (or)  $g(v_i) > 0$ .

$f(N[v_i]) = 1, \forall i, 1 \leq i \leq 3$ .

Therefore,  $v_i \in B_f^s$ .

Suppose  $g(v_i) = 0$ .

If  $g(v_j) = 1$ , for  $j \neq i$ , then  $g(N[v_j]) = 2$ , a contradiction.

Therefore,  $g(v_j) = 1$ , for exactly one  $j, j \neq i, 1 \leq j \leq 3$ .

Therefore,  $g(N[v_i]) = 1$ .

Therefore,  $v_i \in B_g$ . Therefore,  $v_i \in B_f^s \cap B_g^s$ .

Therefore,  $P_f \cup P_g \subseteq B_f^s \cap B_g^s$ .

Hence  $f$  is a UMSIF.

*Definition 1.5:* A function  $f : V(G) \rightarrow [0, 1]$  is **positive** if  $f(u) > 0$  for at least one  $u \in V(G)$ .

*Theorem 1.5:* Any SIF on  $K_3$  which is positive is a UMSIF.

**Proof:**

Let  $f$  be a SIF on  $K_3$  whose vertex set is  $\{v_1, v_2, v_3\}$ .

Suppose  $f$  is positive on  $K_3$ .

Let  $f(v_1) = \alpha > 0, f(v_2) = \beta$  and  $f(v_3) = \gamma$ .

Then  $\beta + \gamma + \alpha = 1$ .

Therefore,  $f(N[v_i]) = 1, \forall i, 1 \leq i \leq 3$ .

Therefore,  $f$  is a MSIF. Let  $g$  be a MSIF.

Suppose  $h_\lambda = \lambda f + (1 - \lambda)g$  is strong independent.

Then  $P_f \cup P_g \subseteq B_f^s \cap B_g^s$ .

Let  $h_\lambda(v_1) = 1$ . Therefore,  $\lambda f(v_1) + (1 - \lambda)g(v_1) = 0$ .

Therefore,  $f(v_1) = 0$  and  $g(v_1) = 0$ .

Therefore,  $f(N[v_1]) \geq 1$  and  $g(N[v_1]) \geq 1$ .

Therefore,  $h_\lambda(N[v_1]) \geq 1$ .

Therefore,  $h_\lambda$  is a MSIF.

Let  $0 < \lambda < 1$ . Let  $u \in P_f \cup P_g$ .

Clearly,  $f(N[u]) = 1$  and  $g(N[u]) = 1, \forall u \in V(K_3)$ .

Therefore,  $u \in B_f^s \cap B_g^s$ . Therefore  $B_f \cap B_g = V(K_3)$ .

Therefore,  $P_f \cup P_g \subseteq B_f^s \cap B_g^s$ .

Therefore, for any  $\lambda, 0 < \lambda < 1, h_\lambda$  is SIF.

Hence  $f$  is a UMSIF.

*Remark 1.7:* Any SIF on  $K_n$  which is positive is a UMSIF.

*Observation 1.6:* If  $f$  is a UMSIF, then  $B_f^s = V(G)$ .

**Proof:**

Let  $u \in V(G)$ .

Then  $\{u\}$  is strong independent and hence is contained in a maximal strong independent set, say  $D$  of  $G$ .

Therefore,  $\chi_D$  is a maximum strong independent function of  $G$ .

Since  $f$  is a UMSIF,  $P_f \cup P_{\chi_D} \subseteq B_f^s \cap B_{\chi_D}^s$ .

Since  $\chi_D(u) = 1, u \in P_{\chi_D}$ .

Therefore,  $u \in P_f \cup P_{\chi_D}$ .

Therefore,  $u \in B_f^s \cap B_{\chi_D}^s$ .

Therefore,  $u \in B_f^s$ .

Therefore,  $V(G) \subseteq B_f^s$ .

But  $B_f^s \subseteq V(G)$ .

Hence  $B_f^s = V(G)$ .

*Observation 1.7:* If there exist two MSIF<sup>s</sup>  $f$  and  $g$  such that  $B_f^s \cap B_g^s = \phi$ , then the graph  $G$  has no UMSIF.

**Proof:**

Suppose  $G$  has a UMSIF  $h$ .

Therefore, any convex combination of  $h$  and  $f$  is a MSIF.

Therefore,  $P_f \cup P_h \subseteq B_f^s \cap B_h^s$ .

Similarly,  $P_h \cup P_g \subseteq B_h^s \cap B_g^s$ .

Therefore,  $P_h \subseteq B_f^s \cap B_g^s$ .

$B_f^s \cap B_g^s = \phi$ , by hypothesis.

Therefore,  $P_h = \phi$ .

Since  $h$  is a MSIF,  $P_h \neq \phi$ , a contradiction.

Hence  $G$  has no UMSIF.

*Observation 1.8:* Let  $G$  be a regular bipartite graph with  $\delta(G) \geq 2$ . Then  $G$  has no UMSIF.

**Proof:**

Let  $G$  be a regular bipartite graph with  $\delta(G) \geq 2$ .

Let  $V_1$  and  $V_2$  be the bipartitions of  $V(G)$ .

Define  $f$  and  $g$  by  $f(v) = 1$ , if  $v \in V_1$  and

$$f(v) = 0, \text{ if } v \in V_2$$

$$g(v) = 1, \text{ if } v \in V_2 \text{ and}$$

$$g(v) = 0, \text{ if } v \in V_1.$$

Then  $f$  and  $g$  are MSIF<sup>s</sup> with  $B_f^s \cap B_g^s = \phi$ .

Therefore, by the above observation,  $G$  has no UMSIF.

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